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Note

CONVEX UNIVERSAL FIXERS

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Abstract

In [1] Burger and Mynhardt introduced the idea of universal fixers. Let $G = (V, E)$ be a graph with n vertices and G' a copy of G. For a bijective function $\pi : V(G) \to V(G')$, define the prism πG of G as follows: $V(\pi G) = V(G) \cup V(G')$ and $E(\pi G) = E(G) \cup E(G') \cup M_\pi$, where $M_\pi = \{u\pi(u) \mid u \in V(G)\}$. Let $\gamma(G)$ be the domination number of G. If $\gamma(\pi G) = \gamma(G)$ for any bijective function π, then G is called a universal fixer. In [9] it is conjectured that the only universal fixers are the edgeless graphs $\overline{K_n}$.

In this work we generalize the concept of universal fixers to the convex universal fixers. In the second section we give a characterization for convex universal fixers (Theorem 6) and finally, we give an in infinite family of convex universal fixers for an arbitrary natural number $n \geq 10$.

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1. Introduction

Let $G = (V, E)$ be an undirected graph. The *neighborhood* of a vertex $v \in V$ in G is the set $N_G(v)$ of all vertices adjacent to v in G. For a set $X \subseteq V$, the *open neighborhood* $N_G(X)$ is defined as $\bigcup_{v\in X} N_G(v)$ and the *closed neighborhood* $N_G[X] = N_G(X) \cup X$.

A set $D \subseteq V$ is a *dominating set* of G if $N_G[D] = V$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G.

The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest uv-path in G. A uv-path of length $d_G(u, v)$ is called $uv-$*geodesic*. A set $X \subseteq V$ is a *convex set* of G if the vertices from all ab-geodesic belong to X for every two vertices $a, b \in X$. A set $X \subseteq V$ is a *convex dominating set* if X is convex and dominating. The *convex domination number* $\gamma_{con}(G)$ of a graph G is equal to the minimum cardinality of a convex dominating set. The convex domination number was defined by Jerzy Topp from the Gdańsk University of Technology in a verbal communication with the first author. In [5], the first results concerning this topic were published and developed in [6] and [7].

Definition 1. *Let* $G = (V, E)$ *be a graph and* G' *a copy of* G. *For a bijective function* $\pi : V(G) \to V(G')$, *define the prism* πG *of* G *as follows:* $V(\pi G) = V(G) \cup V(G')$ *and* $E(\pi G) = E(G) \cup E(G') \cup M_\pi$, *where* $M_\pi = \{u\pi(u) \mid u \in V(G)\}$.

Notice that M_π is a perfect matching of πG. It is clear that every permutation π of $V(G)$ defines a bijective function from $V(G)$ to $V(G')$, so we will indistinctly use the matching M_π, the permutation π of $V(G)$ or the associated bijection $\pi : V(G) \to V(G')$.

The graph G is called a universal fixer if $\gamma(\pi G) = \gamma(G)$ for all permutations π of $V(G)$.

The universal fixers were studied in [9] for several classes of graphs and it was conjectured that the edgeless graphs $\overline{K_n}$ are the only universal fixers. In [2], [3] and [4] it is shown that regular graphs, claw-free graphs and bipartite graphs are not universal fixers. This concept was also generalized for the other types of domination; in [10] the idea of paired domination in prisms was introduced.

We generalize the above definition for the convex domination: if $\gamma_{con}(\pi G) = \gamma_{con}(G)$ for all permutation π of $V(G)$,then we say that G is a *convex universal fixer*.

2. Convex Universal Fixers

From now on we assume that the graph $G = (V, E)$ is a connected undirected graph with n vertices. For $x \in V(G)$, the copy of x in $V(G')$ is denoted by x'. Recall that the diameter of a graph G, denoted by $diam(G)$, is defined to be the maximum distance between any two vertices $x, y \in V(G)$.

Proposition 2. *Let* G *be a connected undirected graph.*

(1) *If* $diam(G) \leq 2$*, then both* $V(G)$ *and* $V(G')$ *are convex dominating sets of* πG *for any permutation* π*.*

(2) *If* $diam(G) \geq 3$*, then there exist permutations* π_1 *and* π_2 *such that* $V(G)$ *is not a convex dominating set of* $\pi_1 G$ *and* $V(G')$ *is not a convex dominating set of* $\pi_2 G$*.*

Proof. (1) It is clear that $V(G)$ and $V(G')$ are dominating sets of πG. Let $x, y \in V(G)$. Since $d_{\pi G}(x, y) \leq d_G(x, y) \leq 2$, any xy-geodesic is contained in G, so $V(G)$ is a convex dominating set of πG. In a similar way, we can prove that $V(G')$ is a convex dominating set of πG.

