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MINIMAL RANKINGS OF THE CARTESIAN PRODUCT $K_n \square K_m$

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Abstract

For a graph $G = (V, E)$, a function $f : V(G) \to \{1, 2, \ldots, k\}$ is a k-ranking if $f(u) = f(v)$ implies that every $u-v$ path contains a vertex w such that $f(w) > f(u)$. A k-ranking is minimal if decreasing any label violates the definition of ranking. The arank number, $\psi_r(G)$, of G is the maximum value of k such that G has a minimal k-ranking. We completely determine the arank number of the Cartesian product $K_n \square K_n$, and we investigate the arank number of $K_n \square K_m$ where $n > m$.

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1. Introduction

Let $G = (V, E)$ be an undirected graph with no loops and no multiple edges. A function $f : V(G) \to \{1, 2, \ldots, k\}$ is a (vertex) *k-ranking* of G if for $u, v \in V(G)$, $f(u) = f(v)$ implies that every $u - v$ path contains a vertex w such that $f(w) > f(u)$. By definition, every ranking is a proper coloring. The *rank number* of G, denoted $\chi_r(G)$, is the minimum value of k such that G has a k-ranking.

If the value of k is not important then f will be referred to simply as a ranking of G. A k-ranking is a *minimal k-ranking* of G if decreasing any label violates the ranking definition. The *arank number*, denoted $\psi_r(G)$, is defined to be the maximum value of k for which G has a minimal k-ranking [4].

Interest in rankings of graphs [2, 6, 7, 15] was sparked by their many applications to other fields including designs of very large scale integration layouts (VLSI), Cholesky factorizations of matrices in parallel, and scheduling problems of assembly steps in manufacturing systems [3, 11, 12, 14]. Many papers have appeared on the topic of minimal rankings. Bodlaender *et al.* [1] established that $\chi_r(P_n) = \lfloor \log_2 n \rfloor + 1$. It has been shown that a k-ranking for $P_n = v_1 v_2 \ldots v_n$, where $k = \chi_r(P_n)$, can be obtained by labeling v_i by $\gamma + 1$ where 2^γ is the largest power of 2 that divides i. In this paper this particular scheme of ranking will be referred as a standard ranking. Laskar and Pillone considered some complexity issues of minimal rankings as well as properties of minimal rankings [5, 9, 10]. Narayan *et al.* studied minimal rankings of paths [8] and more properties of minimal ranking [13].

In this paper we study minimal rankings of the Cartesian product $K_n \Box K_m$. The Cartesian product of two graphs G and H, denoted by $G \Box H$, is the graph with vertex set $V(G) \times V(H)$ and has the property that two vertices (a, b) and (x, y) are adjacent if and only if either $a = x$ and $by \in E(H)$, or $b = y$ and $ax \in E(G)$. We also use the following definitions throughout this paper. For a ranking f, if $f(x) = f(y)$ implies $x = y$ then the label is *distinct*; otherwise it is a *repeated* label. We use a rectangle with n rows and m columns to represent $K_n \Box K_m$. Let P be the path $uz_1 z_2 \ldots z_k v_{i,j} z_{k+1} \ldots z_r v$. We use the notation $P - \{v_{i,j}\}$ to represent the path $uz_1 z_2 \ldots z_k z_{k+1} \ldots z_r v$.

We conclude this section with some known results on minimal rankings.

Lemma 1 [4]**.** *Let f be a minimal k-ranking. Then $|S_1| \geq |S_2| \geq \cdots \geq |S_k|$ where $S_i = \{x | f(x) = i\}$ for $1 \leq i \leq k$.*

Theorem 2 [5]**.** *A k-ranking f is minimal if and only if for all v with $f(v) = a > 1$ and for each p such that $1 \leq p < a$, one of the following is true.*

1. *There exist vertices x and y with $f(x) = f(y) \geq p$ and v is the only vertex on some $x - y$ path such that $f(v) > f(y)$.*
2. *There exists a vertex w with $f(w) = p$ and there exists a $v - w$ path such that for every vertex x on the path, $f(x) \leq f(w)$.*

We completely determine the arank number of $K_n \Box K_n$ and we investigate the arank number of $K_n \Box K_m$ where $n > m$.

2. MINIMAL RANKING OF $K_n \Box K_n$

We start by considering minimal rankings of $K_n \Box K_n$.

