

NOTE

## CONVEX UNIVERSAL FIXERS

MAGDALENA LEMAŃSKA

*Gdańsk University of Technology*  
*Narutowicza 11/12*  
*80-233 Gdańsk, Poland*

**e-mail:** magda@mif.pg.gda.pl

AND

RITA ZUAZUA

*Departamento de Matematicas, Facultad de Ciencias*  
*UNAM, Mexico*

**e-mail:** ritazuazua@gmail.com

### Abstract

In [1] Burger and Mynhardt introduced the idea of universal fixers. Let  $G = (V, E)$  be a graph with  $n$  vertices and  $G'$  a copy of  $G$ . For a bijective function  $\pi : V(G) \rightarrow V(G')$ , define the prism  $\pi G$  of  $G$  as follows:  $V(\pi G) = V(G) \cup V(G')$  and  $E(\pi G) = E(G) \cup E(G') \cup M_\pi$ , where  $M_\pi = \{u\pi(u) \mid u \in V(G)\}$ . Let  $\gamma(G)$  be the domination number of  $G$ . If  $\gamma(\pi G) = \gamma(G)$  for any bijective function  $\pi$ , then  $G$  is called a universal fixer. In [9] it is conjectured that the only universal fixers are the edgeless graphs  $\overline{K_n}$ .

In this work we generalize the concept of universal fixers to the convex universal fixers. In the second section we give a characterization for convex universal fixers (Theorem 6) and finally, we give an infinite family of convex universal fixers for an arbitrary natural number  $n \geq 10$ .

**Keywords:** convex sets, dominating sets, universal fixers.

**2010 Mathematics Subject Classification:** 05C69, 05C99.

### 1. INTRODUCTION

Let  $G = (V, E)$  be an undirected graph. The *neighborhood* of a vertex  $v \in V$  in  $G$  is the set  $N_G(v)$  of all vertices adjacent to  $v$  in  $G$ . For a set  $X \subseteq V$ , the

*open neighborhood*  $N_G(X)$  is defined as  $\bigcup_{v \in X} N_G(v)$  and the *closed neighborhood*  $N_G[X] = N_G(X) \cup X$ .

A set  $D \subseteq V$  is a *dominating set* of  $G$  if  $N_G[D] = V$ . The *domination number* of  $G$ , denoted by  $\gamma(G)$ , is the minimum cardinality of a dominating set in  $G$ .

The *distance*  $d_G(u, v)$  between two vertices  $u$  and  $v$  in a connected graph  $G$  is the length of a shortest  $uv$ -path in  $G$ . A  $uv$ -path of length  $d_G(u, v)$  is called  *$uv$ -geodesic*. A set  $X \subseteq V$  is a *convex set* of  $G$  if the vertices from all  *$ab$ -geodesic* belong to  $X$  for every two vertices  $a, b \in X$ . A set  $X \subseteq V$  is a *convex dominating set* if  $X$  is convex and dominating. The *convex domination number*  $\gamma_{con}(G)$  of a graph  $G$  is equal to the minimum cardinality of a convex dominating set. The convex domination number was defined by Jerzy Topp from the Gdańsk University of Technology in a verbal communication with the first author. In [5], the first results concerning this topic were published and developed in [6] and [7].

**Definition 1.** Let  $G = (V, E)$  be a graph and  $G'$  a copy of  $G$ . For a bijective function  $\pi : V(G) \rightarrow V(G')$ , define the prism  $\pi G$  of  $G$  as follows:  $V(\pi G) = V(G) \cup V(G')$  and  $E(\pi G) = E(G) \cup E(G') \cup M_\pi$ , where  $M_\pi = \{u\pi(u) \mid u \in V(G)\}$ .

Notice that  $M_\pi$  is a perfect matching of  $\pi G$ . It is clear that every permutation  $\pi$  of  $V(G)$  defines a bijective function from  $V(G)$  to  $V(G')$ , so we will indistinctly use the matching  $M_\pi$ , the permutation  $\pi$  of  $V(G)$  or the associated bijection  $\pi : V(G) \rightarrow V(G')$ .

The graph  $G$  is called a *universal fixer* if  $\gamma(\pi G) = \gamma(G)$  for all permutations  $\pi$  of  $V(G)$ .

The universal fixers were studied in [9] for several classes of graphs and it was conjectured that the edgeless graphs  $\overline{K_n}$  are the only universal fixers. In [2], [3] and [4] it is shown that regular graphs, claw-free graphs and bipartite graphs are not universal fixers. This concept was also generalized for the other types of domination; in [10] the idea of paired domination in prisms was introduced.

