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ON THE TOTAL RESTRAINED DOMINATION NUMBER OF DIRECT PRODUCTS OF GRAPHS[1]

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Abstract

Let $G = (V, E)$ be a graph. A total restrained dominating set is a set $S \subseteq V$ where every vertex in $V \setminus S$ is adjacent to a vertex in S as well as to another vertex in $V \setminus S$, and every vertex in S is adjacent to another vertex in S. The total restrained domination number of G, denoted by $\gamma_r^t(G)$, is the smallest cardinality of a total restrained dominating set of G. We determine lower and upper bounds on the total restrained domination number of the direct product of two graphs. Also, we show that these bounds are sharp by presenting some infinite families of graphs that attain these bounds.

Keywords: total domination number, total restrained domination number, direct product of graphs.

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1. Introduction

All graph theory terminology not presented in this paper can be found in [4]. Let $G = (V, E)$ be a graph with $|V| = n$. For any vertex $v \in V$, the *open neighborhood* of v, denoted by $N_G(v)$, is $\{u \in V \mid uv \in E\}$. The *closed neighborhood* of v, denoted by $N_G[v]$, is the set $N_G(v) \cup \{v\}$. Let S be a subset of V. The *neighborhood* of S is the set $N(S) = \bigcup_{v \in S} N(v)$. The *open packing number* $\rho^0(G)$ of G is the maximum cardinality of a set of vertices whose open neighborhoods are pairwise disjoint.

A set S is a *dominating set* of G if for every vertex $u \in V \setminus S$ there exists $v \in S$ such that $uv \in E$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. We call a set S a *γ-set* if S is a dominating set with cardinality $\gamma(G)$.

A set $S \subseteq V$ is a *total dominating set* if $N(S) = V(G)$, and the *total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G.

A set $S \subseteq V$ is a *restrained dominating set* if every vertex in $V \setminus S$ is adjacent to a vertex in S and to another vertex in $V \setminus S$. Let $\gamma_r(G)$ denote the size of a smallest restrained dominating set. A set S is called a *γ_r-set* if S is a restrained dominating set with cardinality $\gamma_r(G)$.

A *total restrained dominating set* is a set $S \subseteq V$ where every vertex in $V \setminus S$ is adjacent to a vertex in S as well as to another vertex in $V \setminus S$, and every vertex in S is adjacent to another vertex in S. The *total restrained domination number* of G, denoted by $\gamma_r^t(G)$, is the smallest cardinality of a total restrained dominating set of G.

It is obvious that $\gamma(G) \le \gamma_t(G) \le \gamma_r^t(G)$.

The *direct product* $G \times H$ (some authors call it the *cross product* [1,3]) of two graphs G and H is the graph with $V(G \times H) = V(G) \times V(H)$ and $(u, v)(u', v') \in E(G \times H)$ if and only if $uu' \in E(G)$ and $vv' \in E(H)$.

In this paper we study the total restrained domination number of the direct product of graphs. In Section 2, we give lower and upper bounds on $\gamma_r^t(G \times H)$ in terms of total and total restrained domination number and maximum degree of G and H. Both bounds are best possible. In Section 3, we further investigate the exact values of the total restrained domination number when one of the factors is a path or cycle. Throughout the rest of the paper, all graphs are assumed to be simple and have no isolated vertices.

2. Upper and Lower Bounds for $\gamma_r^t(G \times H)$

Theorem 2.1. *For any two graphs G and H, we have $\gamma_r^t(G \times H) \le \gamma_r^t(G)\gamma_r^t(H)$.*

Proof. Let D_1 and D_2 be a $\gamma_r^t(G)$-set and a $\gamma_r^t(H)$-set, respectively. We show that $D = D_1 \times D_2$ is a total restrained dominating set of $G \times H$. For any vertex $(x, y) \in V(G \times H)$, we have the following cases.

Case 1. $x \in V(G) \setminus D_1$ and $y \in V(H) \setminus D_2$. Since D_1 is a total restrained dominating set of G, there exist $x_1 \in D_1$ and $x_2 \in V(G) \setminus D_1$ such that $xx_1 \in E(G)$ and $xx_2 \in E(G)$. By a similar way, there exist $y_1 \in D_2$ and $y_2 \in V(H) \setminus D_2$ such that $yy_1 \in E(H)$ and $yy_2 \in E(H)$. Hence, (x, y) is dominated by $(x_1, y_1) \in D$, and (x, y) is adjacent to a vertex $(x_2, y_2) \in V(G \times H) \setminus D$.

Case 2. $x \in D_1$ and $y \in V(H) \setminus D_2$. Since D_1 is a total restrained dominating set of G, there exists $x_1 \in D_1$ such that $xx_1 \in E(G)$. By a similar way, there exist $y_1 \in D_2$ and $y_2 \in V(H) \setminus D_2$ such that $yy_1 \in E(H)$ and $yy_2 \in E(H)$. Hence, (x, y) is dominated by $(x_1, y_1) \in D$, and (x, y) is adjacent to a vertex $(x_1, y_2) \in V(G \times H) \setminus D$.

