## Note

# CONVEX UNIVERSAL FIXERS 

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#### Abstract

In [1] Burger and Mynhardt introduced the idea of universal fixers. Let $G=(V, E)$ be a graph with $n$ vertices and $G^{\prime}$ a copy of $G$. For a bijective function $\pi: V(G) \rightarrow V\left(G^{\prime}\right)$, define the prism $\pi G$ of $G$ as follows: $V(\pi G)=$ $V(G) \cup V\left(G^{\prime}\right)$ and $E(\pi G)=E(G) \cup E\left(G^{\prime}\right) \cup M_{\pi}$, where $M_{\pi}=\{u \pi(u) \mid u \in$ $V(G)\}$. Let $\gamma(G)$ be the domination number of $G$. If $\gamma(\pi G)=\gamma(G)$ for any bijective function $\pi$, then $G$ is called a universal fixer. In [9] it is conjectured that the only universal fixers are the edgeless graphs $\overline{K_{n}}$.

In this work we generalize the concept of universal fixers to the convex universal fixers. In the second section we give a characterization for convex universal fixers (Theorem 6) and finally, we give an in infinite family of convex universal fixers for an arbitrary natural number $n \geq 10$.


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## 1. Introduction

Let $G=(V, E)$ be an undirected graph. The neighborhood of a vertex $v \in V$ in $G$ is the set $N_{G}(v)$ of all vertices adjacent to $v$ in $G$. For a set $X \subseteq V$, the
open neighborhood $N_{G}(X)$ is defined as $\bigcup_{v \in X} N_{G}(v)$ and the closed neighborhood $N_{G}[X]=N_{G}(X) \cup X$.

A set $D \subseteq V$ is a dominating set of $G$ if $N_{G}[D]=V$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in $G$.

The distance $d_{G}(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $u v$-path in $G$. A $u v$-path of length $d_{G}(u, v)$ is called $u v-$ geodesic. A set $X \subseteq V$ is a convex set of $G$ if the vertices from all $a b$ geodesic belong to $X$ for every two vertices $a, b \in X$. A set $X \subseteq V$ is a convex dominating set if $X$ is convex and dominating. The convex domination number $\gamma_{c o n}(G)$ of a graph $G$ is equal to the minimum cardinality of a convex dominating set. The convex domination number was defined by Jerzy Topp from the Gdańsk University of Technology in a verbal communication with the first author. In [5], the first results concerning this topic were published and developed in [6] and [7].

Definition 1. Let $G=(V, E)$ be a graph and $G^{\prime}$ a copy of $G$. For a bijective function $\pi: V(G) \rightarrow V\left(G^{\prime}\right)$, define the prism $\pi G$ of $G$ as follows: $V(\pi G)=$ $V(G) \cup V\left(G^{\prime}\right)$ and $E(\pi G)=E(G) \cup E\left(G^{\prime}\right) \cup M_{\pi}$, where $M_{\pi}=\{u \pi(u) \mid u \in V(G)\}$.

Notice that $M_{\pi}$ is a perfect matching of $\pi G$. It is clear that every permutation $\pi$ of $V(G)$ defines a bijective function from $V(G)$ to $V\left(G^{\prime}\right)$, so we will indistinctly use the matching $M_{\pi}$, the permutation $\pi$ of $V(G)$ or the associated bijection $\pi: V(G) \rightarrow V\left(G^{\prime}\right)$.

The graph $G$ is called a universal fixer if $\gamma(\pi G)=\gamma(G)$ for all permutations $\pi$ of $V(G)$.

The universal fixers were studied in [9] for several classes of graphs and it was conjectured that the edgeless graphs $\overline{K_{n}}$ are the only universal fixers. In [2], [3] and [4] it is shown that regular graphs, claw-free graphs and bipartite graphs are not universal fixers. This concept was also generalized for the other types of domination; in [10] the idea of paired domination in prisms was introduced.

We generalize the above definition for the convex domination: if $\gamma_{\text {con }}(\pi G)=$ $\gamma_{c o n}(G)$ for all permutation $\pi$ of $V(G)$,then we say that $G$ is a convex universal fixer.

## 2. Convex Universal Fixers

From now on we assume that the graph $G=(V, E)$ is a connected undirected graph with $n$ vertices. For $x \in V(G)$, the copy of $x$ in $V\left(G^{\prime}\right)$ is denoted by $x^{\prime}$. Recall that the diameter of a graph $G$, denoted by $\operatorname{diam}(G)$, is defined to be the maximum distance between any two vertices $x, y \in V(G)$.

