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DOUBLE DOMINATION CRITICAL AND STABLE GRAPHS UPON VERTEX REMOVAL [1]

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Abstract

In a graph a vertex is said to dominate itself and all its neighbors. A double dominating set of a graph G is a subset of vertices that dominates every vertex of G at least twice. The double domination number of G, denoted $\gamma_{\times 2}(G)$, is the minimum cardinality among all double dominating sets of G. We consider the effects of vertex removal on the double domination number of a graph. A graph G is $\gamma_{\times 2}$-vertex critical graph ($\gamma_{\times 2}$-vertex stable graph, respectively) if the removal of any vertex different from a support vertex decreases (does not change, respectively) $\gamma_{\times 2}(G)$. In this paper we investigate various properties of these graphs. Moreover, we characterize $\gamma_{\times 2}$-vertex critical trees and $\gamma_{\times 2}$-vertex stable trees.

Keywords: double domination, vertex removal critical graphs, vertex removal stable graphs..

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1. Introduction

For the terminology and notation of graph theory not given here, the reader is referred to [3, 6]. Let $G = (V(G), E(G))$ be a simple graph. The *open neighborhood* of a vertex $v \in V(G)$ is $N(v) = N_G(v) = \{u \in V(G) : uv \in E(G)\}$, and its *closed neighborhood* $N[v] = N_G[v] = N(v) \cup \{v\}$. The *degree* of v, denoted by $\deg_G(v)$, is the size of its open neighborhood. A vertex of degree one is called a *leaf*, and its neighbor is called a *support vertex*. We also denote the set of leaves of a graph G by $L(G)$ and the set of support vertices by $S(G)$. If $D \subseteq V(G)$, then the subgraph induced by D in G is denoted by $G[D]$. A tree T is a *double star* if it contains exactly two vertices that are not leaves. A *subdivided star* $K^*_{1,t}$ is a tree obtained from a star $K_{1,t}$ by replacing each edge uv of $K_{1,t}$ by a vertex w and edges uw and vw. For a vertex v in a rooted tree T, we denote by $C(v)$ and $D(v)$ the set of *children* and *descendants*, respectively, of v. The *maximal subtree* at v is the subtree of T induced by $D(v) \cup \{v\}$, and is denoted by T_v.

A subset S of $V(G)$ is a *double dominating set* (abbreviated DDS) of G if for every vertex $v \in V(G)$, $|N[v] \cap S| \geqslant 2$, that is, v is in S and has at least one neighbor in S or v is in $V(G) - S$ and has at least two neighbors in S. The *double domination number* $\gamma_{\times 2}(G)$ is the minimum cardinality among all double dominating sets of G. If S is a DDS of G of size $\gamma_{\times 2}(G)$, then we call S a $\gamma_{\times 2}(G)$-*set*. Clearly, double domination is defined only for graphs without isolated vertices. Double domination was introduced by Harary and Haynes in [5].

In this paper, we are interested in studying the effect that a graph modification has on the double domination number. More precisely, we first study graphs for which the double domination number decreases on the removal of any vertex. Then we study graphs for which the double domination number remains unchanged on the removal of any vertex. We note that for the same parameter, Khelifi *et al.* studied in [7] graphs that are critical under the deletion of any edge and Chellali and Haynes [4] studied graphs that are stable under the deletion of any edge.

2. Preliminary Results

We begin by some useful observations.

Observation 1. *Every DDS of a graph contains all its leaves and support vertices.*

Observation 2. *Let G be a graph without isolated vertices. Then $\gamma_{\times 2}(G) = |V(G)|$ if and only if every vertex of G is either a leaf or a support vertex.*

Proof. Let G be a graph such that $\gamma_{\times 2}(G) = |V(G)|$. Assume that v is a vertex of G that is neither a leaf nor a support vertex. Then clearly $\deg_G(v) \geq 2$ and $V(G) - \{v\}$ is DDS for G, a contradiction. Hence every vertex of G is either a leaf or a support vertex.

The converse is obvious. ■

Our next result consists of the effect of deleting any vertex in G different from a support vertex on the double domination number of a graph.

Theorem 3. *Let G be a connected graph of order at least three. Then for every vertex v different from a support vertex, $\gamma_{\times 2}(G) - 2 \leq \gamma_{\times 2}(G - v) \leq \gamma_{\times 2}(G) + \deg_G(v) - 1$.*

Proof. We first establish the lower bound. Let D be any $\gamma_{\times 2}(G - v)$-set. If $D \cap N(v) \neq \emptyset$, then $D \cup \{v\}$ is a DDS for G, and if $D \cap N(v) = \emptyset$, then $D \cup \{v, u\}$ is a DDS for G, where u is any vertex in $N(v)$. In both cases, we have $\gamma_{\times 2}(G) \leq |D| + 2$. Now let R be any $\gamma_{\times 2}(G)$-set. Clearly if $v \notin R$, then R is a DDS of $G - v$ and so $\gamma_{\times 2}(G - v) \leq \gamma_{\times 2}(G)$. Thus we may assume that $v \in R$. Let B be the set of vertices in $V - R$ for which v is necessary to be dominated twice, that is, each vertex of B has exactly one neighbor in $R - v$. Let also A be the set of all vertices in R for which v is the unique neighbor in R. Since v is not a support vertex, each vertex of A has a neighbor in $V - R$. Let A' be a smallest subset of $V - R$ that dominates A. Clearly $|A'| \leq |A|$ and $|A'| + |B| \leq \deg_G(v)$. Since $A' \cup B \cup R - \{v\}$ is a DDS for $G - v$, we obtain $\gamma_{\times 2}(G - v) \leq |R| - 1 + |A'| + |B| \leq \gamma_{\times 2}(G) + \deg_G(v) - 1$. ■

The following corollary is immediate.

Corollary 4. *Let G be a connected graph of order at least three and let $v \in V(G) - S(G)$. Then*

(a) $\gamma_{\times 2}(G - v) \leq \gamma_{\times 2}(G) + \Delta(G) - 1$.

(b) *If v is a leaf, then $\gamma_{\times 2}(G - v) \leq \gamma_{\times 2}(G)$.*

The following result shows that there is no graph G such that the deletion of every vertex increases the double domination number.