Theorem 3. $\psi_r(K_n \Box K_n) \geq n^2 - n + 1$.

Proof. Consider the vertex labeling f of $K_n \Box K_n$ defined as

$$f(v_{i,j}) = \begin{cases} j & \text{if } i = 1, \\ n - (i-1) & \text{if } j = n, \\ (i-1)(n-1) + j + 1 & \text{otherwise.} \end{cases}$$

Note that f uses $n^2 - n + 1$ labels. Labels $1, 2, \ldots, n-1$ appear twice, occurring once in the first row and once in the last column. Thus, any path between vertices with the same label will either have the label n or have a label larger than n and hence f is a ranking.

Now we will show that f is minimal. Consider $f(x) > 1$ and let $x = v_{i,j}$. If $f(x) \leq n$, then x is adjacent to vertices labeled $1, 2, \ldots, f(x) - 1$, and hence the second conclusion of Theorem 2 is satisfied. Suppose $f(x) > n$. If $1 \leq p < n$, then $v_{2,n} - v_{i,n} - v_{i,j} - v_{1,j} - v_{1,n-1}$ is a path between $v_{2,n}$ and $v_{1,n-1}$ where $f(v_{2,n}) = f(v_{1,n-1}) = n-1$ and x is the only vertex in the path with $f(x) > n-1$. This satisfies the first conclusion of Theorem 2. Now suppose $n \leq p < f(x)$. The path $v_{k,l} - v_{1,l} - v_{1,j} - v_{i,j}$, where $f(v_{k,l}) = p$, is a path from $v_{k,l}$ to x such that every vertex in the path, other than the end vertices, has a label less than p. This means the second conclusion of Theorem 2 is satisfied.

Therefore, by Theorem 2, f is a minimal ranking and $\psi_r(K_n \Box K_n) \geq |f| = n^2 - n + 1$. An example of this labeling scheme is shown in Figure 1. ∎

1	2	3	4	5
6	7	8	9	4
10	11	12	13	3
14	15	16	17	2
18	19	20	21	1

Figure 1. A minimal ranking with $n^2 - n + 1$ labels for $K_n \Box K_n$ when $n = 5$.

Theorem 4. *Let f be a minimal ranking of $K_n \Box K_n$. Then every row and every column of $K_n \Box K_n$ contains a repeated label and a distinct label under f.*

Proof. First we will show that every row of $K_n \Box K_n$ has a repeated label under f. On the contrary assume $K_n \Box K_n$ has a row i which does not contain a repeated label. That is, for $j = 1, 2, \ldots, n$, we have $f(v_{i,j}) > t$, where t is the largest repeated label. Let $a = f(v_{i,j})$ for some $1 \leq j \leq n$. f is a minimal ranking and thus one of the conclusions of Theorem 2 must be true for every k such that $1 \leq k < a$.

Suppose for some $1 \leq k < a$, the first conclusion of Theorem 2 is true and let P be such a path. Since all vertices of row i have labels greater than t, P does not contain any vertices from row i other than $v_{i,j}$. This implies that $P' = P - \{v_{i,j}\}$ is a path from x to y such that $f(z) \leq f(x)$ for all $z \in V(P')$. This is a contradiction because f is a ranking. Thus the second conclusion of Theorem 2 must be true for all $1 \leq k < a$.

Suppose $k = 1$. Then $v_{i,j}$ must be adjacent to a vertex labeled 1. This implies that for every j such that $1 \leq j \leq n$, $v_{i,j}$ is adjacent to a vertex labeled 1. This is not possible because row i does not have a vertex labeled 1 and no row can have two vertices labeled 1. Thus f is not minimal, which is a contradiction. Hence every row of $K_n \square K_n$ contains a repeated label under f. Using similar arguments we can show that every column of $K_n \square K_n$ has a repeated label.