Case 3. $x \in V(G) \setminus D_1$ and $y \in D_2$. By a similar way as Case 2, it holds.

Case 4. $x \in D_1$ and $y \in D_2$. Since D_1 is a total restrained dominating set of G, there exists $x_1 \in D_1$ such that $xx_1 \in E(G)$. By a similar way, there exists $y_1 \in D_2$ such that $yy_1 \in E(H)$. Hence, (x, y) is adjacent to a vertex $(x_1, y_1) \in D$.

Therefore, D is a total restrained dominating set of $G \times H$. So $\gamma_r^t(G \times H) \leq \gamma_r^t(G)\gamma_r^t(H)$. ■

To present a nontrivial infinite family of graphs that achieve the above bound, we will make use of the following inequality.

Lemma 2.2 [6]**.** *For any two graphs G and H, we have*

$$\gamma_t(G \times H) \geq \max\{\rho^0(G)\gamma_t(H),\ \rho^0(H)\gamma_t(G)\}.$$

Theorem 2.3. *Let G be a graph with $\rho^0(G) = \gamma_r^t(G)$ and H be a graph with $\gamma_t(H) = \gamma_r^t(H)$. Then $\gamma_r^t(G \times H) = \gamma_r^t(G)\gamma_r^t(H)$.*

Proof. By Theorem 2.1 and Lemma 2.2 we obtain

$$\gamma_r^t(G)\gamma_r^t(H) \geq \gamma_r^t(G \times H) \geq \gamma_t(G \times H) \geq \rho^0(G)\gamma_t(H) = \gamma_r^t(G)\gamma_r^t(H),$$

and hence $\gamma_r^t(G \times H) = \gamma_r^t(G)\gamma_r^t(H)$. ■

Since $\rho^0(G) \leq \gamma_t(G)$ holds for an arbitrary graph G (see [6]), the assumption concerning G in the above proposition implies $\rho^0(G) = \gamma_t(G) = \gamma_r^t(G)$. This family of graphs includes paths P_n where $n \equiv 2 \pmod 4$, cycles C_n where $n \equiv 0 \pmod 4$, etc. The complete characterization of trees with equal total and total restrained domination numbers is given in [2].

Apart from the upper bound, we give a lower bound on the total restrained domination number of $G \times H$, which is a direct consequence of the following theorems [3].

Theorem 2.4. *For any two graphs G and H, we have* $\gamma_t(G\times H) \geq \dfrac{|H|}{\Delta(H)}\gamma_t(G)$.

Theorem 2.5. *For any two graphs G and H, we have*

$$\gamma_r^t(G\times H) \geq \max\left\{\frac{|H|}{\Delta(H)}\gamma_t(G), \frac{|G|}{\Delta(G)}\gamma_t(H)\right\}.$$

3. Products of Paths and Cycles

Let P_n and C_n denote respectively a path and a cycle of order n. In this section, we determine the total retrained domination number of direct products of graphs involving paths and cycles. These graphs attain the bounds given in the previous section. From Theorems 2.1 and 2.5, we have:

$$\frac{n\gamma_t(G)}{2} \leq \gamma_r^t(P_n\times G) \leq \gamma_r^t(P_n)\gamma_r^t(G), \tag{1}$$

$$\frac{n\gamma_t(G)}{2} \leq \gamma_r^t(C_n\times G) \leq \gamma_r^t(C_n)\gamma_r^t(G). \tag{2}$$

Before proving Theorems 3.4 and 3.6, the following results should be stated in advance.

Lemma 3.1 [3]**.** *Let P_n and C_n be a path and cycle with n vertices, respectively. Then*

$$\gamma_t(P_n) = \gamma_t(C_n) = \begin{cases} \frac{n}{2} & \text{for } n \equiv 0 \pmod 4, \\ \frac{n+1}{2} & \text{for } n \equiv 1 \pmod 4, \\ \frac{n+2}{2} & \text{for } n \equiv 2 \pmod 4, \\ \frac{n+1}{2} & \text{for } n \equiv 3 \pmod 4. \end{cases}$$

Theorem 3.2 [5]**.** *If $n \geq 3$ and $m \geq 2$, then*

$$\gamma_r^t(C_n) = n - 2\left\lfloor\frac{n}{4}\right\rfloor = \begin{cases} \frac{n}{2} & \text{for } n \equiv 0 \pmod 4, \\ \frac{n+1}{2} & \text{for } n \equiv 1 \pmod 4, \\ \frac{n+2}{2} & \text{for } n \equiv 2 \pmod 4, \\ \frac{n+3}{2} & \text{for } n \equiv 3 \pmod 4. \end{cases}$$

$$\gamma_r^t(P_m) = m - 2\left\lfloor\frac{m-2}{4}\right\rfloor = \begin{cases} \frac{m+4}{2} & \text{for } m \equiv 0 \pmod 4, \\ \frac{m+5}{2} & \text{for } m \equiv 1 \pmod 4, \\ \frac{m+2}{2} & \text{for } m \equiv 2 \pmod 4, \\ \frac{m+3}{2} & \text{for } m \equiv 3 \pmod 4. \end{cases}$$