Proposition 5. *There is no graph G such that $\gamma_{\times 2}(G - v) > \gamma_{\times 2}(G)$ for every vertex $v \in V(G) - S(G)$.*

Proof. Assume that G is a graph such that $\gamma_{\times 2}(G - v) > \gamma_{\times 2}(G)$ for every vertex $v \in V(G) - S(G)$. Then by Corollary 4(b), G contains no leaf and so by Observation 2, $\gamma_{\times 2}(G) < |V(G)|$. Now let D be any $\gamma_{\times 2}(G)$-set and let $w \in V(G) - D$. Then clearly D is a DDS for $G - w$, implying that $\gamma_{\times 2}(G - w) \leq |D| = \gamma_{\times 2}(G)$, a contradiction. ■

According to Theorem 3 and Proposition 5 we define a graph G to be:

- a $\gamma_{\times 2}$-*vertex critical graph* if for every vertex $v \in V(G)-S(G)$, $\gamma_{\times 2}(G-v) < \gamma_{\times 2}(G)$. It follows from Theorem 3 that in a $\gamma_{\times 2}$-vertex critical graph G, $\gamma_{\times 2}(G-v) = \gamma_{\times 2}(G) - 2$ or $\gamma_{\times 2}(G-v) = \gamma_{\times 2}(G) - 1$ for every vertex $v \in V(G) - S(G)$.
- a $\gamma_{\times 2}$-*vertex stable graph* if for every vertex $v \in V(G)-S(G)$, $\gamma_{\times 2}(G-v) = \gamma_{\times 2}(G)$.

3. $\gamma_{\times 2}$-vertex Critical Graphs

We begin by giving some useful properties of $\gamma_{\times 2}$-vertex critical graphs.

Proposition 6. *If G is a $\gamma_{\times 2}$-critical graph and $v \in V(G) - (S(G) \cup L(G))$, then every $\gamma_{\times 2}(G-v)$-set contains at most one neighbor of v.*

Proof. Suppose that D is a $\gamma_{\times 2}(G-v)$-set containing at least two neighbors of v. Then D is a DDS of G and so $\gamma_{\times 2}(G) \le \gamma_{\times 2}(G-v)$, contradicting the fact that G is $\gamma_{\times 2}$-vertex critical. ■

As a consequence of Proposition 6, we have.

Corollary 7. *If G is a path P_n with $n \ge 5$, or a cycle C_n, or a complete graph K_m with $m \ge 3$, then G is not $\gamma_{\times 2}$-vertex critical.*

A vertex of a graph G is said to be *free* if it does not belong to any minimum double dominating set of G.

Proposition 8. *Let G be a $\gamma_{\times 2}$-vertex critical graph and v a vertex of $V(G) - S(G)$. If there is a vertex $w \in N_G(v)$ belonging to every $\gamma_{\times 2}(G-v)$-set, then the following conditions hold.*

(1) $\gamma_{\times 2}(G-v) = \gamma_{\times 2}(G) - 1$.

(2) *Every vertex in $N_G(v) - \{w\}$ is free in $G - v$.*

Proof. Let G be a $\gamma_{\times 2}$-vertex critical graph and v any vertex of $V(G) - S(G)$. Since w is in every $\gamma_{\times 2}(G-v)$-set, then such a set plus v is a DDS of G and so $\gamma_{\times 2}(G-v) < \gamma_{\times 2}(G) \le \gamma_{\times 2}(G-v) + 1$, implying that $\gamma_{\times 2}(G-v) = \gamma_{\times 2}(G) - 1$. Now since w belongs to every $\gamma_{\times 2}(G-v)$-set, then by Proposition 6, every vertex of $N_G(v) - \{w\}$ is free in $G - v$. ■

Proposition 9. *Let G be a graph and $v \in V(G) - (S(G) \cup L(G))$. Then if $\gamma_{\times 2}(G-v) = \gamma_{\times 2}(G) - 2$, then every neighbor of v is free in $G - v$.*

Proof. Suppose that a neighbor of v, say u, is not free in $G-v$. Thus u belongs to some $\gamma_{\times 2}(G-v)$-set D and so $D\cup\{v\}$ is a DDS of G. It follows that $\gamma_{\times 2}(G-v)+2 = \gamma_{\times 2}(G) \leq |D|+1 = \gamma_{\times 2}(G-v)+1$, a contradiction. Hence u is a free vertex in $G-v$. ■

In the next, we will focus on $\gamma_{\times 2}$-vertex critical trees. We begin by showing that the removal of any free vertex from a tree T leaves the double domination number unchanged. We note that free vertices of any tree can be determined in polynomial time [2].

Lemma 10. *If v is a free vertex of a nontrivial tree T, then $\gamma_{\times 2}(T-v) = \gamma_{\times 2}(T)$.*

Proof. Let v be a free vertex of T. Clearly $v \notin S(T) \cup L(T)$ and so $\deg_T(v) = k \geq 2$. We root T at v and let $C(v) = \{u_1, u_2, \ldots, u_k\}$. Since v is free, every $\gamma_{\times 2}(T)$-set is a DDS for $T-v$, and so $\gamma_{\times 2}(T-v) \leq \gamma_{\times 2}(T)$. Furthermore, since every $\gamma_{\times 2}(T-v)$-set can be extended to a $\gamma_{\times 2}(T)$-set by adding v and one of its neighbors, $\gamma_{\times 2}(T) \leq \gamma_{\times 2}(T-v)+2$. Hence we have $\gamma_{\times 2}(T)-2 \leq \gamma_{\times 2}(T-v) \leq \gamma_{\times 2}(T)$. Let us first assume that $\gamma_{\times 2}(T-v) = \gamma_{\times 2}(T)-2$ and let S_1 be any $\gamma_{\times 2}(T-v)$-set. If $N_T(v)\cap S_1 \neq \emptyset$, then $S_1\cup\{v\}$ is a DDS of T with $\gamma_{\times 2}(T)-1$ vertices, which is impossible. Hence $N_T(v)\cap S_1 = \emptyset$ but then $S_1\cup\{u_1, v\}$ is a $\gamma_{\times 2}(T)$-set containing v, contradicting the fact that v is free. Assume now that $\gamma_{\times 2}(T-v) = \gamma_{\times 2}(T)-1$ and let S' be any $\gamma_{\times 2}(T-v)$-set. Clearly S' contains no neighbor of v, otherwise $S'\cup\{v\}$ would be a $\gamma_{\times 2}(T)$-set containing v, a contradiction. Let $S'_i = S'\cap V(T_{u_i})$ for $1 \leq i \leq k$. Then $|S'| = |S'_1|+|S'_2|+\cdots+|S'_k|$ and obviously S'_i is a $\gamma_{\times 2}(T_{u_i})$-set for each i. Now let D be a $\gamma_{\times 2}(T)$-set and $D_i = D\cap V(T_{u_i})$ for every i. Recall that $v \notin D$ since v is free. We claim that D contains exactly two neighbors of v. Suppose to the contrary that $|N_T(v)\cap D| \geq 3$, and assume, without loss of generality, that $u_1, u_2, u_3 \in D$. If for some i, $|D_i| < |S'_i|$, then D_i would be a DDS for T_{u_i} smaller than S'_i which is impossible. Thus $|D_i| \geq |S'_i|$ for every i, implying that $|D_i| = |S'_i|$ for $i \geq 4$. Now using the fact $|D| = |S'|+1$, we deduce that $|D_1|+|D_2|+|D_3| = |S'_1|+|S'_2|+|S'_3|+1$. Without loss of generality, we assume that $|D_1| = |S'_1|+1$. It follows that $S'_1\cup D_2\cup\cdots\cup D_k$ is a DDS of T of cardinality less than $|D|$, a contradiction. Hence $|D\cap N_T(v)| = 2$, say $u_1, u_2 \in D$. As seen before $|D_i| = |S'_i|$ for $i \geq 3$, and so $|D_1|+|D_2| = |S'_1|+|S'_2|+1$. We can suppose that $|D_1| = |S'_1|+1$. Then $S'_1\cup D_2\cup\cdots\cup D_k$ is a DDS of $T-v$ with $|S'_1\cup D_2\cup\cdots\cup D_k| = \gamma_{\times 2}(T)-1$. So $S'' = S'_1\cup D_2\cup\cdots\cup D_k$ is also a $\gamma_{\times 2}(T-v)$-set (containing u_2) but then $S''\cup\{v\}$ would be a $\gamma_{\times 2}(T)$-set that contains v, a contradiction. Therefore, $\gamma_{\times 2}(T-v) = \gamma_{\times 2}(T)$. ■