We will now show that every row and column of $K_n \square K_n$ has a distinct label. Again, on the contrary assume row i contains only repeated labels. Let $v_{i,j}$ have the largest label in row i. Since $f(v_{i,j})$ is a repeated label, let $v_{k,l}$ be such that $f(v_{k,l}) = f(v_{i,j})$. Now, since $v_{i,j}$ has the largest repeated label in row i it follows that $f(v_{i,l}) < f(v_{i,j})$, and thus the path $v_{i,j} - v_{i,l} - v_{k,l}$ does not have any vertex labeled higher than $f(v_{i,j})$. This is a contradiction and hence every row must have a vertex with distinct label. Using similar arguments we can show that every column of $K_n \square K_n$ contains a distinct label. ■

Lemma 5. *Let f be a minimal ranking of $K_n \square K_n$. Also, let t be the largest repeated label in f and $S_i = \{v | f(v) = i\}$. If $t = n - 1 - k$, where $k \geq 0$, then $\sum_{i=1}^{t} |S_i| \geq 2n - (k + 2)$.*

Proof. Let $t = n - 1 - k$, where $k \geq 0$. We want to show that $\sum_{i=1}^{t} |S_i| \geq 2n - (k+2)$. On the contrary, assume that $\sum_{i=1}^{t} |S_i| \leq 2n - (k+3)$. By Theorem 4, every row and every column of $K_n \square K_n$ has a repeated label. Suppose there are δ_r rows with exactly one repeated label. This implies that $n - \delta_r$ rows have at least two repeated label vertices. Thus we have,

$$2n - (k + 3) \geq \sum_{i=1}^{t} |S_i| \geq \delta_r + 2(n - \delta_r) = 2n - \delta_r. \tag{1}$$

It follows from Equation (1) that $\delta_r \geq k + 3$. Similarly, if δ_c is the number of columns with exactly one repeated label, then $\delta_c \geq k + 3$.

Case 1. Among the δ_r rows and δ_c columns, there exists a row i and a column j such that $v_{i,j}$ has a repeated label and $f(v_{i,j}) > 1$.

Note that in this case all other labels in row i and column j are distinct labels. Consider the function g defined as follows:

$$g(v_{k,l}) = \begin{cases} 1 & \text{if } k = i \text{ and } l = j, \\ f(v_{k,l}) & \text{otherwise.} \end{cases}$$

Since f is a minimal ranking, g is not a ranking. This means that there exist $u, v \in V(G)$ and a path P between u and v such that $g(u) = g(v)$ and $g(z) \leq g(u)$ for every $z \in V(P)$. Since f is a ranking and $g(z) = f(z)$ for every $z \neq v_{i,j}$, we have $v_{i,j} \in V(P)$. Let z be the vertex adjacent to $v_{i,j}$ in P. However, z is in row i or column j which means z is a vertex with a distinct label under f and thus $g(z) = f(z) > t \geq f(u) \geq g(u)$. This is a contradiction.

Case 2. Among the δ_r rows and δ_c columns, there does not exist a row i and a column j such that $v_{i,j}$ has a repeated label and $f(v_{i,j}) > 1$.

Note that if $|S_1| \geq k+2$, then $\sum_{i=1}^{t} |S_i| \geq k+2+\sum_{i=2}^{t} |S_i| \geq k+2+2(t-1) = k+2+2(n-2-k) = 2n-(k+2)$, which is a contradiction.

Thus the number of vertices with label 1 is at most $k+1$. We know that there are $\delta_r \geq k+3$ rows and $\delta_c \geq k+3$ columns with exactly one repeated label. Thus there exist at least two rows among the δ_r rows and at least two columns among the δ_c columns with a repeated label greater than 1.

Claim. *Since we assumed that $\sum_{i=1}^{t} |S_i| \leq 2n-(k+3)$ and Case 1 does not hold, it follows that one of the following is true.*

(1) *There exist a row i, among the δ_r rows, and a column j, such that $v_{i,j}$ has a repeated label with $f(v_{i,j}) > 1$ and column j does not contain a vertex labeled 1.*

(2) *There exist a column j, among the δ_c columns, and a row i, such that $v_{i,j}$ has a repeated label with $f(v_{i,j}) > 1$ and row i does not contain a vertex labeled 1.*

Proof. Suppose neither of these statements are true and Case 1 is not true. Then every vertex with a repeated label greater than 1 either is in the same row or column as a vertex with label 1, or is in the same row as another vertex with repeated label greater than 1 and in the same column as another vertex with repeated label greater than 1.

There are $n-|S_1|$ rows and $n-|S_1|$ columns without a vertex labeled 1, because the vertices labeled 1 must be in different rows and different columns. Each of these rows and columns must contain at least one vertex with repeated label greater than 1. Let m_r be the number of such rows that do not contain a vertex with repeated label greater than 1 in the same column as a vertex labeled 1, and m_c be the number of such columns that do not contain a vertex with repeated label greater than 1 in the same row as a vertex labeled 1. Without loss of generality assume $m_r \geq m_c$.

