# THE $i$-CHORDS OF CYCLES AND PATHS 

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#### Abstract

An $i$-chord of a cycle or path is an edge whose endpoints are a distance $i \geq 2$ apart along the cycle or path. Motivated by many standard graph classes being describable by the existence of chords, we investigate what happens when $i$-chords are required for specific values of $i$. Results include the following: A graph is strongly chordal if and only if, for $i \in\{4,6\}$, every cycle $C$ with $|V(C)| \geq i$ has an $(i / 2)$-chord. A graph is a threshold graph if and only if, for $i \in\{4,5\}$, every path $P$ with $|V(P)| \geq i$ has an (i-2)-chord.


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## 1. InTRODUCTION

A chord of a cycle $C$ or path $P$ is an edge $v w$ between two nonconsecutive vertices $v$ and $w$ of $C$ or $P$, and $v w$ is an $i$-chord if the distance between $v$ and $w$ is $i$ within $C$ or $P$. Chords $v w$ and $x y$ are crossing chords of $C$ if the four vertices $v, x, w, y$ come in that order around $C$.

Many graph classes have been characterized by chords existing in long-enough cycles (or, less often, paths). Using $i$-chords for specific $i$ allows finer distinctions to be made. Section 2 will discuss several graph classes in terms of $i$-chords of cycles, with similar-yet only somewhat similar-results in Section 3 for $i$-chords of paths. Sections 4 and 5 will discuss some of the corresponding results for bipartite graphs.

## 2. Chords of Cycles in Graphs

As in $[2,9]$, a graph is chordal if every cycle $C$ with $|V(C)| \geq 4$ has a chord (equivalently, every cycle long enough to have a chord does have a chord). Theorem 1 is a very simple characterization of being chordal.
Theorem 1. A graph is chordal if and only if every cycle $C$ with $|V(C)| \geq 4$ has a 2-chord.

Proof. The 'if direction' is immediate. The 'only if direction' follows immediately from the well-known result that every induced subgraph of a chordal graph has a simplicial vertex, meaning a vertex whose open neighborhood induces a complete subgraph [2, 9].

For comparison with Theorems 2 and 5, note that Theorem 1 could be rephrased as follows: A graph is chordal if and only if, for $i \in\{4\}$, every cycle $C$ with $|V(C)| \geq i$ has an (i/2)-chord.

Theorem 2 characterizes strongly chordal graphs-the chordal graphs in which every cycle of even length at least 6 has an $i$-chord where $i$ is odd [2, 4, 8, 9] (equivalently, for each $i \in\{2,3\}$, every cycle long enough to have an $i$-chord does have an $i$-chord).
Theorem 2. A graph is strongly chordal if and only if, for $i \in\{4,6\}$, every cycle $C$ with $|V(C)| \geq i$ has an (i/2)-chord.

Proof. The 'if direction' is immediate. The 'only if direction' follows from the well-known result that every induced subgraph of a strongly chordal graph has a simple vertex, meaning a vertex $v$ such that the closed neighborhoods of every two neighbors of $v$ are comparable by inclusion $[2,9]$.

The chordal graph formed from a 6 -cycle by inserting three noncrossing 2 -chords shows that cycles of chordal graphs with $|V(C)| \geq 6$ might not have 3 -chords. The strongly chordal graph shown in Figure 1 shows that cycles of strongly chordal graphs with $|V(C)| \geq 8$ might not have 4-chords.

A graph $G$ is distance-hereditary if the distance between vertices in connected induced subgraphs of $G$ always equals the distance between them in $G$. A graph is ptolemaic if it is both chordal and distance-hereditary. Reference [2] contains many other characterizations of these concepts. In particular, from [5], a graph is ptolemaic if and only if it is chordal with no induced subgraph isomorphic to a gem-the graph obtained from a 5 -cycle by inserting two noncrossing 2 -chords. Also, a graph is ptolemaic if and only if it is chordal and every cycle of length at least 5 has crossing chords. Every ptolemaic graph is strongly chordal [2] (but the strongly chordal graph in Figure 1 is not ptolemaic). Theorem 5 will show that, for each $i \in\{2,3,4\}$, every cycle of a ptolemaic graph that is long enough to have an $i$-chord in fact does have an $i$-chord.


Figure 1. A strongly chordal graph spanned by an 8-cycle that has a 2-chord and a 3-chord, but no 4-chord.

Lemma 3 (Howorka [5]). Every 4-cycle, 5-cycle, and 6-cycle in a ptolemaic graph will have, respectively, at least 1, 3, or 4 chords (and so, respectively, at most 1, 2, or 5 nonadjacent pairs of vertices).

Proof. This follows from the characterization of ptolemaic graphs in [5] by every $k$-cycle having at least $\lfloor 3(k-3) / 2\rfloor$ chords (and so having at most $k(k-3) / 2-$ $\lfloor 3(k-3) / 2\rfloor$ nonadjacent pairs of vertices).

Let $v \sim w$ and $v \nsim w$ denote that vertices $v$ and $w$ are, respectively, adjacent or nonadjacent.

Lemma 4. If $a, b, c, d, e, f$ is a path (possibly a closed path with $a=f$ ) in a ptolemaic graph with $b \nsim d$ and $c \nsim e$, then $b \nsim e$. If also $a \neq f$, then $a \nsim e$ and $b \nsim f$ 。

Proof. Inserting an edge be (or $a e$ or bf if $a \neq f$ ) would violate Lemma 3 by creating a cycle with too few chords.

Theorem 5. In a ptolemaic graph, if $i \in\{4,6,8\}$, then every cycle $C$ with $|V(C)| \geq i$ has an (i/2)-chord.

