

## 4-CHROMATIC KOESTER GRAPHS <sup>1</sup>

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### Abstract

Let  $G$  be a simple 4-regular plane graph and let  $S$  be a decomposition of  $G$  into edge-disjoint cycles. Suppose that every two adjacent edges on a face belong to different cycles of  $S$ . Such a graph  $G$  arises as a superposition of simple closed curves in the plane with tangencies disallowed. Studies of coloring of graphs of this kind were originated by Grötzsch. Two 4-chromatic graphs generated by circles in the plane were constructed by Koester in 1984 [10, 11, 12]. Until now, no other examples of such graphs were known. We present fourteen new 4-chromatic graphs generated by circles in the plane.

**Keywords:** planar graph, 4-critical graph, Grötzsch-Sachs graph, Koester graph.

**2010 Mathematics Subject Classification:** 05C10, 05C15.

### 1. INTRODUCTION

A simple graph is called  $k$ -chromatic if its chromatic number is equal to  $k$ . A graph is *edge (vertex)-4-critical* if it is 4-chromatic and the removal of any edge (vertex) decreases its chromatic number. Numerous results and problems related to critical graphs can be found in [9]. Consider a graph  $G = G(S)$  formed by the superposition of a set  $S$  of simple closed curves in the plane, no two of which are tangent and no three of which meet at a point. Vertices and edges of  $G$  correspond to crossing points and arcs of  $S$ , respectively (see, for example, Figure 1). Since, in the plane, every two closed curves have an even number of crossing points,  $G$  is a 4-regular planar graph with even number of vertices. Such 4-regular planar graphs will be called *Grötzsch-Sachs graphs*. If all curves are circles, then such graphs will be referred to as *Koester graphs*.

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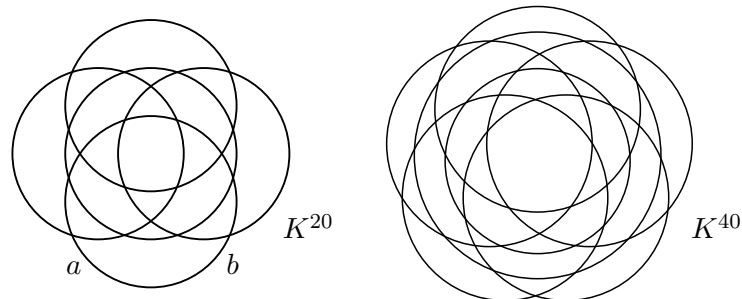


Figure 1. 4-chromatic and edge 4-critical Koester graphs.

The first discussions concerning coloring of graphs generated by curves in the plane are due to Grötzsch. Vertex coloring of these graphs and related problems were studied in [6, 7, 10, 11, 12, 14, 15]. The closed curves in  $S$  can be partitioned into several *parallel* classes, where the curves in each class are pairwise disjoint. The minimum number of parallel classes in  $S$  is called the *class number* of  $S$ . Many results concerning Grötzsch-Sachs graphs were stated in terms of this parameter. Instead of the class number, we shall consider the *characteristic graph*  $H = H(S)$  for a curve set  $S$ . Vertices of  $H(S)$  correspond to curves of  $S$  and two vertices are adjacent if and only if the corresponding curves intersect, *i.e.*  $H(S)$  is the intersection graph of the curves. The chromatic number of  $H(S)$  is equal to the class number of  $S$ . The characteristic graph will be also denoted as  $H(G)$ , where  $G$  is generated by the curves of  $S$ .