Lemma 3.3 [7]**.** *Let G be a graph without isolated vertices and $n \geq 2$. Then $\gamma_t(P_n \times G) = \gamma_t(P_n)\gamma_t(G)$.*

Theorems 3.4 and 3.5 show that the upper bound in (1) is attained when P_n is such that $n \equiv 2 \,(\text{mod } 4)$ and $\gamma_t(G) = \gamma_r^t(G)$, or when G is K_2.

Theorem 3.4. *Let G be a graph with $\gamma_t(G) = \gamma_r^t(G)$ and let P_n be such that $n \equiv 2 \,(\text{mod } 4)$. Then $\gamma_r^t(P_n \times G) = \gamma_r^t(P_n)\gamma_r^t(G)$.*

Proof. From Lemma 3.1 and Theorem 3.2, we have $\gamma_r^t(P_n) = \gamma_t(P_n)$ for $n \equiv 2 \,(\text{mod } 4)$. Since $\gamma_t(P_n)\gamma_t(G) = \gamma_t(P_n \times G) \leq \gamma_r^t(P_n \times G) \leq \gamma_r^t(P_n)\gamma_r^t(G) = \gamma_t(P_n)\gamma_t(G)$, it follows that $\gamma_r^t(P_n \times G) = \gamma_r^t(P_n)\gamma_r^t(G)$. ■

Theorem 3.5. *If $n \geq 2$, then $\gamma_r^t(P_n \times K_2) = 2\gamma_r^t(P_n)$.*

Proof. $P_n \times K_2$ consists of two disjoint copies of P_n, so the result follows. ■

The upper and lower bounds in (2) agree when $n \equiv 0 \,(\text{mod } 4)$ and $\gamma_t(G) = \gamma_r^t(G)$. The following theorem shows that the lower bound in (2) is attained when G is a complete graph K_m for $m \geq 3$, while the upper bound is attained when G is K_2 and $n \not\equiv 3 \,(\text{mod } 4)$. Note that $\gamma_t(K_m) = 2$ for $m \geq 2$.

Throughout this paper we use $\mathbb{Z}_n = \{1, 2, \ldots, n\}$ to be the vertex set of C_n. That means computation involving the name of vertices is taken in modulo n.

Theorem 3.6. *If $m \geq 2$ and $n \geq 3$, then*

$$\gamma_r^t(C_n \times K_m) = \begin{cases} 2\gamma_r^t(C_n) & \text{if } m = 2 \text{ and } n \not\equiv 3 \,(\text{mod } 4), \\ n & \text{if } m \geq 3. \end{cases}$$

Proof. Let $m = 2$. If n is even, then the graph $C_n \times K_2$ consists of two disjoint copies of C_n. So $\gamma_r^t(C_n \times K_2) = 2\gamma_r^t(C_n)$.

If n is odd, then the graph $C_n \times K_2 \cong C_{2n}$ and we have $\gamma_r^t(C_n \times K_2) = \gamma_r^t(C_{2n})$. Since $n \not\equiv 3 \,(\text{mod } 4)$, we have $n \equiv 1 \,(\text{mod } 4)$ and $\gamma_r^t(C_{2n}) = n + 1 = 2\gamma_r^t(C_n)$ by Theorem 3.2.

Suppose $m \geq 3$. We shall construct a total restrained dominating subset D of $C_n \times K_m$ with $|D| = n$. Let the vertex set of K_m be $\{1, 2, \ldots, m\}$. It is possible to choose a mapping $f : \mathbb{Z}_n \to \{1, 2, 3\}$ such that $f(i) \neq f(i + 2)$ for each $i \in \mathbb{Z}_n$. Let $D = \{(i, f(i)) \mid 1 \leq i \leq n\}$. For example, let $D = \{(1, 1), (2, 1), (3, 2), (4, 2), (5, 3), (6, 3), (7, 2)\}$ when $n = 7$.
Consider any vertex (i, j). Since $f(i-1) \neq f(i+1)$ (for $i = 1$, we take n as $i-1$), at least one of them is different from j, say $f(i-1) \neq j$. Then (i, j) is adjacent to $(i-1, f(i-1)) \in D$. This shows that D is a total dominating set of $C_n \times K_m$. For $(i, j) \notin D$, (i, j) is also adjacent to $(i+1, f(i-1)) \notin D$. So D is a total restrained dominating set of $C_n \times K_m$. Since $n = \dfrac{n\gamma_t(K_m)}{2} \leq \gamma_r^t(C_n \times K_m) \leq |D| = n$, $|D|$ is minimum and $\gamma_r^t(C_n \times K_m) = n$. ■