Every row among the m_r rows contains at least two vertices with repeated label greater than 1 (otherwise the first statement in the claim would be true). Also there are at least $n - |S_1| - m_r$ vertices with repeated label greater than 1 that have a vertex labeled 1 in the same column, but not the same row, and at least $n-|S_1|-m_c$ vertices with repeated label greater than 1 that have a vertex labeled 1 in the same row, but not the same column. Therefore,

$$\sum_{i=1}^{t} |S_i| = |S_1| + \sum_{i=2}^{t} |S_i| \geq |S_1| + 2m_r + (n - |S_1| - m_r) + (n - |S_1| - m_c) \geq 2n - |S_1| \geq 2n - (k+1),$$

which is a contradiction. □

Now, without loss of generality, assume condition (1) is true. Note that every vertex in row i other than $v_{i,j}$ is a distinct label vertex. Define g as follows:

$$g(v_{k,l}) = \begin{cases} 1 & \text{if } k = i \text{ and } l = j, \\ f(v_{k,l}) & \text{otherwise.} \end{cases}$$

Since f is a minimal ranking, g is not a ranking. This means there exist $u, v \in V(G)$ and a path P between u and v such that $g(u) = g(v)$ and $g(z) \leq g(u)$ for every $z \in V(P)$. As in Case 1 we must have $v_{i,j} \in V(P)$. Since row i and column j do not contain a vertex labeled 1 under f, row i and column j do not contain a vertex labeled 1 other than $v_{i,j}$ under g.
Therefore, if $u = v_{i,j}$, then P contains at least one vertex with a label greater than 1 which is a contradiction. Therefore, assume $P = uz_1z_2 \ldots z_k v_{i,j} z_{k+1} \ldots z_r v$. Then z_k and z_{k+1} are in column j (because every vertex in row i, except $v_{i,j}$, has a higher label than t and $g(z) \leq g(u)$ for every $z \in V(P)$). Thus $P' = P - \{v_{i,j}\}$ is a path from u to v and $g(z) = f(z)$ for all $z \in V(P')$. Therefore, we have $f(z) = g(z) \leq g(u) = f(u)$ for all $z \in V(P')$, which contradicts that fact that f is a ranking.

Thus, in both cases we get a contradiction, and hence $\sum_{i=1}^{t} |S_i| \geq 2n-(k+2)$. ∎

Theorem 6. $\psi_r(K_n \square K_n) = n^2 - n + 1$.

Proof. Let f be a minimal k-ranking of $K_n \square K_n$ and let t be the largest repeated label in f. Let $S_i = \{v | f(v) = i\}$. If $t > n-1$, then we have $k = n^2 - \sum_{i=1}^{t} |S_i| + t \leq n^2 - 2t + t = n^2 - t < n^2 - (n-1)$.

Suppose $t \leq n-1$. Let $t = n-1-r$, where $r \geq 0$. By Lemma 5, we have $\sum_{i=1}^{t} |S_i| \geq 2n - (r+2)$. Thus, $k = n^2 - \sum_{i=1}^{t} |S_i| + t \leq n^2 - (2n - (r+2)) + n - 1 - r = n^2 - n + 1$.

This means $\psi_r(K_n \square K_n) \leq n^2 - n + 1$, and thus applying Theorem 3 we get $\psi_r(K_n \square K_n) = n^2 - n + 1$. ∎

3. Minimal Ranking of $K_n \square K_m$, where $n > m$

Note that Theorem 4 does not hold for some minimal rankings of $K_n \square K_m$ as shown in Figures 2 and 3. However, for any minimal ranking of $K_n \square K_m$, every row and column has a distinct label.

1	6	5
7	8	1
9	10	2
11	1	3
2	12	4

Figure 2. A minimal 12-ranking of $K_5 \square K_3$ where every row and column has a repeated label.

12	11	10
1	5	6
2	7	1
3	1	8
4	2	9

Figure 3. A minimal 12-ranking of $K_5 \square K_3$ where one row has no repeated labels.