Jaeger proved [6, 7] that if  $\chi(H(G)) \leq 3$ , then  $\chi(G) \leq 3$ . In 1984, Koester constructed two 4-chromatic Grötzsch-Sachs graphs  $K^{20}$  and  $K^{40}$  generated by sets of 5 and 7 circles in the plane, respectively (see Figure 1). Since  $H(K^{20}) \cong K_5$  and  $H(K^{40}) \cong K_7 - e$ , we have  $\chi(H(K^{20})) = 5$  and  $\chi(H(K^{40})) = 6$  [10, 11, 12]. Infinite families of 4-chromatic Grötzsch-Sachs graphs have been recently presented in [1, 2, 3, 4]. These examples disproved Grötzsch-Sachs-Koester's conjecture which stated that if  $\chi(H(G)) = 4$  then  $\chi(G) \leq 3$  [5, 8, 11, 12, 16]. The first Koester graph  $K^{20}$  has order 20 and it is neither vertex critical, nor edge critical. The second graph  $K^{40}$  of order 40 is the first example of a 4-regular edge 4-critical planar graph. Up to the present time, two Koester graphs have been the only known examples of 4-chromatic graphs generated by circles in the plane. Attempts to find similar graphs lead to the following question [13].

**Question.** *Do there exist 4-chromatic Koester graphs except  $K^{20}$  and  $K^{40}$ ?*

In this paper, we answer this question in the affirmative. We present fourteen new 4-chromatic Koester graphs.

2. NEW KOESTER GRAPHS

Consider the family of fourteen Koester graphs of order 28 generated by six circles in the plane shown in Figure 2. Here almost straight lines are arcs of circles with huge radii and several big circles are presented by their arcs. Every curve set has a unique pair of non-crossing circles, *i.e.*  $H(G) \cong K_6 - e$  for all graphs  $G$  of this family.

In order to show that these graphs are pairwise non-isomorphic, we use the notion of black-white signature of graphs. Since every graph is 4-regular, there is a chess 2-coloring of its faces (see Figure 3). Denote by  $f_w$  and  $f_b$  the numbers of white and black faces, respectively. The signature  $sgn(G)$  of a graph  $G$  gives information about sizes of faces and their colors:

$$sgn(G) = \frac{w_1^{n_1}, w_2^{n_2}, \dots, w_r^{n_r}}{b_1^{m_1}, b_2^{m_2}, \dots, b_s^{m_s}},$$

where  $w_i, i = 1, 2, \dots, r$ , and  $b_j, j = 1, 2, \dots, s$ , are the sizes of white and black faces, respectively. Parameters  $n_i, i = 1, 2, \dots, r$ , and  $m_j, j = 1, 2, \dots, s$ , count the numbers of faces of the corresponding sizes,  $n_1 + n_2 + \dots + n_r = f_w$  and  $m_1 + m_2 + \dots + m_s = f_b$ . We choose colors so that  $(w_1, w_2, \dots, w_r) \geq (b_1, b_2, \dots, b_s)$  in the lexicographic order. If  $(w_1, w_2, \dots, w_r) = (b_1, b_2, \dots, b_s)$  then we assume that  $(n_1, n_2, \dots, n_r) \geq (m_1, m_2, \dots, m_s)$ .

**Lemma 1.** *Graphs  $G_1, G_2, \dots, G_{14}$  are pairwise non-isomorphic.*

**Proof.** Koester graphs  $G_1, G_2, \dots, G_{14}$  have the following signatures:

$$\begin{aligned} sgn(G_1) &= \frac{6^1 5^4 4^6 3^2}{5^2 4^1 3^{14}}, & sgn(G_2) &= \frac{6^1 5^5 4^4 3^3}{5^1 4^3 3^{13}}, & sgn(G_3) &= \frac{6^2 5^3 4^5 3^3}{6^1 4^2 3^{14}}, \\ sgn(G_4) &= \frac{7^1 5^3 4^4 3^6}{5^3 4^2 3^{11}}, & sgn(G_5) &= \frac{6^2 5^1 4^6 3^5}{6^1 5^2 4^1 3^{12}}, & sgn(G_6) &= \frac{6^1 5^4 4^3 3^6}{7^1 4^4 3^{11}}, \\ sgn(G_7) &= \frac{6^2 5^2 4^4 3^6}{5^3 4^2 3^{11}}, & sgn(G_8) &= \frac{7^1 5^2 4^3 3^9}{6^1 5^2 4^4 3^8}, & sgn(G_9) &= \frac{6^1 5^3 4^2 3^9}{5^4 4^3 3^8}, \\ sgn(G_{10}) &= \frac{6^1 5^2 4^4 3^8}{5^4 4^3 3^8}, & sgn(G_{11}) &= \frac{6^1 5^2 4^4 3^8}{6^1 5^2 4^4 3^8}, & sgn(G_{12}) &= \frac{6^1 5^2 4^4 3^8}{6^1 5^2 4^4 3^8}, \\ sgn(G_{13}) &= \frac{6^1 5^2 4^4 3^8}{5^4 4^3 3^8}, & sgn(G_{14}) &= \frac{6^1 5^3 4^2 3^9}{6^1 5^3 4^2 3^9}. \end{aligned}$$