After showing that the lower and upper bounds established in Section 2 are the best possible, we are now going to determine the total restrained domination number of direct product of cycles. Consider the direct product of cycles, then (2) becomes

$$\text{(3)} \qquad \max\left\{\left\lceil\frac{n\gamma_t(C_m)}{2}\right\rceil, \left\lceil\frac{m\gamma_t(C_n)}{2}\right\rceil\right\} \le \gamma_r^t(C_n \times C_m) \le \gamma_r^t(C_n)\gamma_r^t(C_m).$$

Theorem 3.12 shows that the lower bound in (3) is attained when $n = m$. Before we present the proof, the following lemma and definition from [3] should be stated.

Consider the graph $G \times H$. For $x \in V(G)$, H_x is the graph induced by $\{(x, y) \mid y \in V(H)\}$.

Lemma 3.7 [3]**.** *Let D be a total dominating set of $G \times H$ and let $u \in V(G)$. We have* $\left|D \cap \left(\bigcup_{v \in N(u)} H_v\right)\right| \ge \gamma_t(H)$.

Definition 3.8 [3]**.** Let D be a total dominating set of $C_n \times C_m$. An S-*sequence* corresponding to D is a sequence $S = (S_1, S_2, \ldots, S_n)$, where $S_i = \{j \mid (i, j) \in D\} \subseteq V(C_m)$ for $i \in \mathbb{Z}_n$.

The above definition together with Lemma 3.7 give the following result:

$$\text{(4)} \qquad S_i \cup S_{i+2} \text{ is a total dominating set for } C_m, \text{ where } i \in \mathbb{Z}_n.$$

The condition in (4) is also sufficient to define a total dominating set for $C_n \times C_m$ in the sense that if $S = (S_1, S_2, \ldots, S_n)$ is a sequence of subsets of $V(C_m)$ such that (4) is satisfied, then $D = \{(i, j) \mid j \in S_i, 1 \le i \le n\}$ is a total dominating set of $C_n \times C_m$.

We define another sequence, the T-sequence, which depends on the S-sequence.

Definition 3.9. Given an S-sequence $S = (S_1, S_2, \ldots, S_n)$, the corresponding T-*sequence* is defined as $T = (S_1, S_3, S_5, \ldots, S_n, S_2, S_4, \ldots, S_{n-1})$ if n is odd. If n is even, then the T-sequence degenerates into two subsequences $T' = (S_1, S_3, \ldots, S_{n-1})$ and $T'' = (S_2, S_4, \ldots, S_n)$.

Let $T = \{T_1, T_2, \ldots, T_n\}$ denote a T-sequence of length n. From (4), we have

$$\text{(5)} \qquad T_i \cup T_{i+1} \text{ is a total dominating set for } C_m, \text{ where } i \in \mathbb{Z}_n.$$

Condition (5) is necessary and sufficient for a sequence $\{T_1, T_2, \ldots, T_n\}$ to be a T-sequence corresponding to a total dominating set for $C_n \times C_m$.

To facilitate the proof of Theorem 3.12, we need the following lemmas about the total domination number of direct product of cycles.

Lemma 3.10 [3]**.** *For odd $m \geq 3$, we have*

$$\gamma_t(C_m \times C_m) = \begin{cases} (4k+3)(k+1) & \textit{if } m = 4k+3, \\ 4k^2+3k+1 & \textit{if } m = 4k+1. \end{cases}$$

Theorem 3.11 [3]**.** *For odd k, we have $\gamma_t(C_{2k} \times C_m) = 2\gamma_t(C_k \times C_m)$.*

Theorem 3.12. *For $m \geq 3$, we have*

$$\gamma_r^t(C_m \times C_m) = \begin{cases} 4k^2+7k+3 & \textit{if } m = 4k+3, \\ 4k^2+3k+1 & \textit{if } m = 4k+1, \\ 4k^2 & \textit{if } m = 4k, \\ 4k^2+6k+4 & \textit{if } m = 4k+2 \textit{ and } k \textit{ is even}, \\ 4k^2+6k+2 & \textit{if } m = 4k+2 \textit{ and } k \textit{ is odd}. \end{cases}$$

Proof. Considering different values of m, we have the following cases.

Case 1. Suppose $m = 4k+3$ for some $k \geq 0$. We define a T-sequence of length $4k+3$ by

(a) $T_1 = \{4j+1 \mid 0 \leq j \leq k\}$ and

(b) $T_{i+1} = \{j+1 \pmod{4k+3} \mid j \in T_i\}$, $1 \leq i \leq 4k+2$.

