

NOTE

CONVEX UNIVERSAL FIXERS

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Abstract

In [1] Burger and Mynhardt introduced the idea of universal fixers. Let $G = (V, E)$ be a graph with n vertices and G' a copy of G . For a bijective function $\pi : V(G) \rightarrow V(G')$, define the prism πG of G as follows: $V(\pi G) = V(G) \cup V(G')$ and $E(\pi G) = E(G) \cup E(G') \cup M_\pi$, where $M_\pi = \{u\pi(u) \mid u \in V(G)\}$. Let $\gamma(G)$ be the domination number of G . If $\gamma(\pi G) = \gamma(G)$ for any bijective function π , then G is called a universal fixer. In [9] it is conjectured that the only universal fixers are the edgeless graphs $\overline{K_n}$.

In this work we generalize the concept of universal fixers to the convex universal fixers. In the second section we give a characterization for convex universal fixers (Theorem 6) and finally, we give an infinite family of convex universal fixers for an arbitrary natural number $n \geq 10$.

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1. INTRODUCTION

Let $G = (V, E)$ be an undirected graph. The *neighborhood* of a vertex $v \in V$ in G is the set $N_G(v)$ of all vertices adjacent to v in G . For a set $X \subseteq V$, the

open neighborhood $N_G(X)$ is defined as $\bigcup_{v \in X} N_G(v)$ and the *closed neighborhood* $N_G[X] = N_G(X) \cup X$.

A set $D \subseteq V$ is a *dominating set* of G if $N_G[D] = V$. The *domination number* of G , denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G .

The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest uv -path in G . A uv -path of length $d_G(u, v)$ is called *uv -geodesic*. A set $X \subseteq V$ is a *convex set* of G if the vertices from all *ab -geodesic* belong to X for every two vertices $a, b \in X$. A set $X \subseteq V$ is a *convex dominating set* if X is convex and dominating. The *convex domination number* $\gamma_{con}(G)$ of a graph G is equal to the minimum cardinality of a convex dominating set. The convex domination number was defined by Jerzy Topp from the Gdańsk University of Technology in a verbal communication with the first author. In [5], the first results concerning this topic were published and developed in [6] and [7].

Definition 1. Let $G = (V, E)$ be a graph and G' a copy of G . For a bijective function $\pi : V(G) \rightarrow V(G')$, define the prism πG of G as follows: $V(\pi G) = V(G) \cup V(G')$ and $E(\pi G) = E(G) \cup E(G') \cup M_\pi$, where $M_\pi = \{u\pi(u) \mid u \in V(G)\}$.

Notice that M_π is a perfect matching of πG . It is clear that every permutation π of $V(G)$ defines a bijective function from $V(G)$ to $V(G')$, so we will indistinctly use the matching M_π , the permutation π of $V(G)$ or the associated bijection $\pi : V(G) \rightarrow V(G')$.

The graph G is called a *universal fixer* if $\gamma(\pi G) = \gamma(G)$ for all permutations π of $V(G)$.

The universal fixers were studied in [9] for several classes of graphs and it was conjectured that the edgeless graphs $\overline{K_n}$ are the only universal fixers. In [2], [3] and [4] it is shown that regular graphs, claw-free graphs and bipartite graphs are not universal fixers. This concept was also generalized for the other types of domination; in [10] the idea of paired domination in prisms was introduced.

We generalize the above definition for the convex domination: if $\gamma_{con}(\pi G) = \gamma_{con}(G)$ for all permutation π of $V(G)$, then we say that G is a *convex universal fixer*.

2. CONVEX UNIVERSAL FIXERS

From now on we assume that the graph $G = (V, E)$ is a connected undirected graph with n vertices. For $x \in V(G)$, the copy of x in $V(G')$ is denoted by x' . Recall that the diameter of a graph G , denoted by $diam(G)$, is defined to be the maximum distance between any two vertices $x, y \in V(G)$.

Proposition 2. Let G be a connected undirected graph.

- (1) If $\text{diam}(G) \leq 2$, then both $V(G)$ and $V(G')$ are convex dominating sets of πG for any permutation π .
- (2) If $\text{diam}(G) \geq 3$, then there exist permutations π_1 and π_2 such that $V(G)$ is not a convex dominating set of $\pi_1 G$ and $V(G')$ is not a convex dominating set of $\pi_2 G$.

Proof. (1) It is clear that $V(G)$ and $V(G')$ are dominating sets of πG . Let $x, y \in V(G)$. Since $d_{\pi G}(x, y) \leq d_G(x, y) \leq 2$, any xy -geodesic is contained in G , so $V(G)$ is a convex dominating set of πG . In a similar way, we can prove that $V(G')$ is a convex dominating set of πG .

(2) Let $x, y \in V(G)$ be such that $d_G(x, y) \geq 3$. Let $wz \in E(G')$ and consider a permutation π_1 such that $\pi_1(x) = w$ and $\pi_1(y) = z$. Then $xwzy$ is an xy -geodesic in $\pi_1 G$ with $z, w \notin V(G)$. In a similar way, we can prove that there exists a permutation π_2 such that $V(G')$ is not a convex dominating set in $\pi_2 G$. ■

From the above proposition we have the following observation.