Now we are ready to characterize $\gamma_{\times 2}$-vertex critical trees.

Theorem 11. *A tree T of order $n \geq 3$ is $\gamma_{\times 2}$-vertex critical if and only if every vertex of T is either a leaf or a support vertex, that is $\gamma_{\times 2}(T) = |V(T)|$.*

Proof. Let T be a $\gamma_{\times 2}$-vertex critical tree. Let v be a support vertex of a tree T having a neighbor that is neither a support nor a leaf. If such a vertex v does not exist, then we are done, that is every vertex of T is either a leaf or a support vertex. So v exists. Let w be a neighbor of v such that $w \notin S(T) \cup L(T)$. Clearly v remains adjacent to a leaf in $T-w$ and so v is in every $\gamma_{\times 2}(T-w)$-set. Thus by Proposition 8, $\gamma_{\times 2}(T-w) = \gamma_{\times 2}(T)-1$, and every vertex in $N_T(w)-\{v\}$ is free in $T-w$. It follows v is the unique support vertex adjacent to w. Let $N_T(w)-\{v\} = \{u_1, u_2, \ldots, u_k\}$. Clearly $k \geq 1$ since w is not a leaf. Let $T_1, T_2, \ldots, T_k$ be the components of $T-w$, where $u_i \in V(T_i)$ for every i, and let T_0 be the remaining component that contains v. Let D_1 be any $\gamma_{\times 2}(T_1)$-set and recall that each u_i is free in $T-w$, that is $u_1 \notin D_1$. Hence D_1 must contain two vertices adjacent to u_1, say x_1 and x_2. Also by Lemma 10, $\gamma_{\times 2}(T_1-u_1) = \gamma_{\times 2}(T_1)$. Now let D be any $\gamma_{\times 2}(T-u_1)$-set and let $T' = T-T_1$. Obviously, $|D| = \gamma_{\times 2}(T_1-u_1)+\gamma_{\times 2}(T')$ and hence $D' = (D \cap T') \cup D_1$ is a DDS of T. Therefore $\gamma_{\times 2}(T) \leq |D'| = \gamma_{\times 2}(T-u_1)$, contradicting the fact that T is $\gamma_{\times 2}$-vertex critical.

The converse is easy to see. ■

4. $\gamma_{\times 2}$-vertex Stable Graphs

We focus in this section to the study of $\gamma_{\times 2}$-vertex stable graphs. We make a useful observation.

Observation 12. *Let G be a $\gamma_{\times 2}$-vertex stable graph. Then every support vertex of G has exactly one neighbor (its leaf) in every $\gamma_{\times 2}(G)$-set.*

Proof. Let G be a $\gamma_{\times 2}$-vertex stable graph and assume that a support vertex v has two neighbors in some $\gamma_{\times 2}(G)$-set D. If v' is a leaf neighbor of v, then $D-\{v'\}$ is a DDS of $G-v'$. Hence $\gamma_{\times 2}(G-v') < \gamma_{\times 2}(G)$, a contradiction. ■

As a consequence of Observation 12, we obtain the following corollary.

Corollary 13. *Let G be a $\gamma_{\times 2}$-vertex stable graph. Then*

(a) *Every support vertex of G is adjacent to exactly one leaf.*

(b) *No two support vertices of G are adjacent.*

The double domination number for cycles C_n and nontrivial paths P_n were given in [5]: $\gamma_{\times 2}(C_n) = \lceil \frac{2n}{3} \rceil$ and [1]: $\gamma_{\times 2}(P_n) = 2\lceil \frac{n}{3} \rceil + 1$ if $n \equiv 0 \pmod 3$ and $\gamma_{\times 2}(P_n) = 2\lceil \frac{n}{3} \rceil$ otherwise.

Using the above results one can see that P_5 is the only $\gamma_{\times 2}$-vertex stable path and cycles C_n are $\gamma_{\times 2}$-vertex stable. Also since $\gamma_{\times 2}(K_n) = 2$ for $n \geq 3$, complete graphs K_n with $n \geq 3$ are $\gamma_{\times 2}$-vertex stable.

A vertex u is said to be *adjacent to a path* P_k in a tree T if there is a neighbor of u, say v, such that the subtree resulting from T by removing the edge uv and which contains the vertex v as a leaf, is a path P_k. The following observation will be useful in the proof of the next lemma.

Observation 14. *Let T be a nontrivial tree and S any subset of vertices containing a free vertex v of T, where every vertex of T except v is dominated twice by S. Then $|S| > \gamma_{\times 2}(T)$.*

Proof. Clearly $\deg_T(v) = k \geq 2$ since v is free. Root T at v and let $v_1, v_2, \ldots, v_k$ be the children of v in $T_v = T$. Let D be any $\gamma_{\times 2}(T)$-set and $D_i = D \cap T_{v_i}$ for every i. By Lemma 10, D is also a $\gamma_{\times 2}(T - v)$-set and so D_i is a $\gamma_{\times 2}(T_{v_i})$-set for every i. Hence $\gamma_{\times 2}(T) = |D_1| + |D_2| + \cdots + |D_k|$. Note that D contains at least two neighbors of v to doubly dominate v. Now assume to the contrary that $|S| = \gamma_{\times 2}(T) = |D|$. Note that S cannot be smaller than D, for otherwise S plus any neighbor of v not in S would be a $\gamma_{\times 2}(T)$-set that contains v, a contradiction with the fact that v is a free vertex. Let $S_i = S \cap T_{v_i}$ for every i. Note that since S does not double dominate v, $v_i \notin S$ for every i. Also since $|S| = |D|$ and $v \in S$, there is an index j such that $|S_j| < |D_j|$. The fact $S_j \cup \{v_j\}$ is a DDS for T_{v_j} implies that $|D_j| \leq |S_j \cup \{v_j\}| < |D_j| + 1$. Hence $|S_j| = |D_j| - 1$ and so the set $S' = (\bigcup_{i=1, i\neq j}^{k} D_i) \cup S_j \cup \{v\}$ double dominate all vertices of T and has size $|D|$. But S' is a $\gamma_{\times 2}(T)$-set that contains v, a contradiction too. Therefore $|S| > |D|$. ∎