Proposition 2. Let $G$ be a connected undirected graph.
(1) If $\operatorname{diam}(G) \leq 2$, then both $V(G)$ and $V\left(G^{\prime}\right)$ are convex dominating sets of $\pi G$ for any permutation $\pi$.
(2) If $\operatorname{diam}(G) \geq 3$, then there exist permutations $\pi_{1}$ and $\pi_{2}$ such that $V(G)$ is not a convex dominating set of $\pi_{1} G$ and $V\left(G^{\prime}\right)$ is not a convex dominating set of $\pi_{2} G$.
Proof. (1) It is clear that $V(G)$ and $V\left(G^{\prime}\right)$ are dominating sets of $\pi G$. Let $x, y \in V(G)$. Since $d_{\pi G}(x, y) \leq d_{G}(x, y) \leq 2$, any $x y$-geodesic is contained in $G$, so $V(G)$ is a convex dominating set of $\pi G$. In a similar way, we can prove that $V\left(G^{\prime}\right)$ is a convex dominating set of $\pi G$.
(2) Let $x, y \in V(G)$ be such that $d_{G}(x, y) \geq 3$. Let $w z \in E\left(G^{\prime}\right)$ and consider a permutation $\pi_{1}$ such that $\pi_{1}(x)=w$ and $\pi_{1}(y)=z$. Then $x w z y$ is an $x y$ geodesic in $\pi_{1} G$ with $z, w \notin V(G)$. In a similar way, we can prove that there exists a permutation $\pi_{2}$ such that $V\left(G^{\prime}\right)$ is not a convex dominating set in $\pi_{2} G$.

From the above proposition we have the following observation.
Observation 3. For any permutation $\pi, \gamma_{c o n}(\pi G) \leq n$ whenever $\operatorname{diam}(G) \leq 2$.
If $D$ is a convex dominating set of $\pi G$, we define $D_{1}$ as $D \cap V(G)$ and $D_{2}$ as $D \cap V\left(G^{\prime}\right)$. Moreover, we write $D_{1}^{c}=V(G)-D_{1}$ and $D_{2}^{c}=V\left(G^{\prime}\right)-D_{2}$.
Proposition 4. Let $D$ be a convex dominating set of $\pi G$.
(1) If $\gamma_{\text {con }}(\pi G)<n$, then $D_{1} \neq \emptyset$ and $D_{2} \neq \emptyset$.
(2) If $D_{1} \neq \emptyset$ and $D_{2} \neq \emptyset$, then there exists at least one edge $x \pi(x) \in M_{\pi}$ with $x \in D_{1}$ and $\pi(x) \in D_{2}$.

Proof. (1) Suppose that $D_{1}=\emptyset$. Then $D=D_{2} \subset V\left(G^{\prime}\right)$. Since $|D|<n, V(G)$ is not dominated by $D$. Similarly, if $D_{2}=\emptyset$, then $V\left(G^{\prime}\right)$ is not dominated by $D$.
(2) Let $x \in D_{1}$ and $\pi(y) \in D_{2}$. Since $D$ is convex, any $x \pi(y)$-geodesic should use the edge $x \pi(x)$ or the edge $y \pi(y)$.

Lemma 5. Suppose that $\operatorname{diam}(G) \leq 2$. Let $D$ be a minimum convex dominating set of $\pi G$. If $D=D_{1} \cup D_{2}$ with $D_{1} \neq \emptyset$ and $D_{2} \neq \emptyset$, then we have the following statements:
(1) if $\pi\left(D_{1}\right) \subseteq D_{2}$, then $D_{2}$ is a convex dominating set of $G^{\prime}$, and
(2) if $\pi^{-1}\left(D_{2}\right) \subseteq D_{1}$, then $D_{1}$ is a convex dominating set of $G$.

Proof. Assume that $\pi\left(D_{1}\right) \subseteq D_{2}$. Then, since $D$ is a dominating set of $\pi G$, every vertex of $D_{2}^{c}$ has a neighbor in $D_{2}$. Moreover, $\operatorname{diam}\left(G^{\prime}\right) \leq 2$ and $d_{\pi G}(a, b) \leq 2$ for every two vertices $a, b \in D_{2}$, so the vertices from all $a b$-geodesics belong to $D_{2}$, because $D$ is convex. Thus $D_{2}$ is a convex dominating set of $G^{\prime}$. Similarly, we can prove the second part of the lemma.

Our main result is the following.
Theorem 6. Let $G$ be a connected undirected graph. If $\gamma_{c o n}(G)=n$ and $\operatorname{diam}(G) \leq 2$, then $\gamma_{c o n}(\pi G)=n$, that is, $G$ is a convex universal fixer.

Proof. By Observation 3, if $\operatorname{diam}(G) \leq 2$, then $\gamma_{c o n}(G) \leq n$ for all permutations $\pi$. By contradiction, suppose that $\gamma_{c o n}(G)=n$ and $\gamma_{c o n}(\pi G)<n$. If $\operatorname{diam}(G)=$ 1 , then $\gamma_{c o n}(G)<n$, so we can assume $\operatorname{diam}(G)=2$.