It is obvious that $T_i \cup T_{i+1}$ is a total dominating set of C_{4k+3}. That is, the condition (5) is satisfied. So $D = \{(i,j) \mid j \in S_i, 1 \leq i \leq 4k+3\}$ is a total dominating set for $C_{4k+3} \times C_{4k+3}$.

Next we will show that D is a total restrained dominating set of $C_{4k+3} \times C_{4k+3}$. For any vertex $(i,j) \notin D$, (i,j) is adjacent to four vertices $(i-1, j-1)$, $(i-1, j+1)$, $(i+1, j-1)$ and $(i+1, j+1)$. Since D is a total dominating set of $C_{4k+3} \times C_{4k+3}$, at least one of them is in D, say $(i-1, j-1) \in D$. By the construction of the T-sequence, we have $(i-1, j+1) \notin D$.

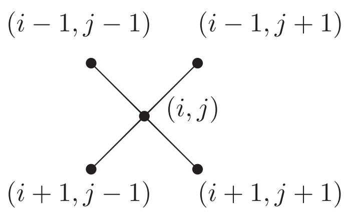

Figure 1. Neighbors of the vertex (i,j).

Thus, D is a total restrained dominating set of $C_{4k+3} \times C_{4k+3}$ and we have $\gamma_r^t(C_{4k+3} \times C_{4k+3}) \leq |D| = (4k+3)(k+1)$.

On the other hand, from Lemma 3.10, we have $\gamma_r^t(C_{4k+3}\times C_{4k+3}) \geq \gamma_t(C_{4k+3}\times C_{4k+3}) = (4k+3)(k+1)$. Hence, $\gamma_r^t(C_{4k+3}\times C_{4k+3}) = (4k+3)(k+1) = 4k^2+7k+3$.

Case 2. Suppose $m = 4k+1$ for some $k \geq 1$. We define a T-sequence of length $4k+1$ by

(a) $T_1 = \{4j+1 \mid\ 0 \leq j \leq k\}$,

(b) $T_2 = \{4j \mid\ 1 \leq j \leq k\}$,

(c) $T_{i+2} = \{j+2 \pmod{4k+1} \mid j \in T_i\}$, $1 \leq i \leq 4k-1$.

With a similar proof as in Case 1, $D = \{(i,j) \mid j \in S_i,\ 1 \leq i \leq 4k+1\}$ is a total restrained dominating set for $C_{4k+1}\times C_{4k+1}$. Thus we have

$$\gamma_r^t(C_{4k+1}\times C_{4k+1}) \leq |D| = \left\lceil \frac{(4k+1)(2k+1)}{2} \right\rceil = \left\lceil 4k^2+3k+\frac{1}{2} \right\rceil = 4k^2+3k+1.$$

Conversely, from Lemma 3.10, we have $\gamma_r^t(C_{4k+1}\times C_{4k+1}) \geq \gamma_t(C_{4k+1}\times C_{4k+1}) = 4k^2+3k+1$. Hence, $\gamma_r^t(C_{4k+1}\times C_{4k+1}) = 4k^2+3k+1$.

Case 3. Suppose $m = 4k$ for some $k \geq 1$. Equation (3) gives $\frac{(4k)\gamma_t(C_{4k})}{2} \leq \gamma_r^t(C_{4k}\times C_{4k}) \leq \gamma_r^t(C_{4k})\gamma_r^t(C_{4k})$, Lemma 3.1 gives $\frac{(4k)\gamma_t(C_{4k})}{2} = 4k\left(\frac{4k}{4}\right) = 4k^2$ and Theorem 3.2 gives $\gamma_r^t(C_{4k})\gamma_r^t(C_{4k}) = \left(\frac{4k}{2}\right)\left(\frac{4k}{2}\right) = 4k^2$. Therefore, $\gamma_r^t(C_{4k}\times C_{4k}) = 4k^2$.

Case 4. Suppose $m = 4k+2$ for some $k \geq 1$. The vertex set of $C_{4k+2}\times C_{4k+2}$ can be divided into four parts as showed in Figure 2. Each part contains $(2k+1)^2$ vertices.

	$1 \quad \cdots \quad 2k+1$	$2k+2 \quad \cdots \quad 4k+2$
1 $\vdots$ $2k+1$	part 1	part 2
$2k+2$ $\vdots$ $4k+2$	part 3	part 4

Figure 2. Vertex distribution of $C_{4k+2}\times C_{4k+2}$.

We divide it into two subcases.

Case 4.1. Suppose $k = 2l$ for some $l \geq 1$. For part 1 and part 3, we define a $T^{(1)}$-sequence of length $4k+2$ by

(a) $T_1^{(1)} = \{4j + 1 \mid 0 \leq j \leq l\}$,

(b) $T_2^{(1)} = \{4j \mid 1 \leq j \leq l\}$,

(c) $T_{i+2}^{(1)} = \{j + 2 \pmod{2k+1} \mid j \in T_i^{(1)}\}$, $1 \leq i \leq 2k - 1$,

(d) $T_{2k+i}^{(1)} = T_{i-1}^{(1)}$ for $2 \leq i \leq 2k + 2$.