Theorem 7. *Let $n > m + \lfloor \log_2 m \rfloor$. Then*

$$\psi_r(K_n \square K_m) \geq nm - \textstyle\sum_{i=0}^{\lfloor \log_2 m \rfloor} \left\lceil \frac{m}{2^i} \right\rceil + \lfloor \log_2 m \rfloor + 1.$$

Proof. Let $k = nm - \sum_{i=0}^{\lfloor \log_2 m \rfloor} \left\lceil \frac{m}{2^i} \right\rceil + \lfloor \log_2 m \rfloor + 1$. Let P be the path $v_{1,1}v_{2,1}v_{2,2}$ $v_{3,2} \ldots v_{m,m}v_{m+1,m}$ on $2m$ vertices. Use the standard ranking of P_{2m} to label the vertices on P. Note that the number of vertices on P with label i is $\left\lfloor \frac{2m+2^{i-1}}{2^i} \right\rfloor = \left\lfloor \frac{m}{2^{i-1}} + \frac{1}{2} \right\rfloor = \left\lceil \frac{m}{2^{i-1}} \right\rceil$ or $\left\lceil \frac{m}{2^{i-1}} \right\rceil - 1$. For $1 < i \leq \lfloor \log_2 m \rfloor + 1$, if the number of times label i appears in P is less than $\left\lceil \frac{m}{2^{i-1}} \right\rceil$, then label $v_{m+i,m}$ with label i. Label the other vertices of $K_n \square K_m$ using labels $\lfloor \log_2 2m \rfloor + 2, \ldots, k$ without repeating any of these labels. This produces a k-ranking (verification left to the reader) of $K_n \square K_m$ with k labels. This ranking has the property that for every $i > 1$, if a vertex v is labeled i then for every $1 \leq j < i$ there is a vertex w labeled j and a $v - w$ path such that every vertex in the path has a label less than j. This means that the second conclusion of Theorem 2 is satisfied, and hence this is a minimal ranking.

An example of such a minimal ranking is shown in Figure 4. ∎

Theorem 8 [4]**.** *Let f be a minimal k-ranking of a graph G. Then $|S_1| \geq |S_2| \geq \cdots \geq |S_k|$, where $S_i = \{v \in V(G) \mid f(v) = i\}$.*

Theorem 9. *Let f be a minimal k-ranking of $K_n \square K_m$ where $n > m\lfloor \log_2 m \rfloor + 1$. If there is a row with no repeated label, then $k \leq nm - \sum_{i=0}^{\lfloor \log_2 m \rfloor} \left\lceil \frac{m}{2^i} \right\rceil + \lfloor \log_2 m \rfloor + 1$.*

1	5	6	7	8
2	1	9	10	11
12	3	1	13	14
15	16	2	1	17
18	19	20	4	1
21	22	23	24	2
25	26	27	28	29
30	31	32	33	3

Figure 4. Minimal ranking of $K_8 \Box K_5$ using the labeling scheme in the proof of Theorem 7.

Proof. Let f be a minimal ranking of $K_n \Box K_m$, such that there is a row with no repeated labels.

Case 1. There is a row r such that r does not have any repeated labels and every label in row r is larger than $\lfloor \log_2 m \rfloor + 1$.

Since f is minimal and row r has no repeated labels, as in the proof of Theorem 4, for every label in row r, the second conclusion of Theorem 2 must be true. This means, by letting $p = 1$ in Theorem 2, every vertex in row r must be adjacent to a vertex labeled 1, which means f must have m 1's, one in each column. Now, (by letting $p = 2$ in Theorem 2), every vertex in row r must be either adjacent to a vertex labeled 2, or must be adjacent to a vertex labeled 1 which is adjacent to a vertex labeled 2. This means a vertex labeled 2 can account for at most 2 vertices in row r. This means, the number of 2's must be at least $\lceil \frac{m}{2} \rceil$. In general, for $1 \leq i \leq \lfloor \log_2 m \rfloor + 1$, f must have at least $\lceil \frac{m}{2^{i-1}} \rceil$ vertices labeled i. Therefore $k \leq nm - \sum_{i=0}^{\lfloor \log_2 m \rfloor} \lceil \frac{m}{2^i} \rceil + \lfloor \log_2 m \rfloor + 1$.