Let $D=D_{1} \cup D_{2}$ be a minimum convex dominating set of $\pi G$ with $|D|<n$. From the first part of Proposition 4, we have that $D_{1} \neq \emptyset$ and $D_{2} \neq \emptyset$. In order to have a partition of $V(\pi G)$, we define the following subsets of vertices:

$$
\begin{gathered}
D_{1}^{+}=\left\{u \in D_{1} \mid \pi(u) \in D_{2}\right\}, \quad D_{2}^{+}=\left\{u^{\prime} \in D_{2} \mid \pi^{-1}\left(u^{\prime}\right) \in D_{1}\right\}=\pi\left(D_{1}^{+}\right), \\
D_{1}^{-}=\left\{u \in D_{1} \mid \pi(u) \notin D_{2}\right\}, \quad D_{2}^{-}=\left\{u^{\prime} \in D_{2} \mid \pi^{-1}\left(u^{\prime}\right) \notin D_{1}\right\}, \\
E_{1}=\pi^{-1}\left(D_{2}^{-}\right), \quad E_{2}=\pi\left(D_{1}^{-}\right), \\
F_{1}=V(G)-D_{1}-E_{1} \text { and } F_{2}=\pi\left(F_{1}\right) .
\end{gathered}
$$

From the second part of Proposition 4, we have that $D_{1}^{+} \neq \emptyset$ and $D_{2}^{+} \neq \emptyset$. If $\pi\left(D_{1}\right) \subseteq D_{2}$, then by Lemma 5 , the set $D_{2}$ is a convex dominating set of $G^{\prime}$, which is a contradiction since $\gamma_{\text {con }}\left(G^{\prime}\right)=n$. Therefore, $D_{1}^{-} \neq \emptyset$. In a similar way, $D_{2}^{-} \neq$ $\emptyset$. In consequence $E_{1} \neq \emptyset$ and $E_{2} \neq \emptyset$. Since $|D|<n,\left|D_{1}^{+} \cup D_{1}^{-} \cup D_{2}^{+} \cup D_{2}^{-}\right|<n$ and $\left|E_{1} \cup E_{2}\right|=\left|D_{1}^{-} \cup D_{2}^{-}\right|<n$. Therefore, $F_{1}$ and $F_{2}$ are nonempty.

We claim that there are no edges between $E_{1}$ and $D_{1}$. Suppose $x \in D_{1}, y \in E_{1}$ and $x y \in E(G)$. Then $d_{\pi G}(x, \pi(y))=2$, and $x, \pi(y) \in D$ implies that $y \in D_{1}$, which leads us to a contradiction.

Let $x$ be a vertex in $D_{1}^{-}$and $y \in E_{1}$. Since $\operatorname{diam}(G)=2, d_{G}(x, y)=2$ and there exists a vertex $z \in F_{1}$ such that $x z \in E(G)$ and $y z \in E(G)$.

If $d_{\pi G}(x, \pi(y)) \geq 3$, then $x z y \pi(y)$ is an $x \pi(y)$-geodesic, which is not possible, since $D$ is a convex dominating set of $\pi G$ and $y, z \notin D$. Thus $d_{\pi G}(x, \pi(y))=2$. But then there exists a vertex $w \in D$ such that $w$ is a common neighbor of $x$ and $\pi(y)$, a contradiction. Therefore, $\gamma_{c o n}(\pi G)=n$.

## 3. An Infinite Family of Convex Universal Fixers

Now we show that for an arbitrarily large $n$, there is a graph $G$ with $n$ vertices such that $G$ is a convex universal fixer. The following family $\mathcal{F}$ of graphs was defined in [8].

Let $G_{1}$ be the cycle of order five, $C_{5}^{1}=\left(v_{1,1}, v_{1,2}, v_{1,3}, v_{1,4}, v_{1,5}, v_{1,1}\right)$. For $i \geq 2$, the graph $G_{i}$ is obtained recursively from $G_{i-1}$ by adding a cycle graph $C_{5}^{i}=\left(v_{i, 1}, v_{i, 2}, v_{i, 3}, v_{i, 4}, v_{i, 5}, v_{i, 1}\right)$ and for every vertex $v_{i, j}, j \in\{1, \cdots, 5\}$ of the
cycle $C_{5}^{i}$ we add edges $v_{i, j} v_{l, j-1}$ and $v_{i, j} v_{l, j+1}$ with $l \in\{1, \cdots, i-1\}$. The sums $j-1, j+1$ are done modulo five.

The authors denoted by $\mathcal{F}$ the family of graphs $G$ obtained by adding to the graph $G_{i}, t \geq 2$ vertices $u_{1}, \ldots, u_{t}$ and edges $u_{k} v_{i, j}$, with $k \in\{1, \ldots, t\}$ and $j \in\{1, \ldots, 5\}$.


Figure 1. A graph belonging to the family $\mathcal{F}$ with $n=12, t=2$ and $i=2$.
The following result was proved in [8].
Theorem 7. If $G$ belongs to the family $\mathcal{F}$, then $\gamma_{c o n}(G)=n$ and $\operatorname{diam}(G)=2$.
From the above theorem and our main result we can conclude the following
Corollary 8. For every natural number $n \geq 10$, there is a graph $G$ with $n$ vertices such that $G$ is a convex universal fixer.

## 4. Acknowledgments and Conjectures

We conclude this paper with the following two conjectures.
Conjecture 9. If $G$ is a convex universal fixer, then $\gamma_{c o n}(G)=n \operatorname{and} \operatorname{diam}(G)=$ 2.

Conjecture 10. If $G$ is a convex universal fixer, then the only minimum convex dominating sets of $\pi G$ are $V(G)$ and $V\left(G^{\prime}\right)$.

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