We generalize the above definition for the convex domination: if  $\gamma_{con}(\pi G) = \gamma_{con}(G)$  for all permutation  $\pi$  of  $V(G)$ , then we say that  $G$  is a *convex universal fixer*.

## 2. CONVEX UNIVERSAL FIXERS

From now on we assume that the graph  $G = (V, E)$  is a connected undirected graph with  $n$  vertices. For  $x \in V(G)$ , the copy of  $x$  in  $V(G')$  is denoted by  $x'$ . Recall that the diameter of a graph  $G$ , denoted by  $diam(G)$ , is defined to be the maximum distance between any two vertices  $x, y \in V(G)$ .

**Proposition 2.** Let  $G$  be a connected undirected graph.

- (1) If  $\text{diam}(G) \leq 2$ , then both  $V(G)$  and  $V(G')$  are convex dominating sets of  $\pi G$  for any permutation  $\pi$ .
- (2) If  $\text{diam}(G) \geq 3$ , then there exist permutations  $\pi_1$  and  $\pi_2$  such that  $V(G)$  is not a convex dominating set of  $\pi_1 G$  and  $V(G')$  is not a convex dominating set of  $\pi_2 G$ .

**Proof.** (1) It is clear that  $V(G)$  and  $V(G')$  are dominating sets of  $\pi G$ . Let  $x, y \in V(G)$ . Since  $d_{\pi G}(x, y) \leq d_G(x, y) \leq 2$ , any  $xy$ -geodesic is contained in  $G$ , so  $V(G)$  is a convex dominating set of  $\pi G$ . In a similar way, we can prove that  $V(G')$  is a convex dominating set of  $\pi G$ .

(2) Let  $x, y \in V(G)$  be such that  $d_G(x, y) \geq 3$ . Let  $wz \in E(G')$  and consider a permutation  $\pi_1$  such that  $\pi_1(x) = w$  and  $\pi_1(y) = z$ . Then  $xwzy$  is an  $xy$ -geodesic in  $\pi_1 G$  with  $z, w \notin V(G)$ . In a similar way, we can prove that there exists a permutation  $\pi_2$  such that  $V(G')$  is not a convex dominating set in  $\pi_2 G$ . ■

From the above proposition we have the following observation.

**Observation 3.** For any permutation  $\pi$ ,  $\gamma_{\text{con}}(\pi G) \leq n$  whenever  $\text{diam}(G) \leq 2$ .

If  $D$  is a convex dominating set of  $\pi G$ , we define  $D_1$  as  $D \cap V(G)$  and  $D_2$  as  $D \cap V(G')$ . Moreover, we write  $D_1^c = V(G) - D_1$  and  $D_2^c = V(G') - D_2$ .

**Proposition 4.** Let  $D$  be a convex dominating set of  $\pi G$ .

- (1) If  $\gamma_{\text{con}}(\pi G) < n$ , then  $D_1 \neq \emptyset$  and  $D_2 \neq \emptyset$ .
- (2) If  $D_1 \neq \emptyset$  and  $D_2 \neq \emptyset$ , then there exists at least one edge  $x\pi(x) \in M_\pi$  with  $x \in D_1$  and  $\pi(x) \in D_2$ .

**Proof.** (1) Suppose that  $D_1 = \emptyset$ . Then  $D = D_2 \subset V(G')$ . Since  $|D| < n$ ,  $V(G)$  is not dominated by  $D$ . Similarly, if  $D_2 = \emptyset$ , then  $V(G')$  is not dominated by  $D$ .

(2) Let  $x \in D_1$  and  $\pi(y) \in D_2$ . Since  $D$  is convex, any  $x\pi(y)$ -geodesic should use the edge  $x\pi(x)$  or the edge  $y\pi(y)$ . ■

**Lemma 5.** Suppose that  $\text{diam}(G) \leq 2$ . Let  $D$  be a minimum convex dominating set of  $\pi G$ . If  $D = D_1 \cup D_2$  with  $D_1 \neq \emptyset$  and  $D_2 \neq \emptyset$ , then we have the following statements:

- (1) if  $\pi(D_1) \subseteq D_2$ , then  $D_2$  is a convex dominating set of  $G'$ , and
- (2) if  $\pi^{-1}(D_2) \subseteq D_1$ , then  $D_1$  is a convex dominating set of  $G$ .