Case 2. Every row that does not have any repeated labels has a label less than or equal to $\lfloor \log_2 m \rfloor + 1$.
Among all rows without repeated labels, let r be the row that has the largest label z such that $z \leq \lfloor \log_2 m \rfloor + 1$. This means that every row other than r must have a repeated label or a label less than z. However, by Theorem 8, any repeated label must be less than z. Therefore every row other than r has at least one label less than z. However, since there are only m columns, there are at most m vertices with any label l. Therefore, the number of vertices with label less than z is at most $m(z-1)$. This means, since every row other than r has a label less than z, we have, $n \leq m(z-1) + 1 \leq m\lfloor \log_2 m \rfloor + 1$.

However, we assumed that $n > m\lfloor \log_2 m \rfloor + 1$. Therefore we have a contradiction, and thus Case 2 does not exist. ∎

Theorem 10. *Let $n \geq 4m$ and $n > m\lfloor \log_2 m \rfloor + 1$. Then*

$$\psi_r(K_n \Box K_m) = nm - \sum_{i=0}^{\lfloor \log_2 m \rfloor} \left\lceil \tfrac{m}{2^i} \right\rceil + \lfloor \log_2 m \rfloor + 1.$$

Proof. Let f be a minimal k-ranking of $G = K_n \Box K_m$. Suppose f has repeated labels in every row of G. Since we are trying to maximize the number of labels used, in the best case, f has two or three vertices with label 1, and two vertices with each of the labels $2, 3, \ldots, t$ where t is the largest repeated label under f, and also has exactly one repeated label in each row. Then $k \le mn - \lfloor n/2 \rfloor$. However,

$$\begin{aligned}\sum_{i=0}^{\lfloor \log_2 m \rfloor} \left\lceil \tfrac{m}{2^i} \right\rceil - \lfloor \log_2 m \rfloor - 1 \le\ & \sum_{i=0}^{\lfloor \log_2 m \rfloor} \left(\tfrac{m}{2^i} + 1 \right) - \lfloor \log_2 m \rfloor - 1 \\ =\ & \sum_{i=0}^{\lfloor \log_2 m \rfloor} \tfrac{m}{2^i} + \lfloor \log_2 m \rfloor + 1 - \lfloor \log_2 m \rfloor - 1 \\ =\ & \sum_{i=0}^{\lfloor \log_2 m \rfloor} \tfrac{m}{2^i} \le 2m \le \lfloor n/2 \rfloor, \text{ because } n \ge 4m.\end{aligned}$$

Therefore, we have $k \le mn - \lfloor n/2 \rfloor \le nm - \sum_{i=0}^{\lfloor \log_2 m \rfloor} \left\lceil \frac{m}{2^i} \right\rceil + \lfloor \log_2 m \rfloor + 1$.

Hence, $\psi_r(K_n \Box K_m) \le nm - \sum_{i=0}^{\lfloor \log_2 m \rfloor} \left\lceil \frac{m}{2^i} \right\rceil + \lfloor \log_2 m \rfloor + 1$ and by applying Theorem 7, we get $\psi_r(K_n \Box K_m) = nm - \sum_{i=0}^{\lfloor \log_2 m \rfloor} \left\lceil \frac{m}{2^i} \right\rceil + \lfloor \log_2 m \rfloor + 1$, if $n \ge 4m$ and $n > m\lfloor \log_2 m \rfloor + 1$. ■

The cases where $m < n < 4m$ or $m < n \le m\lfloor \log_2 m \rfloor + 1$ seems to be more difficult to solve. When $G = K_7 \Box K_6$ we have a minimal k-ranking where $k = 36$, as shown in Figure 5, thus making the bound in Theorem 9 not valid for this case. To show that $\psi_r(K_7 \Box K_6) = 36$, we will have to consider many cases depending on the number of vertices with each label and the positions where each of these labels appear. This approach does not appear to be feasible for $K_n \Box K_m$ as the number of cases increases rapidly as n increases.

8	1	9	10	11	7
1	2	3	4	5	6
12	13	14	15	16	5
17	18	19	20	21	4
22	23	24	25	26	3
27	28	29	30	31	2
32	33	34	35	36	1

Figure 5. A minimal ranking of $K_7 \Box K_6$ using 36 labels.

We state an improvement of Theorem 10 in the following conjecture.

Conjecture 11. *Let $n > m + \lfloor \log_2 m \rfloor$. Then*

$$\psi_r(K_n \Box K_m) = nm - \sum_{i=0}^{\lfloor \log_2 m \rfloor} \left\lceil \tfrac{m}{2^i} \right\rceil + \lfloor \log_2 m \rfloor + 1.$$

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