For part 2 and part 4, we define a $T^{(2)}$-sequence of length $4k + 2$ by

$$T_i^{(2)} = \{j + (2k + 1) \mid j \in T_i^{(1)}\}, \text{ for } 1 \leq i \leq 4k + 2.$$

For example, when $k = 2$, the $T^{(1)}$-sequence and $T^{(2)}$-sequence are

$$\begin{array}{llll}
T_1^{(1)} = \{1,5\}, & T_1^{(2)} = \{6,10\}, & T_6^{(1)} = T_1^{(1)}, & T_6^{(2)} = T_1^{(2)}, \\
T_2^{(1)} = \{4\}, & T_2^{(2)} = \{9\}, & T_7^{(1)} = T_2^{(1)}, & T_7^{(2)} = T_2^{(2)}, \\
T_3^{(1)} = \{3,2\}, & T_3^{(2)} = \{8,7\}, & T_8^{(1)} = T_3^{(1)}, & T_8^{(2)} = T_3^{(2)}, \\
T_4^{(1)} = \{1\}, & T_4^{(2)} = \{6\}, & T_9^{(1)} = T_4^{(1)}, & T_9^{(2)} = T_4^{(2)}, \\
T_5^{(1)} = \{5,4\}, & T_5^{(2)} = \{10,9\}, & T_{10}^{(1)} = T_5^{(1)}, & T_{10}^{(2)} = T_5^{(2)}.
\end{array}$$

Let T-sequence be $T = \{T_1, T_2, \ldots, T_{4k+2}\}$, where $T_i = T_i^{(1)} \cup T_i^{(2)}$, $1 \leq i \leq 4k+2$. The T-sequence degenerates into two subsequences:

$$\begin{aligned}
T' &= \{T_1, T_2, \ldots, T_{2k+1}\} = \{S_1, S_3, \ldots, S_{4k+1}\}, \\
T'' &= \{T_{2k+2}, T_{2k+3}, \ldots, T_{4k+2}\} \\
&= \{S_{2k+2}, S_{2k+4}, \ldots, S_{4k+2}, S_2, S_4, \ldots, S_{2k}\}.
\end{aligned}$$

It is obvious that $T_i \cup T_{i+1}$ is a total dominating set of C_{4k+2}. So $D = \{(i,j) \mid j \in S_i, 1 \leq i \leq 4k + 2\}$ is a total dominating set for $C_{4k+2} \times C_{4k+2}$ (see Figure 3 for $k = 2$).

	1	2	3	4	5	6	7	8	9	10
1	•				•	•				•
2	•					•				
3				•					•	
4				•	•				•	•
5		•	•				•	•		
6	•				•	•				•
7	•					•				
8				•					•	
9				•	•				•	•
10		•	•				•	•		

Figure 3. A total dominating set of $C_{10} \times C_{10}$.

Similarly to the proof of Case 1, D is also a total restrained dominating set of $C_{4k+2} \times C_{4k+2}$. Therefore, $\gamma_r^t(C_{4k+2} \times C_{4k+2}) \leq |D| = 4\left\lceil \frac{(2k+1)(2l+1)}{2} \right\rceil = 4\left\lceil \frac{(2k+1)(k+1)}{2} \right\rceil = 4k^2 + 6k + 4$.

On the other hand, by Theorem 3.11 and Lemma 3.10, we have

$$\begin{aligned}\gamma_r^t(C_{4k+2} \times C_{4k+2}) &\geq \gamma_t(C_{4k+2} \times C_{4k+2}) = 4\gamma_t(C_{2l+1} \times C_{2l+1}) \\ &= 4(4l^2 + 3l + 1) = 4k^2 + 6k + 4.\end{aligned}$$

Hence, $\gamma_r^t(C_{4k+2} \times C_{4k+2}) = 4k^2 + 6k + 4$.

Case 4.2. Suppose $k = 2l+1$ for some $l \geq 0$. For part 1 and part 3, we define a $T^{(1)}$-sequence of length $4k+2$ by

(a) $T_1^{(1)} = \{4j + 1 \mid 0 \leq j \leq l\}$,

(b) $T_{i+1}^{(1)} = \{j + 1 \pmod{2k+1} \mid j \in T_i^{(1)}\}$, $1 \leq i \leq 2k$,

(c) $T_{2k+i}^{(1)} = T_{i-1}^{(1)}$ for $2 \leq i \leq 2k+2$.