Lemma 15. *Let T be a $\gamma_{\times 2}$-vertex stable tree and v a vertex of T different from a support vertex. Then v is not adjacent to a path P_3.*

Proof. Let v be a vertex of T different from a support vertex and assume that v is adjacent to a path $P_3 = u_1$-x_1-y_1 by the edge vu_1. Root tree T at v and let $C(v) = \{u_1, u_2, \ldots, u_k\}$ be the set of children of v in the rooted tree. Note that since T is $\gamma_{\times 2}$-vertex stable and v is not a support vertex, T has order $n \geq 6$ and so $k \geq 2$. Clearly x_1 and y_1 are in every $\gamma_{\times 2}(T)$-set and by Observation 12, u_1 belongs to no $\gamma_{\times 2}(T)$-set. It follows that v is in every $\gamma_{\times 2}(T)$-set. Now let D be a $\gamma_{\times 2}(T - v)$-set. Note that $|D| = \gamma_{\times 2}(T)$ since T is $\gamma_{\times 2}$-vertex stable. It is clear that $\{u_1, x_1, y_1\} \subset D$. Now if for some $i \neq 1$, $u_i \in D$, then v is dominated twice by u_1 and u_i, and so D is $\gamma_{\times 2}(T)$-set containing u_1, contradicting Observation 12. Hence every $u_i \neq u_1$ is a free vertex in $T - v$. Note that if any vertex is free in $T - v$, then it is also free in the component that contains such a vertex. Let T_{u_j} denote the component of $T - v$ such that $u_j \in V(T_{u_j})$. Then $\gamma_{\times 2}(T) = \gamma_{\times 2}(T - v) = \gamma_{\times 2}(T_{u_1}) + \sum_{i=2}^{k} \gamma_{\times 2}(T_{u_i}) = 3 + \sum_{i=2}^{k} \gamma_{\times 2}(T_{u_i})$. If A is any $\gamma_{\times 2}(T)$-set, then A contains v and some $u_i \neq u_1$, say u_2, to double dominate v. Hence $|A| = |\{x_1, y_1\}| + |\{v\}| + \sum_{i=2}^{k} |A \cap T_{u_i}| = 3 + \sum_{i=2}^{k} |A \cap T_{u_i}|$. Now if for some $i \geq 3$, $|A \cap T_{u_i}| < |D \cap T_{u_i}|$, then $A \cup \{u_i\}$ would be a $\gamma_{\times 2}(T_{u_i})$-set containing u_i, a contradiction since u_i is free in its component. Thus we must have $|A \cap T_{u_i}| = |D \cap T_{u_i}|$ for each $i \geq 3$. Using this fact and $|D| = |A|$, we conlude that $|A \cap T_{u_2}| = \gamma_{\times 2}(T_{u_2})$. Observe that $A \cap T_{u_2}$ is a set of vertices containing u_2 which is a free vertex in T_{u_2}. Now if $A \cap T_{u_2}$ is a DDS for T_{u_2}, then u_2 is not free, a contradiction. Thus $A \cap T_{u_2}$ double dominates all vertices of T_{u_2} except u_2 but then by Observation 14, we must have $|A \cap T_{u_2}| > \gamma_{\times 2}(T_{u_2})$, a contradiction too, and the proof of Lemma 15 is complete. ∎

Before presenting our next result, we give another definition. Let u be a support vertex of T with a unique leaf v and let T' be the forest obtained from T by removing u and v. Then we call u a *good support vertex* if $\gamma_{\times 2}(T') = \gamma_{\times 2}(T) - 2$ and every neighbor of u except v is free in T. For example every support vertex of a subdivided star of order at least seven is good. However, no support vertex of the path P_5 is good since $\gamma_{\times 2}(T') > \gamma_{\times 2}(P_5) - 2$.

In the aim to characterize $\gamma_{\times 2}$-vertex stable trees, we define the family $\mathcal{H}$ of all trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_j$ $(j \geqslant 1)$ of trees such that T_1 is a subdivided star $K^*_{1,r}$ with $r \geq 2$, $T = T_j$, and if $j \geqslant 2$, T_{i+1} can be obtained recursively from T_i by one of the operations listed below. Let $A(T_1) = S(T_1) \cup L(T_1)$

- **Operation $\mathcal{O}_1$:** Add a subdivided star $K^*_{1,k}$, $k \geq 2$, centered at a vertex x and join x by an edge to a vertex y of T_i, with the condition that if y is a leaf, then its support vertex is a good one in T_i. Let $A(T_{i+1}) = A(T_i) \cup S(K^*_{1,k}) \cup L\left(K^*_{1,k}\right)$.
- **Operation $\mathcal{O}_2$:** Add a path $P_3 = x\text{-}y\text{-}z$ attached by an edge xv at any support vertex v of T_i. Let $A(T_{i+1}) = A(T_i) \cup \{y, z\}$.

We will use the following observations.

Observation 16. *Let T be a tree obtained from a nontrivial tree T' by adding a path $P_3 = u\text{-}v\text{-}z$ attached by an edge ux at any vertex x of T'. Then $\gamma_{\times 2}(T') \leq \gamma_{\times 2}(T) - 2$, with equality if x belongs to some $\gamma_{\times 2}(T')$-set.*

Proof. Let D be a $\gamma_{\times 2}(T)$-set. By Observation 1, D contains v, z. If $u \in D$, then we can replace it in D by x or any neighbor of x in T'. Hence we may suppose that $u \notin D$ and so $D \cap V(T')$ is a DDS for T'. It follows that $\gamma_{\times 2}(T') \leq |D \cap V(T')| = \gamma_{\times 2}(T) - 2$. Now if x belongs to some $\gamma_{\times 2}(T')$-set, then such a set can be extended to a DDS for T, implying that $\gamma_{\times 2}(T) \leq \gamma_{\times 2}(T') - 2$. Therefore we obtain equality. ∎

Observation 17. *Let T be a tree obtained from a nontrivial tree T' by adding a subdivided star $K^*_{1,k}$ $(k \geq 2)$ centered at u, attached by an edge uw at any vertex w of T'. Then $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2k$.*