Ten of these graphs have pairwise distinct signatures. Graphs of pairs  $\{G_{11}, G_{12}\}$  and  $\{G_{10}, G_{13}\}$  have the same signatures. There is an edge of  $G_{11}$  such that its incident vertices belong to black and white faces of size 6. Graph  $G_{12}$  does not have such an edge. Graphs  $G_{10}$  and  $G_{13}$  have a unique white face of size 6. This exterior 6-face is adjacent with several 3-faces and with precisely one 5-face in both these graphs. Further, this 5-face has a common vertex with another 5-face in graph  $G_{13}$  but graph  $G_{10}$  does not contain such a fragment. Therefore, we can conclude that Koester graphs  $G_1, G_2, \dots, G_{14}$  are pairwise non-isomorphic. ■

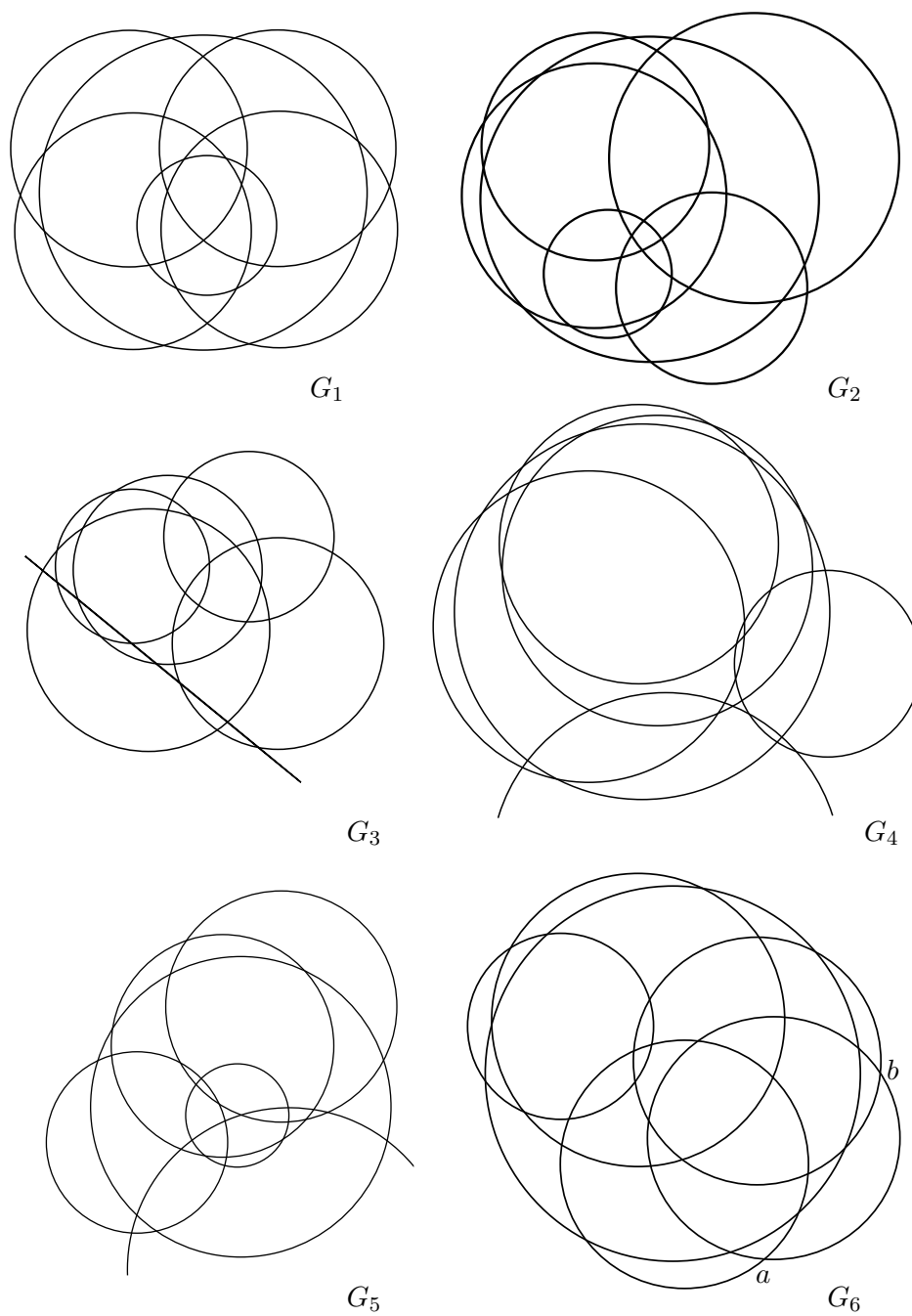


Figure 2. 4-chromatic Koester graphs of order 28.