(2) Let $x, y \in V(G)$ be such that $d_G(x, y) \geq 3$. Let $wz \in E(G')$ and consider a permutation π_1 such that $\pi_1(x) = w$ and $\pi_1(y) = z$. Then $xwzy$ is an xy-geodesic in $\pi_1 G$ with $z, w \notin V(G)$. In a similar way, we can prove that there exists a permutation π_2 such that $V(G')$ is not a convex dominating set in $\pi_2 G$. ■

From the above proposition we have the following observation.

Observation 3. *For any permutation* π*,* $\gamma_{con}(\pi G) \leq n$ *whenever* $diam(G) \leq 2$*.*

If D is a convex dominating set of πG, we define D_1 as $D \cap V(G)$ and D_2 as $D \cap V(G')$. Moreover, we write $D_1^c = V(G) - D_1$ and $D_2^c = V(G') - D_2$.

Proposition 4. *Let* D *be a convex dominating set of* πG*.*

(1) *If* $\gamma_{con}(\pi G) < n$*, then* $D_1 \neq \emptyset$ *and* $D_2 \neq \emptyset$*.*

(2) *If* $D_1 \neq \emptyset$ *and* $D_2 \neq \emptyset$*, then there exists at least one edge* $x\pi(x) \in M_\pi$ *with* $x \in D_1$ *and* $\pi(x) \in D_2$*.*

Proof. (1) Suppose that $D_1 = \emptyset$. Then $D = D_2 \subset V(G')$. Since $|D| < n$, $V(G)$ is not dominated by D. Similarly, if $D_2 = \emptyset$, then $V(G')$ is not dominated by D.

(2) Let $x \in D_1$ and $\pi(y) \in D_2$. Since D is convex, any $x\pi(y)$-geodesic should use the edge $x\pi(x)$ or the edge $y\pi(y)$. ■

Lemma 5. *Suppose that* $diam(G) \leq 2$*. Let* D *be a minimum convex dominating set of* πG*. If* $D = D_1 \cup D_2$ *with* $D_1 \neq \emptyset$ *and* $D_2 \neq \emptyset$*, then we have the following statements:*

(1) *if* $\pi(D_1) \subseteq D_2$*, then* D_2 *is a convex dominating set of* G'*, and*

(2) *if* $\pi^{-1}(D_2) \subseteq D_1$*, then* D_1 *is a convex dominating set of* G*.*

Proof. Assume that $\pi(D_1) \subseteq D_2$. Then, since D is a dominating set of πG, every vertex of D_2^c has a neighbor in D_2. Moreover, $diam(G') \leq 2$ and $d_{\pi G}(a, b) \leq 2$ for every two vertices $a, b \in D_2$, so the vertices from all ab-geodesics belong to D_2, because D is convex. Thus D_2 is a convex dominating set of G'. Similarly, we can prove the second part of the lemma. ■

Our main result is the following.

Theorem 6. *Let G be a connected undirected graph. If $\gamma_{con}(G) = n$ and $diam(G) \leq 2$, then $\gamma_{con}(\pi G) = n$, that is, G is a convex universal fixer.*

Proof. By Observation 3, if $diam(G) \leq 2$, then $\gamma_{con}(G) \leq n$ for all permutations π. By contradiction, suppose that $\gamma_{con}(G) = n$ and $\gamma_{con}(\pi G) < n$. If $diam(G) = 1$, then $\gamma_{con}(G) < n$, so we can assume $diam(G) = 2$.

Let $D = D_1 \cup D_2$ be a minimum convex dominating set of πG with $|D| < n$. From the first part of Proposition 4, we have that $D_1 \neq \emptyset$ and $D_2 \neq \emptyset$. In order to have a partition of $V(\pi G)$, we define the following subsets of vertices:

$$D_1^+ = \{u \in D_1 | \pi(u) \in D_2\}, \;\; D_2^+ = \{u' \in D_2 | \pi^{-1}(u') \in D_1\} = \pi(D_1^+),$$

$$D_1^- = \{u \in D_1 | \pi(u) \notin D_2\}, \;\; D_2^- = \{u' \in D_2 | \pi^{-1}(u') \notin D_1\},$$