Observation 3. For any permutation π , $\gamma_{\text{con}}(\pi G) \leq n$ whenever $\text{diam}(G) \leq 2$.

If D is a convex dominating set of πG , we define D_1 as $D \cap V(G)$ and D_2 as $D \cap V(G')$. Moreover, we write $D_1^c = V(G) - D_1$ and $D_2^c = V(G') - D_2$.

Proposition 4. Let D be a convex dominating set of πG .

- (1) If $\gamma_{\text{con}}(\pi G) < n$, then $D_1 \neq \emptyset$ and $D_2 \neq \emptyset$.
- (2) If $D_1 \neq \emptyset$ and $D_2 \neq \emptyset$, then there exists at least one edge $x\pi(x) \in M_\pi$ with $x \in D_1$ and $\pi(x) \in D_2$.

Proof. (1) Suppose that $D_1 = \emptyset$. Then $D = D_2 \subset V(G')$. Since $|D| < n$, $V(G)$ is not dominated by D . Similarly, if $D_2 = \emptyset$, then $V(G')$ is not dominated by D .

(2) Let $x \in D_1$ and $\pi(y) \in D_2$. Since D is convex, any $x\pi(y)$ -geodesic should use the edge $x\pi(x)$ or the edge $y\pi(y)$. ■

Lemma 5. Suppose that $\text{diam}(G) \leq 2$. Let D be a minimum convex dominating set of πG . If $D = D_1 \cup D_2$ with $D_1 \neq \emptyset$ and $D_2 \neq \emptyset$, then we have the following statements:

- (1) if $\pi(D_1) \subseteq D_2$, then D_2 is a convex dominating set of G' , and
- (2) if $\pi^{-1}(D_2) \subseteq D_1$, then D_1 is a convex dominating set of G .

Proof. Assume that $\pi(D_1) \subseteq D_2$. Then, since D is a dominating set of πG , every vertex of D_2^c has a neighbor in D_2 . Moreover, $\text{diam}(G') \leq 2$ and $d_{\pi G}(a, b) \leq 2$ for every two vertices $a, b \in D_2$, so the vertices from all ab -geodesics belong to D_2 , because D is convex. Thus D_2 is a convex dominating set of G' . Similarly, we can prove the second part of the lemma. ■

Our main result is the following.

Theorem 6. *Let G be a connected undirected graph. If $\gamma_{con}(G) = n$ and $\text{diam}(G) \leq 2$, then $\gamma_{con}(\pi G) = n$, that is, G is a convex universal fixer.*

Proof. By Observation 3, if $\text{diam}(G) \leq 2$, then $\gamma_{con}(G) \leq n$ for all permutations π . By contradiction, suppose that $\gamma_{con}(G) = n$ and $\gamma_{con}(\pi G) < n$. If $\text{diam}(G) = 1$, then $\gamma_{con}(G) < n$, so we can assume $\text{diam}(G) = 2$.

Let $D = D_1 \cup D_2$ be a minimum convex dominating set of πG with $|D| < n$. From the first part of Proposition 4, we have that $D_1 \neq \emptyset$ and $D_2 \neq \emptyset$. In order to have a partition of $V(\pi G)$, we define the following subsets of vertices:

$$D_1^+ = \{u \in D_1 \mid \pi(u) \in D_2\}, \quad D_2^+ = \{u' \in D_2 \mid \pi^{-1}(u') \in D_1\} = \pi(D_1^+),$$

$$D_1^- = \{u \in D_1 \mid \pi(u) \notin D_2\}, \quad D_2^- = \{u' \in D_2 \mid \pi^{-1}(u') \notin D_1\},$$

$$E_1 = \pi^{-1}(D_2^-), \quad E_2 = \pi(D_1^-),$$

$$F_1 = V(G) - D_1 - E_1 \quad \text{and} \quad F_2 = \pi(F_1).$$

From the second part of Proposition 4, we have that $D_1^+ \neq \emptyset$ and $D_2^+ \neq \emptyset$. If $\pi(D_1) \subseteq D_2$, then by Lemma 5, the set D_2 is a convex dominating set of G' , which is a contradiction since $\gamma_{con}(G') = n$. Therefore, $D_1^- \neq \emptyset$. In a similar way, $D_2^- \neq \emptyset$. In consequence $E_1 \neq \emptyset$ and $E_2 \neq \emptyset$. Since $|D| < n$, $|D_1^+ \cup D_1^- \cup D_2^+ \cup D_2^-| < n$ and $|E_1 \cup E_2| = |D_1^- \cup D_2^-| < n$. Therefore, F_1 and F_2 are nonempty.

We claim that there are no edges between E_1 and D_1 . Suppose $x \in D_1, y \in E_1$ and $xy \in E(G)$. Then $d_{\pi G}(x, \pi(y)) = 2$, and $x, \pi(y) \in D$ implies that $y \in D_1$, which leads us to a contradiction.