**Proof.** Assume that  $\pi(D_1) \subseteq D_2$ . Then, since  $D$  is a dominating set of  $\pi G$ , every vertex of  $D_2^c$  has a neighbor in  $D_2$ . Moreover,  $\text{diam}(G') \leq 2$  and  $d_{\pi G}(a, b) \leq 2$  for every two vertices  $a, b \in D_2$ , so the vertices from all  $ab$ -geodesics belong to  $D_2$ , because  $D$  is convex. Thus  $D_2$  is a convex dominating set of  $G'$ . Similarly, we can prove the second part of the lemma. ■

Our main result is the following.

**Theorem 6.** *Let  $G$  be a connected undirected graph. If  $\gamma_{con}(G) = n$  and  $\text{diam}(G) \leq 2$ , then  $\gamma_{con}(\pi G) = n$ , that is,  $G$  is a convex universal fixer.*

**Proof.** By Observation 3, if  $\text{diam}(G) \leq 2$ , then  $\gamma_{con}(G) \leq n$  for all permutations  $\pi$ . By contradiction, suppose that  $\gamma_{con}(G) = n$  and  $\gamma_{con}(\pi G) < n$ . If  $\text{diam}(G) = 1$ , then  $\gamma_{con}(G) < n$ , so we can assume  $\text{diam}(G) = 2$ .

Let  $D = D_1 \cup D_2$  be a minimum convex dominating set of  $\pi G$  with  $|D| < n$ . From the first part of Proposition 4, we have that  $D_1 \neq \emptyset$  and  $D_2 \neq \emptyset$ . In order to have a partition of  $V(\pi G)$ , we define the following subsets of vertices:

$$D_1^+ = \{u \in D_1 \mid \pi(u) \in D_2\}, \quad D_2^+ = \{u' \in D_2 \mid \pi^{-1}(u') \in D_1\} = \pi(D_1^+),$$

$$D_1^- = \{u \in D_1 \mid \pi(u) \notin D_2\}, \quad D_2^- = \{u' \in D_2 \mid \pi^{-1}(u') \notin D_1\},$$

$$E_1 = \pi^{-1}(D_2^-), \quad E_2 = \pi(D_1^-),$$

$$F_1 = V(G) - D_1 - E_1 \quad \text{and} \quad F_2 = \pi(F_1).$$

From the second part of Proposition 4, we have that  $D_1^+ \neq \emptyset$  and  $D_2^+ \neq \emptyset$ . If  $\pi(D_1) \subseteq D_2$ , then by Lemma 5, the set  $D_2$  is a convex dominating set of  $G'$ , which is a contradiction since  $\gamma_{con}(G') = n$ . Therefore,  $D_1^- \neq \emptyset$ . In a similar way,  $D_2^- \neq \emptyset$ . In consequence  $E_1 \neq \emptyset$  and  $E_2 \neq \emptyset$ . Since  $|D| < n$ ,  $|D_1^+ \cup D_1^- \cup D_2^+ \cup D_2^-| < n$  and  $|E_1 \cup E_2| = |D_1^- \cup D_2^-| < n$ . Therefore,  $F_1$  and  $F_2$  are nonempty.

We claim that there are no edges between  $E_1$  and  $D_1$ . Suppose  $x \in D_1, y \in E_1$  and  $xy \in E(G)$ . Then  $d_{\pi G}(x, \pi(y)) = 2$ , and  $x, \pi(y) \in D$  implies that  $y \in D_1$ , which leads us to a contradiction.

Let  $x$  be a vertex in  $D_1^-$  and  $y \in E_1$ . Since  $\text{diam}(G) = 2$ ,  $d_G(x, y) = 2$  and there exists a vertex  $z \in F_1$  such that  $xz \in E(G)$  and  $yz \in E(G)$ .

If  $d_{\pi G}(x, \pi(y)) \geq 3$ , then  $xzy\pi(y)$  is an  $x\pi(y)$ -geodesic, which is not possible, since  $D$  is a convex dominating set of  $\pi G$  and  $y, z \notin D$ . Thus  $d_{\pi G}(x, \pi(y)) = 2$ . But then there exists a vertex  $w \in D$  such that  $w$  is a common neighbor of  $x$  and  $\pi(y)$ , a contradiction. Therefore,  $\gamma_{con}(\pi G) = n$ . ■

### 3. AN INFINITE FAMILY OF CONVEX UNIVERSAL FIXERS

Now we show that for an arbitrarily large  $n$ , there is a graph  $G$  with  $n$  vertices such that  $G$  is a convex universal fixer. The following family  $\mathcal{F}$  of graphs was defined in [8].

