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THE i-CHORDS OF CYCLES AND PATHS

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Abstract

An i-chord of a cycle or path is an edge whose endpoints are a distance $i \geq 2$ apart along the cycle or path. Motivated by many standard graph classes being describable by the existence of chords, we investigate what happens when i-chords are required for specific values of i. Results include the following: A graph is strongly chordal if and only if, for $i \in \{4, 6\}$, every cycle C with $|V(C)| \geq i$ has an $(i/2)$-chord. A graph is a threshold graph if and only if, for $i \in \{4, 5\}$, every path P with $|V(P)| \geq i$ has an $(i-2)$-chord.

Keywords: chord, chordal graph, strongly chordal graph, ptolemaic graph, trivially perfect graph, threshold graph.

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1. Introduction

A *chord* of a cycle C or path P is an edge vw between two nonconsecutive vertices v and w of C or P, and vw is an *i-chord* if the distance between v and w is i within C or P. Chords vw and xy are *crossing chords* of C if the four vertices v, x, w, y come in that order around C.

Many graph classes have been characterized by chords existing in long-enough cycles (or, less often, paths). Using i-chords for specific i allows finer distinctions to be made. Section 2 will discuss several graph classes in terms of i-chords of cycles, with similar—yet only somewhat similar—results in Section 3 for i-chords of paths. Sections 4 and 5 will discuss some of the corresponding results for bipartite graphs.

2. Chords of Cycles in Graphs

As in [2, 9], a graph is *chordal* if every cycle C with $|V(C)| \geq 4$ has a chord (equivalently, every cycle long enough to have a chord does have a chord). Theorem 1 is a very simple characterization of being chordal.

Theorem 1. *A graph is chordal if and only if every cycle C with $|V(C)| \geq 4$ has a 2-chord.*

Proof. The 'if direction' is immediate. The 'only if direction' follows immediately from the well-known result that every induced subgraph of a chordal graph has a *simplicial vertex*, meaning a vertex whose open neighborhood induces a complete subgraph [2, 9]. ∎

For comparison with Theorems 2 and 5, note that Theorem 1 could be rephrased as follows: *A graph is chordal if and only if, for $i \in \{4\}$, every cycle C with $|V(C)| \geq i$ has an $(i/2)$-chord.*

Theorem 2 characterizes *strongly chordal graphs*—the chordal graphs in which every cycle of even length at least 6 has an i-chord where i is odd [2, 4, 8, 9] (equivalently, for each $i \in \{2,3\}$, every cycle long enough to have an i-chord does have an i-chord).

Theorem 2. *A graph is strongly chordal if and only if, for $i \in \{4,6\}$, every cycle C with $|V(C)| \geq i$ has an $(i/2)$-chord.*

Proof. The 'if direction' is immediate. The 'only if direction' follows from the well-known result that every induced subgraph of a strongly chordal graph has a *simple vertex*, meaning a vertex v such that the closed neighborhoods of every two neighbors of v are comparable by inclusion [2, 9]. ∎

The chordal graph formed from a 6-cycle by inserting three noncrossing 2-chords shows that cycles of chordal graphs with $|V(C)| \geq 6$ might not have 3-chords. The strongly chordal graph shown in Figure 1 shows that cycles of strongly chordal graphs with $|V(C)| \geq 8$ might not have 4-chords.

A graph G is *distance-hereditary* if the distance between vertices in connected induced subgraphs of G always equals the distance between them in G. A graph is *ptolemaic* if it is both chordal and distance-hereditary. Reference [2] contains many other characterizations of these concepts. In particular, from [5], a graph is ptolemaic if and only if it is chordal with no induced subgraph isomorphic to a *gem*—the graph obtained from a 5-cycle by inserting two noncrossing 2-chords. Also, a graph is ptolemaic if and only if it is chordal and every cycle of length at least 5 has crossing chords. Every ptolemaic graph is strongly chordal [2] (but the strongly chordal graph in Figure 1 is not ptolemaic). Theorem 5 will show that, for each $i \in \{2,3,4\}$, every cycle of a ptolemaic graph that is long enough to have an i-chord in fact does have an i-chord.