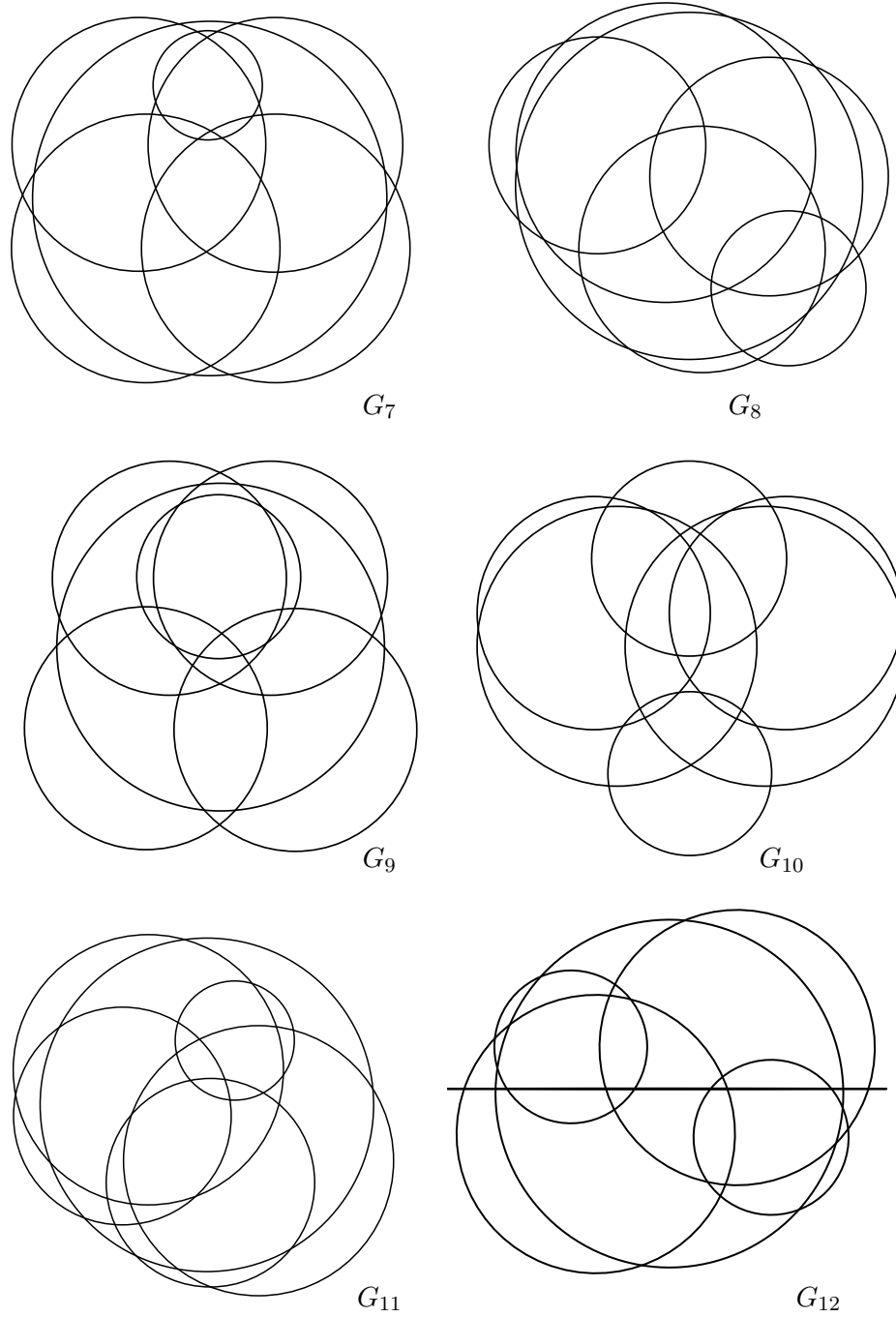


Figure 2. 4-chromatic Koester graphs of order 28 (*continued*).

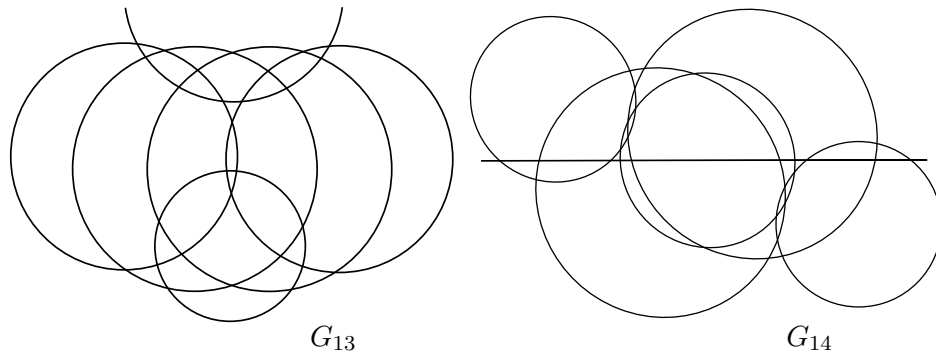


Figure 2. 4-chromatic Koester graphs of order 28 (*conclusion*).

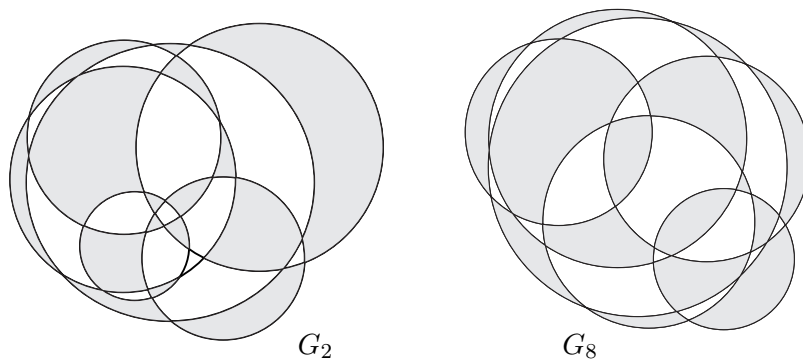


Figure 3. Chess coloring of faces in graphs  $G_2$  and  $G_8$ .

### 3. COLORING OF GRAPHS

Now we prove that Koester graph  $G_1$  is 4-chromatic. The described technique can be applied to all graphs  $G_2, G_3, \dots, G_{14}$  in Figure 2 (a proof for all graphs is available from the authors).

**Theorem 2.** *Graph  $G_1$  is 4-chromatic.*

**Proof.** First consider graph  $G_1$  with the vertex numbering depicted in Figure 4. By Brooks theorem (see, e.g., [9]),  $\chi(G_1) \leq 4$ . Since  $G_1$  contains triangles,  $\chi(G_1) \geq 3$ . Suppose that  $G_1$  is a 3-chromatic graph and try to color  $G_1$ . The initial step of the coloring procedure is to assign colors to vertices of some 5-face in all possible ways. Then we show that any extension of the initial coloring implies that it is impossible to color  $G_1$  by three colors.