For part 2 and part 4, we define a $T^{(2)}$-sequence of length $4k+2$ by

$$T_i^{(2)} = \{j + (2k+1) \mid j \in T_i^{(1)}\} \text{ for } 1 \leq i \leq 4k+2.$$

For example, when $k = 1$, the $T^{(1)}$-sequence and $T^{(2)}$-sequence are

$$\begin{array}{llll} T_1^{(1)} = \{1\}, & T_1^{(2)} = \{4\}, & T_4^{(1)} = T_1^{(1)}, & T_4^{(2)} = T_1^{(2)}, \\ T_2^{(1)} = \{2\}, & T_2^{(2)} = \{5\}, & T_5^{(1)} = T_2^{(1)}, & T_5^{(2)} = T_2^{(2)}, \\ T_3^{(1)} = \{3\}, & T_3^{(2)} = \{6\}, & T_6^{(1)} = T_3^{(1)}, & T_6^{(2)} = T_3^{(2)}. \end{array}$$

Let T-sequence be $T = \{T_1, T_2, \ldots, T_{4k+2}\}$, where $T_i = T_i^{(1)} \cup T_i^{(2)}$, $1 \leq i \leq 4k+2$. The T-sequence degenerates into two subsequences:

$$\begin{aligned} T' &= \{T_1, T_2, \ldots, T_{2k+1}\} = \{S_1, S_3, \ldots, S_{4k+1}\}, \\ T'' &= \{T_{2k+2}, T_{2k+3}, \ldots, T_{4k+2}\} \\ &= \{S_{2k+2}, S_{2k+4}, \ldots, S_{4k+2}, S_2, S_4, \ldots, S_{2k}\}. \end{aligned}$$

It is obvious that $T_i \cup T_{i+1}$ is a total dominating set of C_{4k+2}. Thus $D = \{(i,j) \mid j \in S_i, 1 \leq i \leq 4k+2\}$ is a total dominating set for $C_{4k+2} \times C_{4k+2}$ (see Figure 4 for $k = 1$).

Similarly to the proof of Case 1, D is a total restrained dominating set of $C_{4k+2} \times C_{4k+2}$. Therefore, $\gamma_r^t(C_{4k+2} \times C_{4k+2}) \leq |D| = 4(2k+1)(l+1) =$

$4(2k+1)(\frac{k-1}{2}+1) = (4k+2)(k+1)$. Moreover, by Lemma 3.10 and Theorem 3.11, we have

$$\begin{aligned}\gamma_r^t(C_{4k+2} \times C_{4k+2}) \geq \gamma_t(C_{4k+2} \times C_{4k+2}) &= 4\gamma_t(C_{2k+1} \times C_{2k+1}) \\ &= 4(2k+1)(l+1) = 4(2k+1)\left(\frac{k-1}{2}+1\right) \\ &= (4k+2)(k+1).\end{aligned}$$

Hence, $\gamma_r^t(C_{4k+2} \times C_{4k+2}) = (4k+2)(k+1)$ if k is odd. ■

	1	2	3	4	5	6
1	•			•		
2			•			•
3		•			•	
4	•			•		
5			•			•
6		•			•	

Figure 4. A total dominating set of $C_6 \times C_6$.

Theorem 3.13. *Let n and m be odd and $n > m$. Then* $\gamma_r^t(C_n \times C_m) = \left\lceil \dfrac{n\gamma_t(C_m)}{2} \right\rceil$.

Proof. Consider the following two possibilities.

Case 1. Suppose $m = 4k+3$. Let $T = \{T_1, T_2, \ldots, T_m\}$ be the T-sequence of $C_m \times C_m$ defined in the proof of Case 1 of Theorem 3.12. It is obvious that $\sum_{i=1}^m |T_i| = m \cdot (k+1) = \left\lceil \frac{m\gamma_t(C_m)}{2} \right\rceil$.

From the construction of the T-sequence, we have $|T_1| + |T_2| = \gamma_t(C_m)$. We insert $n-m$ terms between T_1 and T_2 which are alternately equal to T_2 and T_1. We obtain a sequence

$$S = \{S_1, \ldots, S_n\} = \{T_1, T_2, T_1, T_2, \ldots, T_1, T_2, T_3, T_4, \ldots, T_m\}$$

of length n. Thus $D = \{(i,j) \mid j \in S_i,\ 1 \leq i \leq n\}$ is a total dominating set for $C_n \times C_m$. Similarly as in Case 1 of Theorem 3.12, D is a total restrained dominating set for $C_n \times C_m$. As a consequence, $\gamma_r^t(C_n \times C_m) \leq |D| = \left\lceil \frac{n\gamma_t(C_m)}{2} \right\rceil$.