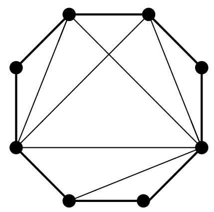

Figure 1. A strongly chordal graph spanned by an 8-cycle that has a 2-chord and a 3-chord, but no 4-chord.

Lemma 3 (Howorka [5]). *Every* 4*-cycle,* 5*-cycle, and* 6*-cycle in a ptolemaic graph will have, respectively, at least* 1*,* 3*, or* 4 *chords* (*and so, respectively, at most* 1*,* 2*, or* 5 *nonadjacent pairs of vertices*)*.*

Proof. This follows from the characterization of ptolemaic graphs in [5] by every k-cycle having at least $\lfloor 3(k-3)/2 \rfloor$ chords (and so having at most $k(k-3)/2 - \lfloor 3(k-3)/2 \rfloor$ nonadjacent pairs of vertices). ∎

Let $v \sim w$ and $v \not\sim w$ denote that vertices v and w are, respectively, adjacent or nonadjacent.

Lemma 4. *If* a, b, c, d, e, f *is a path* (*possibly a closed path with* $a = f$) *in a ptolemaic graph with* $b \not\sim d$ *and* $c \not\sim e$*, then* $b \not\sim e$*. If also* $a \neq f$*, then* $a \not\sim e$ *and* $b \not\sim f$*.*

Proof. Inserting an edge be (or ae or bf if $a \neq f$) would violate Lemma 3 by creating a cycle with too few chords. ∎

Theorem 5. *In a ptolemaic graph, if* $i \in \{4, 6, 8\}$*, then every cycle* C *with* $|V(C)| \geq i$ *has an* $(i/2)$*-chord.*

Proof. Suppose G is ptolemaic (and so is strongly chordal). By Theorem 2, every cycle C with $|V(C)| \geq 4$ has a 2-chord and every C with $|V(C)| \geq 6$ has a 3-chord. Suppose $C = v_1, v_2, \ldots, v_k, v_1$ is a k-cycle in G. Argue by induction on $k \geq 8$ that C has a 4-chord.

For the basis step, suppose $k = 8$, but C has no 4-chord (arguing by contradiction); thus $v_1 \not\sim v_5$, $v_2 \not\sim v_6$, $v_3 \not\sim v_7$, and $v_4 \not\sim v_8$. Since G is strongly chordal, C has a 3-chord; without loss of generality, say v_3v_6 is a chord. Lemma 4 on the path $v_1, v_2, v_3, v_6, v_7, v_8$ implies $v_1 \not\sim v_7 \not\sim v_2 \not\sim v_8$. Therefore, the 6-cycle $C^- = v_1, v_2, v_3, v_6, v_7, v_8, v_1$ has five nonadjacent pairs of vertices and so, by Lemma 3, C^- has all the other four possible chords v_1v_3, v_1v_6, v_3v_8 and v_6v_8. Lemma 4 on the path $v_4, v_5, v_6, v_1, v_2, v_3$ now implies $v_3 \not\sim v_5$. Similarly, the path $v_5, v_4, v_3, v_8, v_7, v_6$ implies $v_4 \not\sim v_6$. But the chordless 4-cycle v_3, v_4, v_5, v_6, v_3 would now contradict Lemma 3.