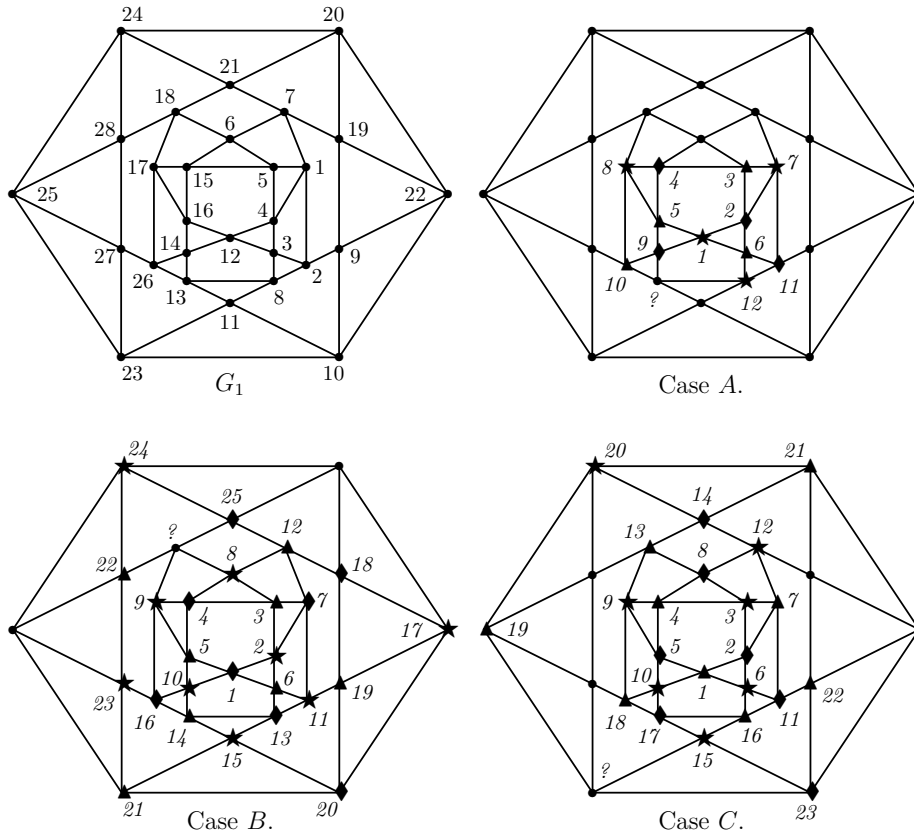


Figure 4. Extensions of 3-colorings for the graph  $G_1$ .

Consider the 5-face (12,4,5,15,16) of the graph  $G_1$  in Figure 4. We will depict vertex color as a star, a triangle, or a diamond. To color vertices of any 5-face in a 3-chromatic graph, one needs exactly 3 colors. One vertex of a 5-face has a color that is distinct from the colors of the other four vertices. Assume that this color is depicted as a star. The total number of 3-colorings of the initial 5-face is five. Because of symmetry, it is sufficient to examine only three different colorings of this face.

Three cases of a coloring of the initial face are shown in Figure 4. The first colored vertex is 12, 4, and 5 for Case A, B, and C, respectively. The unique possible extension of the initial coloring for every case is also presented in Figure 4. The italic number near every vertex indicates the number of the step at which this vertex gets a forced color during the coloring procedure. The question mark indicates a vertex that cannot be properly colored.