$$E_1 = \pi^{-1}(D_2^-), \;\; E_2 = \pi(D_1^-),$$

$$F_1 = V(G) - D_1 - E_1 \text{ and } F_2 = \pi(F_1).$$

From the second part of Proposition 4, we have that $D_1^+ \neq \emptyset$ and $D_2^+ \neq \emptyset$. If $\pi(D_1) \subseteq D_2$,then by Lemma 5, the set D_2 is a convex dominating set of G', which is a contradiction since $\gamma_{con}(G') = n$. Therefore, $D_1^- \neq \emptyset$. In a similar way, $D_2^- \neq \emptyset$. In consequence $E_1 \neq \emptyset$ and $E_2 \neq \emptyset$. Since $|D| < n$, $|D_1^+ \cup D_1^- \cup D_2^+ \cup D_2^-| < n$ and $|E_1 \cup E_2| = |D_1^- \cup D_2^-| < n$. Therefore, F_1 and F_2 are nonempty.

We claim that there are no edges between E_1 and D_1. Suppose $x \in D_1, y \in E_1$ and $xy \in E(G)$. Then $d_{\pi G}(x, \pi(y)) = 2$, and $x, \pi(y) \in D$ implies that $y \in D_1$, which leads us to a contradiction.

Let x be a vertex in D_1^- and $y \in E_1$. Since $diam(G) = 2$, $d_G(x, y) = 2$ and there exists a vertex $z \in F_1$ such that $xz \in E(G)$ and $yz \in E(G)$.

If $d_{\pi G}(x, \pi(y)) \geq 3$, then $xzy\pi(y)$ is an $x\pi(y)$-geodesic, which is not possible, since D is a convex dominating set of πG and $y, z \notin D$. Thus $d_{\pi G}(x, \pi(y)) = 2$. But then there exists a vertex $w \in D$ such that w is a common neighbor of x and $\pi(y)$, a contradiction. Therefore, $\gamma_{con}(\pi G) = n$. ■

3. An Infinite Family of Convex Universal Fixers

Now we show that for an arbitrarily large n, there is a graph G with n vertices such that G is a convex universal fixer. The following family $\mathcal{F}$ of graphs was defined in [8].

Let G_1 be the cycle of order five, $C_5^1 = (v_{1,1}, v_{1,2}, v_{1,3}, v_{1,4}, v_{1,5}, v_{1,1})$. For $i \geq 2$, the graph G_i is obtained recursively from G_{i-1} by adding a cycle graph $C_5^i = (v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4}, v_{i,5}, v_{i,1})$ and for every vertex $v_{i,j}$, $j \in \{1, \cdots, 5\}$ of the cycle C_5^i we add edges $v_{i,j}v_{l,j-1}$ and $v_{i,j}v_{l,j+1}$ with $l \in \{1, \cdots, i-1\}$. The sums $j-1, j+1$ are done modulo five.

The authors denoted by $\mathcal{F}$ the family of graphs G obtained by adding to the graph G_i, $t \geq 2$ vertices $u_1, \ldots, u_t$ and edges $u_k v_{i,j}$, with $k \in \{1, \ldots, t\}$ and $j \in \{1, \ldots, 5\}$.

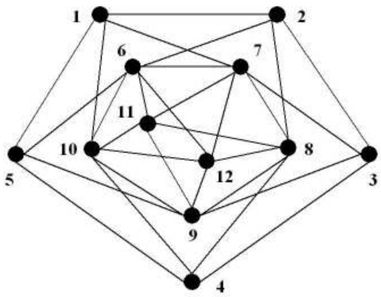

Figure 1. A graph belonging to the family $\mathcal{F}$ with $n = 12$, $t = 2$ and $i = 2$.

The following result was proved in [8].

Theorem 7. *If G belongs to the family $\mathcal{F}$, then $\gamma_{con}(G) = n$ and $diam(G) = 2$.*

From the above theorem and our main result we can conclude the following

Corollary 8. *For every natural number $n \geq 10$, there is a graph G with n vertices such that G is a convex universal fixer.*

4. Acknowledgments and Conjectures

We conclude this paper with the following two conjectures.

Conjecture 9. *If G is a convex universal fixer, then $\gamma_{con}(G) = n$ and $diam(G) = 2$.*

Conjecture 10. *If G is a convex universal fixer, then the only minimum convex dominating sets of πG are $V(G)$ and $V(G')$.*

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