Proof. Let D be a $\gamma_{\times 2}(T)$-set. Then by Observation 1, D contains all vertices of $K^*_{1,k}$ except u (else replace u by w or a neighbor of w in T'). Hence $D \cap V(T')$ is a DDS of T' and $\gamma_{\times 2}(T') \leq \gamma_{\times 2}(T) - 2k$. Also, if D' is any $\gamma_{\times 2}(T')$-set, then $D' \cup \Big(V(K^*_{1,k}) - \{u\}\Big)$ is a DDS of T' and so $\gamma_{\times 2}(T) \leq \gamma_{\times 2}(T') + 2k$. It follows that $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2k$. ∎

Lemma 18. *If $T \in \mathcal{H}$, then*

(a) *$A(T)$ is the unique $\gamma_{\times 2}(T)$-set.*

(b) *T is a $\gamma_{\times 2}$-vertex stable tree.*

Proof. Let $T \in \mathcal{H}$. Then from the way in which T is constructed, $A(T)$ is a DDS of T. Now to show that $A(T)$ is the unique $\gamma_{\times 2}(T)$-set and T is $\gamma_{\times 2}$-vertex stable, we use an induction on the total number of operations $\mathcal{O}_i$ performed to construct T. We use the terminology of the construction for set $A(T)$. Clearly the two properties are true for $T_1 = K^*_{1,r}$ with $r \geq 2$. Assume that both properties are true for all trees of $\mathcal{H}$ constructed with $j - 1 \geq 0$ operations, and let T be a tree of $\mathcal{H}$ constructed with j operations. Thus T is obtained by performing operation $\mathcal{O}_1$ or $\mathcal{O}_2$ on a tree T' obtained by $j - 1$ operations. Let D be a $\gamma_{\times 2}(T)$-set. We examine the following two cases.

Case 1. T is obtained from T' using Operation $\mathcal{O}_1$. By Observation 17, $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2k$ and so $A(T) = A(T') \cup \Big(V(K^*_{1,k}) - \{x\}\Big)$ is a $\gamma_{\times 2}(T)$-set. Now assume that $A(T)$ is not the unique $\gamma_{\times 2}(T)$-set and let R be a second $\gamma_{\times 2}(T)$-set. Clearly $V(K^*_{1,k}) - \{x\} \subset D$. Now if $x \notin R$, then $R \cap V(T')$ is a $\gamma_{\times 2}(T')$-set and since $R \neq A(T') \cup \Big(V(K^*_{1,k}) - \{x\}\Big)$ it follows that $R \cap V(T') \neq A(T')$, a contradiction to the uniqueness of $A(T')$. Hence $x \in R$. Clearly x is in R to double dominate y and so $R \cap V(T')$ is not a DDS for T'. If y is not a leaf of T', that is $\deg_{T'}(y) \geq 2$, we can replace x by y or any neighbor of y in T', which gives different $\gamma_{\times 2}(T')$-sets, a contradiction too. Hence we can assume that y is a leaf in T'. Let z be the support vertex of y in T'. There are two situations: either $y \in R$ and $z \notin R$ or $y \notin R$ and $z \in R$. In both cases, z must have a neighbor $z' \neq y$ in R. Let $R \cap V(T')$ plus z for the first situation and $R \cap V(T')$ plus y. Both sets are $\gamma_{\times 2}(T')$-set, however z is not a good support vertex of T' since z' is not free in T', which contradicts the construction. We conclude that $A(T)$ is the unique $\gamma_{\times 2}(T)$-set.

Now let us prove item (b). Recall that by the inductive hypothesis on T', T' is a $\gamma_{\times 2}$-vertex stable tree. Also since T has a unique $\gamma_{\times 2}(T)$-set that does not contain x; x is a free vertex in T. Now let $y_1, \ldots, y_k$ denote the support vertices of $K^*_{1,k}$ and $z_1, \ldots, z_k$ the leaves, with edges $y_i z_i$ for every i. Let w be any vertex of T different from a support vertex. We shall show that $\gamma_{\times 2}(T - w) = \gamma_{\times 2}(T)$.

If $w = x$, then since x is a free vertex in T, by Lemma 10, $\gamma_{\times 2}(T - x) = \gamma_{\times 2}(T)$.

Suppose now that $w = z_i$ for some i. Then by Theorem 3, $\gamma_{\times 2}(T) - 2 \leq \gamma_{\times 2}(T - z_i) \leq \gamma_{\times 2}(T)$. Let D be any $\gamma_{\times 2}(T - z_i)$-set. Then x is a support vertex for y_i and so $V(K^*_{1,k}) - \{z_i\} \subset D$. First assume that $\gamma_{\times 2}(T - z_i) = \gamma_{\times 2}(T) - 2$. Then clearly $D \cup \{z_1\}$ would be a DDS for T of size less than $\gamma_{\times 2}(T)$, a contradiction. Now if $\gamma_{\times 2}(T - z_i) = \gamma_{\times 2}(T) - 1$, then $D \cup \{z_1\}$ is a $\gamma_{\times 2}(T)$-set that contains x, contradicting the fact that x is a free vertex. Therefore $\gamma_{\times 2}(T - z_i) = \gamma_{\times 2}(T)$.

Finally, suppose that w is a vertex of $V(T')$. First let $w = y$ and $t = \deg_{T'}(y) \geq 1$. Then $T' - y$ provides t subtrees T'_1, T'_2, $T'_3, \ldots, T'_t$ and $T - y$ provides the same subtrees and in addition the added subdivided star $K^*_{1,k}$. Now since T' is $\gamma_{\times 2}$-vertex stable, $\gamma_{\times 2}(T' - y) = \sum_{i=1}^{t} \gamma_{\times 2}(T_i) = \gamma_{\times 2}(T')$. Also $\gamma_{\times 2}(T - y) = \gamma_{\times 2}(K^*_{1,k}) + \gamma_{\times 2}(T' - y) = 2k + \gamma_{\times 2}(T') = \gamma_{\times 2}(T)$. Second, suppose that $w \neq y$ is any leaf of T'. Then $T' - w$ is nontrivial and so by Observation 17, $\gamma_{\times 2}(T - w) = \gamma_{\times 2}(T' - w) + 2k$. Using the fact that T' is $\gamma_{\times 2}$-vertex stable with the equality $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2k$ we obtain $\gamma_{\times 2}(T - w) = \gamma_{\times 2}(T' - w) + 2k = \gamma_{\times 2}(T') + 2k = \gamma_{\times 2}(T)$. Now assume that $w \neq y$ is any vertex of T different from z (when z is the support vertex of y in T'). Then w has the same degree in T as in T', say $p \geq 2$. Clearly $T' - w$ provides p subtrees T'_1, T'_2, $T'_3, \ldots, T'_p$, where $y \in V(T_1)$. Likewise $T - w$ provides p subtrees T_1, T'_2, $T'_3, \ldots, T'_p$, where T_1 is the subtree that contains vertices of the added subdivided star $K^*_{1,k}$ and those of T'_1. By Observation 17, $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T'_1) + 2k$. We know that $\gamma_{\times 2}(T' - w) = \sum_{i=1}^{p} \gamma_{\times 2}(T_i) = \gamma_{\times 2}(T')$ and $\gamma_{\times 2}(T - w) = \gamma_{\times 2}(T'_1) + 2k + \sum_{i=2}^{p} \gamma_{\times 2}(T_i)$. Therefore $\gamma_{\times 2}(T - w) = \gamma_{\times 2}(T)$. It remains to examine the last situation when $w = z$ and y is the leaf neighbor of z in T'. Let $T'' = T' - \{y, z\}$. Then $T - z$ is formed by T'' and $T(y)$, where $T(y)$ is obtained from $K^*_{1,k}$ by adding an edge to the single vertex y. Hence $\gamma_{\times 2}(T(y)) = 2k + 2$ and $\gamma_{\times 2}(T - z) = \gamma_{\times 2}(T'') + \gamma_{\times 2}(T(y))$. Now since z is a good support vertex in T', $\gamma_{\times 2}(T'') = \gamma_{\times 2}(T') - 2$. It follows that $\gamma_{\times 2}(T - z) = \gamma_{\times 2}(T'') + \gamma_{\times 2}(T(y)) = \gamma_{\times 2}(T') - 2 + 2k + 2 = \gamma_{\times 2}(T)$.