Therefore suppose $k \geq 9$ and every k'-cycle with $8 \leq k' < k$ has a 4-chord, but also suppose that C has no 4-chord (arguing by contradiction); thus $v_1 \not\sim v_{k-3}$, $v_2 \not\sim v_{k-2}$, $v_3 \not\sim v_{k-1}$, $v_4 \not\sim v_k$, $v_1 \not\sim v_5$, $v_2 \not\sim v_6$, and $v_3 \not\sim v_7$. Since G is chordal, C has a 2-chord; without loss of generality, say v_1v_{k-1} is a chord. Let C' be the cycle with edge set $E(C) - \{v_{k-1}v_k, v_1v_k\} \cup \{v_1v_{k-1}\}$ and length $k-1 \geq 8$. The inductive hypothesis implies that C' has a 4-chord (that is not a 4-chord of C), and so $k \geq 10$ and either $v_1 \sim v_{k-4}$ or $v_2 \sim v_{k-3}$ or $v_3 \sim v_{k-2}$ or $v_4 \sim v_{k-1}$. Observe that $v_2 \not\sim v_{k-3}$; otherwise Lemma 4 on the path $v_k, v_1, v_2, v_{k-3}, v_{k-2}, v_{k-1}$ would contradict $v_1 \sim v_{k-1}$. Similarly, $v_3 \not\sim v_{k-2}$. Therefore, either $v_1 \sim v_{k-4}$ or $v_4 \sim v_{k-1}$; without loss of generality, suppose $v_4 \sim v_{k-1}$.

Lemma 4 on the path $v_2, v_3, v_4, v_{k-1}, v_k, v_1$ implies $v_1 \not\sim v_3 \not\sim v_k \not\sim v_2$. Lemma 3 on the 5-cycle $v_1, v_2, v_3, v_4, v_{k-1}, v_1$ with the two nonadjacent pairs $\{v_1, v_3\}$ and $\{v_3, v_{k-1}\}$ implies $v_1 \sim v_4 \sim v_2 \sim v_{k-1}$. Lemma 4 on the path $v_{k-3}, v_{k-2}, v_{k-1}, v_2, v_3, v_4$ then implies $v_4 \not\sim v_{k-2}$, and also, now on the path $v_{k-1}, v_k, v_1, v_4, v_5, v_6$, implies $v_{k-1} \not\sim v_5 \not\sim v_k \not\sim v_6$. Finally, Lemma 4 on the path $v_{k-3}, v_{k-2}, v_{k-1}, v_4, v_5, v_6$ implies $v_5 \not\sim v_{k-2}$.

Let C'' be the cycle with edge set $E(C) - \{v_1v_2, v_2v_3, v_3v_4\} \cup \{v_1v_4\}$ and length $k-2 \geq 8$. The inductive hypothesis implies that C'' has a 4-chord (that is not a 4-chord of C), and so $k \geq 11$ and either $v_4 \sim v_{k-2}$ or $v_5 \sim v_{k-1}$ or $v_6 \sim v_k$ or $v_1 \sim v_7$. Since we have proved that $v_4 \not\sim v_{k-2}$ and $v_5 \not\sim v_{k-1}$ and $v_6 \not\sim v_k$, it follows that $v_1 \sim v_7$. Observe that $v_1 \not\sim v_6$, since otherwise Lemma 4 on the path $v_4, v_5, v_6, v_1, v_2, v_3$ would contradict $v_2 \sim v_4$. Lemma 3 on the 5-cycle $v_1, v_4, v_5, v_6, v_7, v_1$ with the two nonadjacent pairs $\{v_1, v_5\}$ and $\{v_1, v_6\}$ then implies $v_5 \sim v_7 \sim v_4 \sim v_6$. Note that $v_2 \not\sim v_5$, since otherwise the 5-cycle $v_1, v_2, v_5, v_6, v_7, v_1$ with the three nonadjacent pairs $\{v_1, v_5\}$, $\{v_1, v_6\}$ and $\{v_2, v_6\}$ would contradict Lemma 3. Furthermore, Lemma 3 on the 5-cycle $v_1, v_2, v_4, v_5, v_7, v_1$ with the two nonadjacent pairs $\{v_1, v_5\}$ and $\{v_2, v_5\}$ implies $v_2 \sim v_7$. But then Lemma 4 on the path $v_5, v_6, v_7, v_2, v_3, v_4$ would contradict $v_4 \sim v_6$. ∎

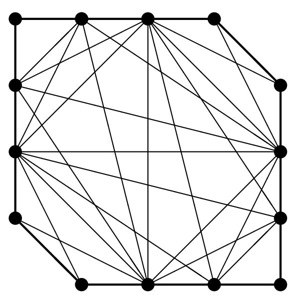

Figure 2. A ptolemaic graph spanned by a 14-cycle that has a 2-chord, a 3-chord, and a 4-chord (and a 6-chord and a 7-chord), but no 5-chord.