Let x be a vertex in D_1^- and $y \in E_1$. Since $\text{diam}(G) = 2$, $d_G(x, y) = 2$ and there exists a vertex $z \in F_1$ such that $xz \in E(G)$ and $yz \in E(G)$.

If $d_{\pi G}(x, \pi(y)) \geq 3$, then $xzy\pi(y)$ is an $x\pi(y)$ -geodesic, which is not possible, since D is a convex dominating set of πG and $y, z \notin D$. Thus $d_{\pi G}(x, \pi(y)) = 2$. But then there exists a vertex $w \in D$ such that w is a common neighbor of x and $\pi(y)$, a contradiction. Therefore, $\gamma_{con}(\pi G) = n$. ■

3. AN INFINITE FAMILY OF CONVEX UNIVERSAL FIXERS

Now we show that for an arbitrarily large n , there is a graph G with n vertices such that G is a convex universal fixer. The following family \mathcal{F} of graphs was defined in [8].

Let G_1 be the cycle of order five, $C_5^1 = (v_{1,1}, v_{1,2}, v_{1,3}, v_{1,4}, v_{1,5}, v_{1,1})$. For $i \geq 2$, the graph G_i is obtained recursively from G_{i-1} by adding a cycle graph $C_5^i = (v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4}, v_{i,5}, v_{i,1})$ and for every vertex $v_{i,j}$, $j \in \{1, \dots, 5\}$ of the

cycle C_5^i we add edges $v_{i,j}v_{l,j-1}$ and $v_{i,j}v_{l,j+1}$ with $l \in \{1, \dots, i-1\}$. The sums $j-1, j+1$ are done modulo five.

The authors denoted by \mathcal{F} the family of graphs G obtained by adding to the graph G_i , $t \geq 2$ vertices u_1, \dots, u_t and edges $u_kv_{i,j}$, with $k \in \{1, \dots, t\}$ and $j \in \{1, \dots, 5\}$.

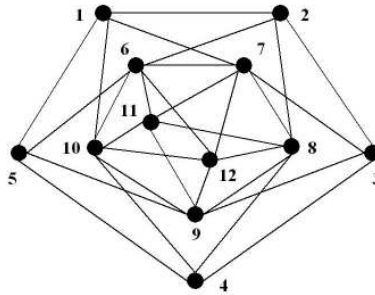


Figure 1. A graph belonging to the family \mathcal{F} with $n = 12$, $t = 2$ and $i = 2$.

The following result was proved in [8].

Theorem 7. *If G belongs to the family \mathcal{F} , then $\gamma_{con}(G) = n$ and $diam(G) = 2$.*

From the above theorem and our main result we can conclude the following

Corollary 8. *For every natural number $n \geq 10$, there is a graph G with n vertices such that G is a convex universal fixer.*

4. ACKNOWLEDGMENTS AND CONJECTURES

We conclude this paper with the following two conjectures.

Conjecture 9. *If G is a convex universal fixer, then $\gamma_{con}(G) = n$ and $diam(G) = 2$.*

Conjecture 10. *If G is a convex universal fixer, then the only minimum convex dominating sets of πG are $V(G)$ and $V(G')$.*

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REFERENCES

- [1] A.P. Burger, C.M. Mynhardt and W.D. Weakley, *On the domination number of prisms of graphs*, Discuss. Math. Graph Theory **24** (2004) 303–318.
doi:10.7151/dmgt.1233
- [2] A.P. Burger and C.M. Mynhardt, *Regular graphs are not universal fixers*, Discrete Math. **310** (2010) 364–368.
doi:10.1016/j.disc.2008.09.016
- [3] E.J. Cockayne, R.G. Gibson and C.M. Mynhardt, *Claw-free graphs are not universal fixers*, Discrete Math. **309** (2009) 128–133.
doi:10.1016/j.disc.2007.12.053
- [4] R.G. Gibson, *Bipartite graphs are not universal fixers*, Discrete Math. **308** (2008) 5937–5943.
doi:10.1016/j.disc.2007.11.006
- [5] M. Lemańska, *Weakly convex and convex domination numbers*, Opuscula Math. **24** (2004) 181–188.
- [6] J. Cyman, M. Lemańska and J. Raczek, *Graphs with convex domination number close to their order*, Discuss. Math. Graph Theory **26** (2006) 307–316.
doi:10.7151/dmgt.1322
- [7] J. Raczek and M. Lemańska, *A note of the weakly convex and convex domination numbers of a torus*, Discrete Appl. Math. **158** (2010) 1708–1713.
doi:10.1016/j.dam.2010.06.001
- [8] M. Lemańska, I. González Yero and J.A. Rodríguez-Velázquez, *Nordhaus-Gaddum results for a convex domination number of a graph*, Acta Math. Hungar., to appear (2011).
- [9] C.M. Mynhardt and Z. Xu, *Domination in Prisms of Graphs: Universal Fixers*, Util. Math. **78** (2009) 185–201.
- [10] C.M. Mynhardt and M. Schurch, *Paired domination in prisms of graphs*, Discuss. Math. Graph Theory **31** (2011) 5–23.
doi:10.7151/dmgt.1526

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