Case 2. Suppose $m = 4k+1$. Let $T = \{T_1, T_2, \ldots, T_m\}$ be the T-sequence for $C_m \times C_m$ defined in the proof of Case 2 of Theorem 3.12. Similarly to the proof of Case 1, we have $\gamma_r^t(C_n \times C_m) \leq |D| = \left\lceil \frac{n\gamma_t(C_m)}{2} \right\rceil$. The result follows from (2). ■

Theorem 3.14. *Let n and m be even and $n > m$. Then*
(A) $\gamma_r^t(C_n \times C_m) = \frac{n}{2}\gamma_t(C_m)$ *if* $m = 4k$.
(B) $\gamma_r^t(C_n \times C_m) = \frac{n}{2}\gamma_t(C_m)$ *if* $m = 4k+2$ *for odd* k.
(C) $\frac{n}{2}\gamma_t(C_m) \le \gamma_r^t(C_n \times C_m) \le \frac{n}{2}\gamma_t(C_m) + 2$ *if* $m = 4k+2$ *for even* k.

Proof. Consider the following three cases.

(A) Suppose $m = 4k$. Let $T = \{T_1, T_2, \ldots, T_m\}$ be a T-sequence for $C_m \times C_m$, where

(a) $T_1 = \{4j+1 \mid 0 \le j \le k-1\}$ and

(b) $T_{i+1} = \{j+1 \pmod{4k} \mid j \in T_i\}$, $1 \le i \le 4k-1$.

Similarly as in Case 1 of Theorem 3.13, we have $\gamma_r^t(C_n \times C_m) \le \frac{n}{2}\gamma_t(C_m)$ and the result follows from (2).

(B) Suppose $m = 4k+2$ for odd k. Let $T = \{T_1, T_2, \ldots, T_m\}$ be the T-sequence of $C_m \times C_m$ defined in the proof of Case 4.2 of Theorem 3.12. Similarly as in Case 1 of Theorem 3.13, we have $\gamma_r^t(C_n \times C_m) \le \frac{n}{2}\gamma_t(C_m)$ and the result follows from (2).

(C) Suppose $m = 4k+2$ for even k. Let $T = \{T_1, T_2, \ldots, T_m\}$ be the T-sequence of $C_m \times C_m$ defined in the proof of Case 4.1 of Theorem 3.12. Similarly as in Case 1 of Theorem 3.13, we have

$$\begin{aligned}\gamma_r^t(C_n \times C_m) &\le \frac{n-(4k+2)}{2} \cdot \gamma_t(C_{4k+2}) + (4k^2+6k+4)\\ &= [n-(4k+2)](k+1) + (4k^2+6k+4)\\ &= n(k+1) + 2 = \frac{n}{2}\gamma_t(C_m) + 2.\end{aligned}$$

The result follows from (2). ■

We conclude this section with an upper bound for $\gamma_r^t(G \times K_n)$ which is in terms of the total 2-tuple domination number.

$S \subseteq V$ is a *k-tuple dominating set* of G if for every vertex $v \in V$, $|N[v] \cap S| \ge k$. The *k-tuple domination number* $\gamma^k(G)$ is the minimum cardinality of a k-tuple dominating set of G. $S \subseteq V$ is a *total k-tuple dominating set* of G if for every vertex $v \in V$, $|N(v) \cap S| \ge k$, that is, v is dominated by at least k neighbors in S. The *total k-tuple domination number* $\gamma_t^k(G)$ is the minimum cardinality of a total k-tuple dominating set of G.

Theorem 3.15. *Let G be a graph with $\delta(G) \ge 2$ and let $n \ge \max\{3, \gamma_t^2(G)\}$. Then $\gamma_r^t(G \times K_n) \le \gamma_t^2(G)$.*

Proof. Let $S = \{s_1, \ldots, s_k\}$ be a minimum total 2-tuple dominating set of G and let $V(K_n) = \{v_1, \ldots, v_n\}$. We claim that $T = \{(s_i, v_i) \mid 1 \leq i \leq k\}$ is a minimum total restrained dominating set of $G \times K_n$. Note that T is well-defined since $n \geq \gamma_t^2(G) = k$. Let (x, v_l) be an arbitrary vertex of $G \times K_n$ and assume that x is dominated by vertices s_i and s_j. Then s_i, s_j and x are distinct vertices. Suppose without loss of generality that $l \neq i$. Then (x, v_l) is dominated by (s_i, v_i) and hence T is a total dominating set of $G \times K_n$. Suppose $(x, v_l) \in V(G \times K_n) \setminus T$. Since $n \geq 3$, there is a vertex $v_h \in V(K_n) \setminus \{v_i, v_l\}$. Thus (x, v_l) is dominated by $(s_i, v_h) \in V(G \times K_n) \setminus T$ and hence T is a total restrained dominating set of $G \times K_n$. We conclude that $\gamma_r^t(G \times K_n) \leq \gamma_t^2(G)$. ∎

Theorem 3.6 is a direct consequence of Theorem 3.15 and we give another proof of it.

Theorem 3.16. *$\gamma_r^t(C_n \times K_m) = n$ for $n, m \geq 3$.*

Proof. Clearly, $\gamma_t^2(C_n) = n$, hence $\gamma_r^t(C_n \times K_m) \leq n$ by Theorem 3.15. On the other hand, the lower bound directly follows from Theorem 2.5. ∎

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