The gem graph shows that the converse of Theorem 5 fails. The graph in Figure 2 is ptolemaic—Corollary 6 of [1] is an easy way to verify this—and shows that cycles of ptolemaic graphs with $|V(C)| \geq 10$ might not have 5-chords. (This is a minimum-order counterexample: cycles with $10 \leq |V(C)| \leq 13$ in ptolemaic graphs turn out to always have 5-chords.)

3. Chords of Paths in Graphs

Let C_n and P_n denote, respectively, a cycle and path on n vertices (so C_n has length n and P_n has length $n-1$). For any graph H, a graph G is *H-free* if G contains no induced subgraph isomorphic to H.

Theorem 6. *For every $i \geq 3$, a graph is both P_i-free and chordal if and only if every path P with $|V(P)| \geq i$ has a 2-chord.*

Proof. First suppose G is P_i-free and chordal with a path P where $|V(P)| \geq i \geq 3$. Consider the minimum j such that P has a j-chord. If $j \geq 3$, then that j-chord would combine with P to form a chordless $(j+1)$-cycle where $j+1 \geq 4$, contradicting that G is chordal. Therefore, $j = 2$.

Conversely, suppose every path P of G with $|V(P)| \geq i \geq 3$ has a 2-chord. Therefore, G contains no induced subgraph isomorphic to any such P_i or to any C_n with $n \geq 4$, and so G is P_i-free chordal. ∎

The *trivially perfect graphs* have many names and characterizations [2, 9], one of which is that they are precisely the P_4-free chordal graphs. Corollary 7 is the path analog of Theorem 1.

Corollary 7. *A graph has complete components if and only if every path P with $|V(P)| \geq 3$ has a 2-chord. A graph is trivially perfect if and only if every path P with $|V(P)| \geq 4$ has a 2-chord.*

Proof. These are the $i = 3$ and $i = 4$ cases of Theorem 6. ∎

The *threshold graphs* also have many characterizations [2, 7, 9], one of which is that they are precisely the $2K_2$-free trivially perfect graphs ($2K_2$ is the complement of C_4). For a connected graph, this is equivalent—see Theorem 1.2.4 of [7] for the history—to being constructible from a single vertex by recursively appending either an isolated vertex or a *dominating vertex* (often called a *universal vertex*, meaning a vertex adjacent to all the previously-existing vertices). Thus, a graph is a threshold graph if and only if every induced subgraph has either an isolated vertex or a dominating vertex. Theorem 8 is a path analog of Theorem 2.

Theorem 8. *The following are equivalent for all graphs G:*

(8.1) *G is a threshold graph.*

(8.2) *For all $i \geq 4$, every path P with $|V(P)| \geq i$ has an $(i-2)$-chord.*

(8.3) *For $i \in \{4,5\}$, every path P with $|V(P)| \geq i$ has an $(i-2)$-chord.*

Proof. Suppose G is any connected graph.

(8.1) $\Rightarrow$ (8.2): Suppose G is a threshold graph (and so is trivially perfect) and $i \geq 4$. Corollary 7 implies that every path P with $|V(P)| \geq 4$ has a 2-chord. Therefore assume path $P = v_1, \ldots, v_p$ has $p = |V(P)| \geq i > 4$ with subpath $Q = v_1, \ldots, v_i$. Suppose P has no $(i-2)$-chord (arguing by contradiction), and so $v_1 \not\sim v_{i-1}$ and $v_2 \not\sim v_i$. But then the subgraph induced by $\{v_1, v_2, v_{i-1}, v_i\}$ would be isomorphic to $2K_2$ or P_4 or C_4 (contradicting that G is a threshold graph).

(8.2) $\Rightarrow$ (8.3): This implication is immediate.