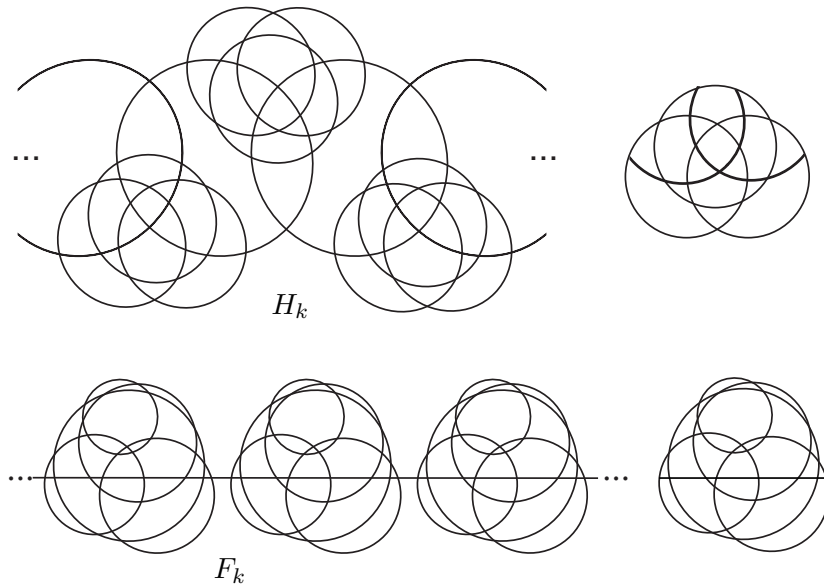


Figure 5. Two infinite families of 4-chromatic Koester graphs.

For Case *A*, the structure of the graph forces a simple coloring: every uncolored vertex will always have two previously colored neighbors. In order to extend the initial coloring for Cases *B* and *C*, the following helpful simple observations have been used.

1) Let a graph  $G$  be obtained from  $P_4$  by joining a new vertex  $v$  with the non-pendant vertices of  $P_4$ . If the pendant vertices of  $G$  have the same color in some proper 3-coloring, then  $v$  always has this color. For example, the pendant vertices 2 and 11 in the path (2,9,10,11) of  $G_1$  get the same color (star) at Steps 11 and 15 in Case *B*. Therefore, vertex 22 gets this color at Step 17. For Case *C*, vertex 25 gets its color from the path (26,27,28,18) at Step 19.

2) In any proper 3-coloring of  $K_4 - e$ , the non-adjacent vertices of it have the same color. This rule is applied in Steps 22 and 24 for coloring vertices 28 and 24 in Case *B* and in step 22 for coloring vertex 9 in Case *C*.

The considered cases *A–C* imply the equality  $\chi(G_1) = 4$ . ■

All graphs in Figure 2 are vertex-critical but not edge 4-critical. For example, graph  $G_1$  has exactly two non-critical symmetrical edges. After removal edge (2,3) or (14,26), graph  $G_1$  becomes edge 4-critical.



## 4. INFINITE FAMILIES OF KOESTER GRAPHS

The existence of non-critical edges in Koester graphs allows the construction of infinite families of 4-chromatic graphs. Any number of circles can be added to a 4-chromatic Koester graph  $G$  in various ways such that they cross only non-critical edges of  $G$ . It is obvious that the new graphs are always vertex non-critical.

Two examples of infinite families are presented in Figure 5. A member of the first family,  $H_k$ ,  $k \geq 1$ , is obtained from  $k$  copies of Koester graph  $K^{20}$  in which exterior edge  $(a, b)$  is non-critical (see Figure 1). A 4-chromatic fragment of  $H_k$  is depicted near the graph. Graphs of the second family,  $F_k$ ,  $k \geq 1$ , are constructed from  $k$  copies of graph  $G_6$  with exterior non-critical edge  $(a, b)$  (see Figure 2). A circle with this arc is presented as an almost straight line in Figure 5. A 4-chromatic fragment of  $F_k$  is also shown.

## 5. OPEN PROBLEMS

Graph  $K^{20}$  is the minimal known 4-chromatic Koester graph. What is the minimal number of vertices in such graphs other than  $K^{20}$ ? Our intensive search gives graphs only with 28 vertices.

**Problem 3.** Find minimal Koester graphs in the following classes: a) 4-chromatic graphs; b) vertex 4-critical graphs; c) edge 4-critical graphs.

The graph  $K^{40}$  is the only edge 4-critical graph among all known Koester graphs.

**Problem 4.** Find edge (vertex) 4-critical Koester graphs generated by 5, 6, or 7 circles.

There are no known examples of infinite families consisting of 4-critical graphs.

**Problem 5.** Find infinite families of edge (vertex) 4-critical Koester graphs.

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