We saw that the deletion of any vertex of T different from a support vertex does not change $\gamma_{\times 2}(T)$; so T is $\gamma_{\times 2}$-vertex stable.

Case 2. T is obtained from T' using Operation $\mathcal{O}_2$. By Observation 16, $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2$ and so $A(T) = A(T') \cup \{y, z\}$ is a $\gamma_{\times 2}(T)$-set. Observe that for every $\gamma_{\times 2}(T)$-set D, $D \cap V(T')$ is a $\gamma_{\times 2}(T')$-set, implying that $A(T)$ is the unique $\gamma_{\times 2}(T)$-set. Hence we have item (a).

Now let us prove item (b). Let u be the leaf neighbor of v and let w be any vertex of T different from a support vertex. If $w = x$, then since $A(T)$ is the unique $\gamma_{\times 2}(T)$-set with $x \notin A(T)$; x is a free vertex in T. Hence by Lemma 10, $\gamma_{\times 2}(T - x) = \gamma_{\times 2}(T)$. Also if $w = z$, then it is easy to see that $\gamma_{\times 2}(T - z) = \gamma_{\times 2}(T') + 2 = \gamma_{\times 2}(T)$. Assume now that $w \neq u$ and let $k = \deg_{T'}(w)$. Clearly we also have $k = \deg_T(w)$. Then $T' - w$ provides k subtrees T'_1, T'_2, $T'_3, \ldots, T'_k$, where $v \in V(T'_1)$. Also $T - w$ provides k subtrees T_1, T'_2, $T'_3, \ldots, T'_k$, where T_1 is obtained from T_1' by attaching the path $P_3 = x\text{-}y\text{-}z$. By Observation 16, $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T_1') + 2$. Now we know that $\gamma_{\times 2}(T' - w) = \sum_{i=1}^{k} \gamma_{\times 2}(T_i) = \gamma_{\times 2}(T')$ and $\gamma_{\times 2}(T - w) = \gamma_{\times 2}(T_1) + \sum_{i=2}^{k} \gamma_{\times 2}(T_i) = 2 + \sum_{i=1}^{k} \gamma_{\times 2}(T_i) = 2 + \gamma_{\times 2}(T' - w)$. Therefore $\gamma_{\times 2}(T - w) = \gamma_{\times 2}(T') + 2 = \gamma_{\times 2}(T)$. Finally assume that $w = u$. We first show that v is not a free vertex in $T' - u$. We know that $A(T')$ is the unique $\gamma_{\times 2}(T')$-set and every neighbor of v in T' besides u is free (by Observation 12). Now since $\gamma_{\times 2}(T' - u) = \gamma_{\times 2}(T')$, the set $\{a\} \cup (A(T') - \{u\})$ is a $\gamma_{\times 2}(T' - u)$-set that contains v, where a is any neighbor of v in $T' - u$. Thus indeed v is not free in $T' - u$ and belongs to some $\gamma_{\times 2}(T' - u)$-set. Using this fact and the fact that $T - u$ is obtained from $T' - u$ by attaching the path $P_3 = x$-y-z, then by Observation 16, $\gamma_{\times 2}(T - u) = \gamma_{\times 2}(T' - u) + 2$. It follows that $\gamma_{\times 2}(T - u) = \gamma_{\times 2}(T' - u) + 2 = \gamma_{\times 2}(T') + 2 = \gamma_{\times 2}(T)$. Consequently the removing of any vertex of T different from a support vertex does not change $\gamma_{\times 2}(T)$; so T is $\gamma_{\times 2}$-vertex stable. ∎

We now are ready to prove the following.

Theorem 19. *A tree T is $\gamma_{\times 2}$-vertex stable if and only if $T \in \mathcal{H}$.*

Proof. If $T \in \mathcal{H}$, then by Lemma 18, T is a $\gamma_{\times 2}$-vertex stable tree. Now let T be a $\gamma_{\times 2}$-vertex stable tree. To prove that $T \in \mathcal{H}$ we proceed by induction on the order of T. Since stars and double stars are not $\gamma_{\times 2}$-vertex stable (by Observation 12), T has diameter at least four. Clearly the smallest tree of diameter four is the path $P_5 = K_{1,2}^*$ that belongs to $\mathcal{H}$. Assume that every $\gamma_{\times 2}$-vertex stable tree of order $n' < n$ is in $\mathcal{H}$.

Let T be a $\gamma_{\times 2}$-vertex stable tree of order n. If T has diameter 4, then by Observation 12, every support vertex is adjacent to exactly one leaf and no two support vertices are adjacent. So T is a subdivided star $K_{1,r}^*$ with $r \geq 2$ and such trees are in $\mathcal{H}$. Hence we can assume T has diameter at least 5.

We now root T at leaf r of a longest path. Let u be a vertex at distance $\operatorname{diam}(T) - 2$ from r on a longest path starting at r. We further assume that among all such vertices u has maximum degree. Let x_1 be the child of u and y_1 the (unique) child of x_1 on this path. We also let v be the parent of u and z the parent of v. Since x_1 is a support vertex, by Observation 12, u cannot be a support vertex. Also since x_1 and y_1 are in every $\gamma_{\times 2}(T)$-set, u does not belong to any $\gamma_{\times 2}(T)$-set, otherwise we contradict Observation 12. Hence u is a free vertex in T. We now consider the following two cases.