(8.3) $\Rightarrow$ (8.1): Suppose every path with $|V(P)| \geq 4$ has a 2-chord and every path with $|V(P)| \geq 5$ has a 3-chord, yet G is not a threshold graph (arguing by contradiction). By Corollary 7, G is trivially perfect. Since G is not a threshold graph, G must contain two edges vw and $v'w'$ in an induced $2K_2$. Suppose P is a minimum-length path that contains both vw and $v'w'$. But $v \not\sim v' \not\sim w$ and $v \not\sim w' \not\sim w$ imply $V(P) \geq 5$. Therefore, P would have a 3-chord (contradicting the minimality of P). ∎

4. Chords of Cycles in Bipartite Graphs

As in [2, 9], a graph G is *chordal bipartite* if G is bipartite and every cycle C with $|V(C)| \geq 6$ has a chord (equivalently, every cycle long enough to have a chord does have a chord).

Theorem 9. *A graph is chordal bipartite if and only if every cycle C with $|V(C)| \geq 6$ has a 3-chord.*

Proof. The 'if direction' is immediate. The 'only if direction' follows from the well-known result that every induced subgraph of a chordal bipartite graph has a *simplicial edge*, meaning an edge vw such that the union of the neighborhoods of v and w induce a complete subgraph [2, 9]. ∎

A graph is *bipartite distance-hereditary* if it is both bipartite and distance-hereditary; see section 6 of [1]. This is equivalent to being chordal bipartite with no induced subgraph isomorphic to a *domino*—the graph obtained from a 6-cycle by inserting one 3-chord. Theorem 10 is a bipartite analog of Theorems 2 and 5 simultaneously.

Theorem 10. *In a bipartite distance-hereditary graph, if* $i \in \{6, 10\}$*, then every cycle* C *with* $|V(C)| \geq i$ *has an* $(i/2)$*-chord.*

Proof. Suppose G is a bipartite distance-hereditary graph. By [1], G is chordal bipartite with no induced subgraph isomorphic to a domino. By Theorem 9, every cycle C with $|V(C)| \geq 6$ has a 3-chord. Suppose $C = v_1, v_2, \ldots, v_k, v_1$ is a k-cycle in G (k is even, of course). Argue by induction on even $k \geq 10$ that C has a 5-chord.

For the basis step, suppose $k = 10$, but C has no 5-chord (arguing by contradiction); thus $v_1 \not\sim v_6$, $v_2 \not\sim v_7$, $v_3 \not\sim v_8$, $v_4 \not\sim v_9$, and $v_5 \not\sim v_{10}$. Since G is chordal bipartite, C has a 3-chord; without loss of generality, say v_3v_6 is a chord. Thus $v_1 \not\sim v_8$ (to avoid $v_1, v_2, v_3, v_6, v_7, v_8, v_1$ being a 6-cycle with no chords), $v_2 \not\sim v_9$ (to avoid $v_2, v_3, v_6, v_7, v_8, v_9, v_2$ either being a 6-cycle with no chords or spanning an induced domino), and $v_7 \not\sim v_{10}$ (similarly). Therefore, the 8-cycle $v_1, v_2, v_3, v_6, v_7, v_8, v_9, v_{10}, v_1$ must have both of the only possible 3-chords v_3v_{10} and v_6v_9 (*both* of them, to avoid inducing a domino). But then $v_1, v_2, v_3, v_6, v_9, v_{10}, v_1$ would span an induced domino, a contradiction.

Therefore suppose $k \geq 12$ and every k'-cycle with $10 \leq k' < k$ has a 5-chord, and again suppose that C has no 5-chord arguing by contradiction. By Theorem 9, C has a 3-chord; without loss of generality, say v_3v_6 is a chord. Let C' be the cycle with edge set $E(C) - \{v_3v_4, v_4v_5, v_5v_6\} \cup \{v_3v_6\}$ and length $k - 2 \geq 10$. The inductive hypothesis implies that C' has a 5-chord (that is not a 5-chord of C), and so either $v_1 \sim v_8$ or $v_2 \sim v_9$ or $v_3 \sim v_{10}$ or $v_6 \sim v_{k-1}$ or $v_7 \sim v_k$. Thus $k \geq 14$ (since those five edges would be 5-chords of C if $k = 12$). By the same argument used in the basis step, $v_1 \not\sim v_8$ and $v_2 \not\sim v_9$ and $v_7 \not\sim v_k$. Therefore either $v_3 \sim v_{10}$ or $v_6 \sim v_{k-1}$; without loss of generality, suppose $v_3 \sim v_{10}$. Since $v_3 \not\sim v_8$ (because C has no 5-chords) and $v_3, v_6, v_7, v_8, v_9, v_{10}, v_3$ must have a 3-chord without spanning an induced domino, it follows that $v_6 \sim v_9$ and $v_7 \sim v_{10}$. But then the cycle $v_3, v_4, v_5, v_6, v_9, v_{10}, v_3$ would span an induced domino [a contradiction]. ∎