Let  $G_1$  be the cycle of order five,  $C_5^1 = (v_{1,1}, v_{1,2}, v_{1,3}, v_{1,4}, v_{1,5}, v_{1,1})$ . For  $i \geq 2$ , the graph  $G_i$  is obtained recursively from  $G_{i-1}$  by adding a cycle graph  $C_5^i = (v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4}, v_{i,5}, v_{i,1})$  and for every vertex  $v_{i,j}$ ,  $j \in \{1, \dots, 5\}$  of the

cycle  $C_5^i$  we add edges  $v_{i,j}v_{l,j-1}$  and  $v_{i,j}v_{l,j+1}$  with  $l \in \{1, \dots, i-1\}$ . The sums  $j-1, j+1$  are done modulo five.

The authors denoted by  $\mathcal{F}$  the family of graphs  $G$  obtained by adding to the graph  $G_i$ ,  $t \geq 2$  vertices  $u_1, \dots, u_t$  and edges  $u_kv_{i,j}$ , with  $k \in \{1, \dots, t\}$  and  $j \in \{1, \dots, 5\}$ .

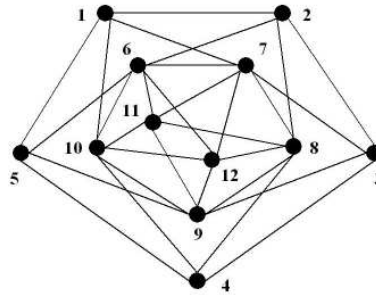


Figure 1. A graph belonging to the family  $\mathcal{F}$  with  $n = 12$ ,  $t = 2$  and  $i = 2$ .

The following result was proved in [8].

**Theorem 7.** *If  $G$  belongs to the family  $\mathcal{F}$ , then  $\gamma_{con}(G) = n$  and  $diam(G) = 2$ .*

From the above theorem and our main result we can conclude the following

**Corollary 8.** *For every natural number  $n \geq 10$ , there is a graph  $G$  with  $n$  vertices such that  $G$  is a convex universal fixer.*

#### 4. ACKNOWLEDGMENTS AND CONJECTURES

We conclude this paper with the following two conjectures.

**Conjecture 9.** *If  $G$  is a convex universal fixer, then  $\gamma_{con}(G) = n$  and  $diam(G) = 2$ .*

**Conjecture 10.** *If  $G$  is a convex universal fixer, then the only minimum convex dominating sets of  $\pi G$  are  $V(G)$  and  $V(G')$ .*

#### Acknowledgements

We thank Ilán Goldfeder, Bernardo Llano and the anonymous referees for useful comments. The authors thank the financial support received from Grant UNAM-PAPIIT IN-111309 and by Grant SEP-CONACyT 83856-E.

## REFERENCES

- [1] A.P. Burger, C.M. Mynhardt and W.D. Weakley, *On the domination number of prisms of graphs*, Discuss. Math. Graph Theory **24** (2004) 303–318.  
doi:10.7151/dmgt.1233
- [2] A.P. Burger and C.M. Mynhardt, *Regular graphs are not universal fixers*, Discrete Math. **310** (2010) 364–368.  
doi:10.1016/j.disc.2008.09.016
- [3] E.J. Cockayne, R.G. Gibson and C.M. Mynhardt, *Claw-free graphs are not universal fixers*, Discrete Math. **309** (2009) 128–133.  
doi:10.1016/j.disc.2007.12.053
- [4] R.G. Gibson, *Bipartite graphs are not universal fixers*, Discrete Math. **308** (2008) 5937–5943.  
doi:10.1016/j.disc.2007.11.006
- [5] M. Lemańska, *Weakly convex and convex domination numbers*, Opuscula Math. **24** (2004) 181–188.
- [6] J. Cyman, M. Lemańska and J. Raczek, *Graphs with convex domination number close to their order*, Discuss. Math. Graph Theory **26** (2006) 307–316.  
doi:10.7151/dmgt.1322
- [7] J. Raczek and M. Lemańska, *A note of the weakly convex and convex domination numbers of a torus*, Discrete Appl. Math. **158** (2010) 1708–1713.  
doi:10.1016/j.dam.2010.06.001
- [8] M. Lemańska, I. González Yero and J.A. Rodríguez-Velázquez, *Nordhaus-Gaddum results for a convex domination number of a graph*, Acta Math. Hungar., to appear (2011).
- [9] C.M. Mynhardt and Z. Xu, *Domination in Prisms of Graphs: Universal Fixers*, Util. Math. **78** (2009) 185–201.
- [10] C.M. Mynhardt and M. Schurch, *Paired domination in prisms of graphs*, Discuss. Math. Graph Theory **31** (2011) 5–23.  
doi:10.7151/dmgt.1526

Received 25 August 2011

Revised 14 November 2011

Accepted 18 November 2011