Case 1. $\deg_T(u) \geq 3$. Hence every child of u is a support vertex and so $T_u = K_{1,k}^*$, where $k = \deg_T(u) - 1$. Let $\{x_1, x_2, \ldots, x_k\}$ be the set of all children of u and let y_i be the leaf neighbor of x_i for every i. Now consider the tree $T' = T - D[u]$. Note that T' has order $n' > 3$. Indeed T' is nontrivial since $\operatorname{diam}(T) \geq 5$; also if $n' = 3$, then T' would be a path P_3 centered at z, where all its vertices are in every $\gamma_{\times 2}(T)$-set which contradicts Observation 12. Moreover, by Observation 17, $\gamma_{\times 2}(T') = \gamma_{\times 2}(T) - |D(u)|$. Next we shall show that T' is a $\gamma_{\times 2}$-vertex stable tree. Let $w \neq v$ be any vertex of T' different from a support vertex. Let $b = \deg_{T'}(w) \geq 1$. Clearly the removing of w from T' produces a forest with b components $T'_1, T'_2, \ldots, T'_b$, where without loss of generality, $v \in V(T'_1)$. Also the removing of w from T produces a forest with b components $T_1, T_2, \ldots, T_b$, with $T_i = T'_i$ for every $i \geq 2$ and T_1 is the component that contains T'_1 and the vertices of $D[u]$. Since T'_1 is nontrivial, by Observation 17, $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T'_1) + |D(u)|$. Clearly $\gamma_{\times 2}(T' - w) = \sum_{i=1}^{b} \gamma_{\times 2}(T'_i)$ and $\gamma_{\times 2}(T - w) = \gamma_{\times 2}(T_1) + \sum_{i=2}^{b} \gamma_{\times 2}(T'_i)$. Using the facts $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + |D(u)|$ and T is a $\gamma_{\times 2}$-vertex stable tree, we get

$$\begin{aligned}\gamma_{\times 2}(T) &= \gamma_{\times 2}(T - w) = \gamma_{\times 2}(T_1) + \textstyle\sum_{i=2}^{b} \gamma_{\times 2}(T'_i) \\ &= \gamma_{\times 2}(T'_1) + |D(u)| + \textstyle\sum_{i=2}^{b} \gamma_{\times 2}(T'_i) \\ &= \textstyle\sum_{i=1}^{b} \gamma_{\times 2}(T'_i) + |D(u)| = \gamma_{\times 2}(T' - w) + |D(u)|\,,\end{aligned}$$

and so $\gamma_{\times 2}(T' - w) = \gamma_{\times 2}(T')$. Hence the deletion of such a vertex w from T' does not change the double domination number of T'. It remains to examine the case $w = v$ and of course v is not a support vertex. We consider two possibilities depending on the degree of v.

Subcase 1.1. $\deg_T(v) \geq 3$. Let $j = \deg_T(v)$. Clearly $T - v$ is a forest with j components $T_1 = T_u, T_2, T_3, \ldots, T_j$ and so $T_2, T_3, \ldots, T_j$ are the components of $T' - v$. Note that $\gamma_{\times 2}(T_u) = |D(u)|$, $\gamma_{\times 2}(T - v) = \gamma_{\times 2}(T_1) + \sum_{i=2}^{j} \gamma_{\times 2}(T_i)$ and $\gamma_{\times 2}(T' - v) = \sum_{i=2}^{j} \gamma_{\times 2}(T_i)$. Since T is $\gamma_{\times 2}$-vertex stable and $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + |D(u)|$, we obtain $\gamma_{\times 2}(T') + |D(u)| = \gamma_{\times 2}(T) = \gamma_{\times 2}(T_1) + \sum_{i=2}^{j} \gamma_{\times 2}(T_i) = |D(u)| + \gamma_{\times 2}(T' - v)$, and hence $\gamma_{\times 2}(T' - v) = \gamma_{\times 2}(T')$.

Subcase 1.2. $\deg_T(v) = 2$, that is v is a leaf in T'. First we show that z is good support vertex in T'. Since u is a free vertex in T, it follows that v and z belong to every $\gamma_{\times 2}(T)$-set. Hence every neighbor of z besides v is a free vertex in T; indeed if z has another neighbor in some $\gamma_{\times 2}(T)$-set X, then $\{u\} \cup X - \{v\}$ is a $\gamma_{\times 2}(T)$-set containing y_1, x_1 and u, a contradiction to Observation 12. Clearly this implies that every neighbor of z in T' besides v remain free. It remains to see that $\gamma_{\times 2}(T'') = \gamma_{\times 2}(T') - 2$, where T'' is the tree resulting from T' by removing v and z. Observe that the removing of z from T provides a forest with at least two components $T_1 = T_v$, T_2, $T_3, \ldots, T_t$, where T_2, $T_3, \ldots, T_t$ are precisely the components of T''. Note that T_v is a tree, where every vertex is either a leaf or a support vertex; so $\gamma_{\times 2}(T_v) = |D(u)| + 2$. It follows that $\gamma_{\times 2}(T - z) = \gamma_{\times 2}(T_v) + \sum_{i=2}^{t} \gamma_{\times 2}(T_i) = |D(u)| + 2 + \sum_{i=2}^{t} \gamma_{\times 2}(T_i)$. On the other hand, $\gamma_{\times 2}(T'') = \sum_{i=2}^{t} \gamma_{\times 2}(T_i) = \gamma_{\times 2}(T - z) - |D(u)| - 2$. Now since T is $\gamma_{\times 2}$-vertex stable and $\gamma_{\times 2}(T') = \gamma_{\times 2}(T) - |D(u)|$, we get $\gamma_{\times 2}(T'') = \gamma_{\times 2}(T) - |D(u)| - 2 = \gamma_{\times 2}(T') - 2$. Therefore z is a good support vertex in T'.

Now let us see that the removing of v in T' does not change $\gamma_{\times 2}(T')$. Clearly $T - v$ is a forest with two components T_u and $T' - v$. It follows that $\gamma_{\times 2}(T - v) = \gamma_{\times 2}(T_u) + \gamma_{\times 2}(T' - v) = |D(u)| + \gamma_{\times 2}(T' - v)$. Using the equalities $\gamma_{\times 2}(T - v) = \gamma_{\times 2}(T)$ and $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + |D(u)|$, we obtain $\gamma_{\times 2}(T' - v) = \gamma_{\times 2}(T')$.

According to all previous situations, we conclude that T' is a $\gamma_{\times 2}$-vertex stable tree. By induction on T', we have $T' \in \mathcal{H}$. It follows that $T \in \mathcal{H}$ because it is obtained from T' by using Operation $\mathcal{O}_1$.