The domino graph shows that the converse of Theorem 10 fails. The graph in Figure 3 is bipartite distance-hereditary—Corollary 3 of [1] is an easy way to verify this—and shows that cycles of bipartite distance-hereditary graphs with $|V(C)| \geq 14$ might not have 7-chords.

5. Chords of Paths in Bipartite Graphs

Theorem 11 is the bipartite analog of Theorem 6.

Theorem 11. *For every* $i \geq 4$*, a bipartite graph is both* P_i*-free and chordal bipartite if and only if every path* P *with* $|V(P)| \geq i$ *has a* 3*-chord.*

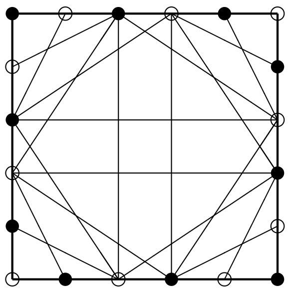

Figure 3. A bipartite distance-hereditary graph spanned by a 20-cycle that has a 3-chord and a 5-chord (and a 9-chord), but no 7-chord.

Proof. This is proved in the same way as Theorem 6 (with $i \geq 4$ and $j \geq 4$ in the first paragraph and with $n \geq 5$ in the second). ■

The bipartite graphs that are P_5-free chordal bipartite graphs have been characterized—see Corollary 3.2 of [3] or Theorem 4 of [6]—by every connected induced subgraph of G having either a dominating vertex or a *dominating edge* (meaning an edge vw such that every vertex is adjacent to v or w). Thus, a bipartite graph is P_5-free chordal bipartite if and only if every induced subgraph has an isolated vertex or a dominating edge.

Corollary 12. *Every connected bipartite graph is complete bipartite if and only if every path P with $|V(P)| \geq 4$ has a 3-chord. Every connected induced subgraph of a bipartite graph has a dominating vertex or edge if and only if every path P with $|V(P)| \geq 5$ has a 3-chord.*

Proof. These are the $i = 4$ and $i = 5$ cases of Theorem 11. ■

Comparing Corollary 12 to Corollary 7, reference [10] shows that a graph is trivially perfect if and only if every connected induced subgraph of G has a dominating vertex.

The *difference graphs*—these are close relatives of threshold graphs and are also called *chain graphs*, see [7, 9]—are the P_5-free bipartite graphs. The *chordal bipartite difference graphs* are the $2K_2$-free (and so P_5-free) chordal bipartite graphs. Theorem 13 is a bipartite analog of Theorem 8. When comparing it to Corollary 12, note that the second condition in the second part of Corollary 12 could be rephrased as *for $i \in \{5\}$, every path P with $|V(P)| \geq i$ has an $(i-2)$-chord.*

Theorem 13. *The following are equivalent for all bipartite graphs G:*

(13.1) *G is a chordal bipartite difference graph.*

(13.2) *For all odd $i \geq 5$, every path P with $|V(P)| \geq i$ has an $(i-2)$-chord.*

(13.3) *For* $i \in \{5, 7\}$, *every path* P *with* $|V(P)| \geq i$ *has an* $(i-2)$*-chord.*

Proof. This is proved in the same way as Theorem 8 (for (13.1) ⇒ (13.2), the subgraph induced by $\{v_1, v_2, v_{i-1}, v_i\}$ would be isomorphic to $2K_2$). ■

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