Case 2. $\deg_T(u) = 2$. Clearly v is a vertex adjacent to the path $P_3 = u\text{-}x_1\text{-}y_1$. Since T is $\gamma_{\times 2}$-vertex stable, by Lemma 15, v is a support vertex in T. Let v' be its unique leaf. Let T' be the tree obtained from T by removing u, x_1, y_1. Note that v belongs to every $\gamma_{\times 2}(T')$-set. By Observation 16, $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2$. Next we shall show that T' is a $\gamma_{\times 2}$-vertex stable tree. First let w be any vertex of T' such that $w \notin L(T') \cup S(T')$. The removing of w in T' provides a forest with at least two components, say $T'_1, T'_2, \ldots, T'_m$. Without loss of generality, we can assume that T'_1 is the component that contains v and so v'. Also the removing of w from T provides a forest with m components $T_1, T'_2, \ldots, T'_m$, where T_1 is the tree obtained from T'_1 by attaching the path $P_3 = u\text{-}x_1\text{-}y_1$ with the edge vu. Clearly since v, v' belong to every $\gamma_{\times 2}(T'_1)$-set, by Observation 16, $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T'_1) + 2$. On the other hand we have $\gamma_{\times 2}(T' - w) = \sum_{i=1}^{m} \gamma_{\times 2}(T'_i)$ and $\gamma_{\times 2}(T - w) = \gamma_{\times 2}(T_1) + \sum_{i=2}^{m} \gamma_{\times 2}(T'_i)$. Now since T is $\gamma_{\times 2}$-vertex stable, we obtain

$$\begin{aligned}\gamma_{\times 2}(T) &= \gamma_{\times 2}(T - w) = \gamma_{\times 2}(T_1) + \textstyle\sum_{i=2}^{m} \gamma_{\times 2}(T'_i)\\ &= \gamma_{\times 2}(T'_1) + 2 + \textstyle\sum_{i=2}^{m} \gamma_{\times 2}(T'_i)\\ &= \textstyle\sum_{i=1}^{m} \gamma_{\times 2}(T'_i) + 2 = \gamma_{\times 2}(T' - w) + 2.\end{aligned}$$

Now using the fact $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2$ we obtain $\gamma_{\times 2}(T') = \gamma_{\times 2}(T' - w)$. Therefore the removing of w from T' does not change the double domination number of T'.

Now we assume that w is a leaf of T' and let $T'' = T' - w$. Suppose that $w \neq v'$. We note here that $T - w$ can be seen as the tree obtained from T'' by attaching the path $P_3 = u\text{-}x_1\text{-}y_1$ by the edge uv. Since v, v' are in every $\gamma_{\times 2}(T'')$-set, by Observation 16, $\gamma_{\times 2}(T - w) = \gamma_{\times 2}(T'') + 2$. Since T is $\gamma_{\times 2}$-vertex stable, $\gamma_{\times 2}(T) = \gamma_{\times 2}(T - w)$. Combining the previous equalities with the fact $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2$ we get $\gamma_{\times 2}(T') + 2 = \gamma_{\times 2}(T) = \gamma_{\times 2}(T - w) = \gamma_{\times 2}(T'') + 2$ and so $\gamma_{\times 2}(T') = \gamma_{\times 2}(T'') = \gamma_{\times 2}(T' - w)$. Therefore the removing of any leaf of T' different from v' does not change the double domination number of T'. Finally assume that $w = v'$ and recall that by Corollary 4, $\gamma_{\times 2}(T - v') \leq \gamma_{\times 2}(T')$. Let us consider two subcases.

Subcase 2.1. $\deg_{T'}(v) = 2$, that is, v is a leaf in $T' - v'$. Recall that z is the parent of v in T and note that $T' - v'$ has order at least three since $\operatorname{diam}(T) \geq 5$. Now suppose that $\gamma_{\times 2}(T' - v') < \gamma_{\times 2}(T')$ and let D' be any

$\gamma_{\times 2}(T'-v')$-set. Then clearly $v, z \in D'$ and $D' \cup \{v', x_1, y_1\}$ is a DDS for T, implying that $\gamma_{\times 2}(T) \leq |D'| + 3$. Using the equality $\gamma_{\times 2}(T') = \gamma_{\times 2}(T) - 2$ we obtain

$$\gamma_{\times 2}(T) \leq \left|D'\right| + 3 = \gamma_{\times 2}(T'-v') + 3 < \gamma_{\times 2}(T') + 3 = \gamma_{\times 2}(T) + 1,$$

and so $\gamma_{\times 2}(T) = |D'| + 3$. It follows that $D' \cup \{v', x_1, y_1\}$ is a $\gamma_{\times 2}(T)$-set but in such a set v would be adjacent to z and v', contradicting Observation 12. Therefore $\gamma_{\times 2}(T-v') = \gamma_{\times 2}(T')$.

Subcase 2.2. $\deg_{T'}(v) \geq 3$. Clearly v should have in T at least one other child different from u and v'. Such a child cannot be a support vertex (by Observation 12). Thus let u' be a child of v, x_1' a child of u' and y_1' a child of x_1'. By our choice of u, u' must have degree two. We shall show that v is not a free vertex in $T'-v'$. Suppose to the contrary that v belongs to no $\gamma_{\times 2}(T'-v')$-set and let X be any $\gamma_{\times 2}(T'-v')$-set. Then $x_1', y_1' \in X$ and since $v \notin X$ we have $u' \in X$. To doubly dominate v by X, X contains at least one other neighbor of v but then $\{v\} \cup X - \{u'\}$ is a $\gamma_{\times 2}(T'-v')$-set that contains v, a contradiction. Hence v belongs to some $\gamma_{\times 2}(T'-v')$-set Y and such a set can be extended to DDS for $T-v'$ by adding x_1, y_1, implying that $\gamma_{\times 2}(T-v') \leq \gamma_{\times 2}(T'-v') + 2$. Equality is obtained from the fact that there is a $\gamma_{\times 2}(T-v')$-set containing x_1, y_1, v and not u. Now assume that $\gamma_{\times 2}(T'-v') < \gamma_{\times 2}(T')$. Observe that $Y \cup \{x_1, y_1, v'\}$ is a DDS for T. Now using the fact $\gamma_{\times 2}(T') = \gamma_{\times 2}(T) - 2$ we obtain

$$\gamma_{\times 2}(T) \leq \left|Y \cup \{x_1, y_1, v'\}\right| = \gamma_{\times 2}(T'-v') + 3 < \gamma_{\times 2}(T') + 3 = \gamma_{\times 2}(T) - 2 + 3,$$

and so $\gamma_{\times 2}(T) = |Y \cup \{x_1, y_1, v'\}|$, that is $Y \cup \{x_1, y_1, v'\}$ is a $\gamma_{\times 2}(T)$-set in which v has two neighbors, contradicting Observation 12. Hence $\gamma_{\times 2}(T'-v') = \gamma_{\times 2}(T')$.

According to all previous situations, we conclude that the removing of any vertex of T' different from a support vertex does not change the double domination number of T', that is, T' is a $\gamma_{\times 2}$-vertex stable tree. Applying the inductive hypothesis on T', we have $T' \in \mathcal{H}$. It follows that $T \in \mathcal{H}$ because it is obtained from T' by using Operation $\mathcal{O}_2$. ■

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