Proof. Suppose $G$ is ptolemaic (and so is strongly chordal). By Theorem 2, every cycle $C$ with $|V(C)| \geq 4$ has a 2 -chord and every $C$ with $|V(C)| \geq 6$ has a 3 -chord. Suppose $C=v_{1}, v_{2}, \ldots, v_{k}, v_{1}$ is a $k$-cycle in $G$. Argue by induction on $k \geq 8$ that $C$ has a 4-chord.

For the basis step, suppose $k=8$, but $C$ has no 4 -chord (arguing by contradiction); thus $v_{1} \nsim v_{5}, v_{2} \nsim v_{6}, v_{3} \nsim v_{7}$, and $v_{4} \nsim v_{8}$. Since $G$ is strongly chordal, $C$ has a 3 -chord; without loss of generality, say $v_{3} v_{6}$ is a chord. Lemma 4 on the path $v_{1}, v_{2}, v_{3}, v_{6}, v_{7}, v_{8}$ implies $v_{1} \nsim v_{7} \nsim v_{2} \nsim v_{8}$. Therefore, the 6 -cycle $C^{-}=v_{1}, v_{2}, v_{3}, v_{6}, v_{7}, v_{8}, v_{1}$ has five nonadjacent pairs of vertices and so, by Lemma 3, $C^{-}$has all the other four possible chords $v_{1} v_{3}, v_{1} v_{6}, v_{3} v_{8}$ and $v_{6} v_{8}$. Lemma 4 on the path $v_{4}, v_{5}, v_{6}, v_{1}, v_{2}, v_{3}$ now implies $v_{3} \nsim v_{5}$. Similarly, the path $v_{5}, v_{4}, v_{3}, v_{8}, v_{7}, v_{6}$ implies $v_{4} \nsim v_{6}$. But the chordless 4 -cycle $v_{3}, v_{4}, v_{5}, v_{6}, v_{3}$ would now contradict Lemma 3.

Therefore suppose $k \geq 9$ and every $k^{\prime}$-cycle with $8 \leq k^{\prime}<k$ has a 4 -chord, but also suppose that $C$ has no 4 -chord (arguing by contradiction); thus $v_{1} \nsim v_{k-3}$, $v_{2} \nsim v_{k-2}, v_{3} \nsim v_{k-1}, v_{4} \nsim v_{k}, v_{1} \nsim v_{5}, v_{2} \nsim v_{6}$, and $v_{3} \nsim v_{7}$. Since $G$ is chordal, $C$ has a 2 -chord; without loss of generality, say $v_{1} v_{k-1}$ is a chord. Let $C^{\prime}$ be the cycle with edge set $E(C)-\left\{v_{k-1} v_{k}, v_{1} v_{k}\right\} \cup\left\{v_{1} v_{k-1}\right\}$ and length $k-1 \geq 8$. The inductive hypothesis implies that $C^{\prime}$ has a 4 -chord (that is not a 4-chord of $C$ ), and so $k \geq 10$ and either $v_{1} \sim v_{k-4}$ or $v_{2} \sim v_{k-3}$ or $v_{3} \sim v_{k-2}$ or $v_{4} \sim v_{k-1}$. Observe that $v_{2} \nsim v_{k-3}$; otherwise Lemma 4 on the path $v_{k}, v_{1}, v_{2}, v_{k-3}, v_{k-2}, v_{k-1}$ would contradict $v_{1} \sim v_{k-1}$. Similarly, $v_{3} \nsim v_{k-2}$. Therefore, either $v_{1} \sim v_{k-4}$ or $v_{4} \sim v_{k-1}$; without loss of generality, suppose $v_{4} \sim v_{k-1}$.

Lemma 4 on the path $v_{2}, v_{3}, v_{4}, v_{k-1}, v_{k}, v_{1}$ implies $v_{1} \nsim v_{3} \nsim v_{k} \nsim v_{2}$. Lemma 3 on the 5 -cycle $v_{1}, v_{2}, v_{3}, v_{4}, v_{k-1}, v_{1}$ with the two nonadjacent pairs $\left\{v_{1}, v_{3}\right\}$ and $\left\{v_{3}, v_{k-1}\right\}$ implies $v_{1} \sim v_{4} \sim v_{2} \sim v_{k-1}$. Lemma 4 on the path $v_{k-3}, v_{k-2}, v_{k-1}, v_{2}, v_{3}, v_{4}$ then implies $v_{4} \nsim v_{k-2}$, and also, now on the path $v_{k-1}, v_{k}, v_{1}, v_{4}, v_{5}, v_{6}$, implies $v_{k-1} \nsim v_{5} \nsim v_{k} \nsim v_{6}$. Finally, Lemma 4 on the path $v_{k-3}, v_{k-2}, v_{k-1}, v_{4}, v_{5}, v_{6}$ implies $v_{5} \nsim v_{k-2}$.

Let $C^{\prime \prime}$ be the cycle with edge set $E(C)-\left\{v_{1} v_{2}, v_{2} v_{3}, v_{3} v_{4}\right\} \cup\left\{v_{1} v_{4}\right\}$ and length $k-2 \geq 8$. The inductive hypothesis implies that $C^{\prime \prime}$ has a 4 -chord (that is not a 4 -chord of $C$ ), and so $k \geq 11$ and either $v_{4} \sim v_{k-2}$ or $v_{5} \sim v_{k-1}$ or $v_{6} \sim v_{k}$ or $v_{1} \sim v_{7}$. Since we have proved that $v_{4} \nsim v_{k-2}$ and $v_{5} \nsim v_{k-1}$ and $v_{6} \nsim v_{k}$, it follows that $v_{1} \sim v_{7}$. Observe that $v_{1} \nsim v_{6}$, since otherwise Lemma 4 on the path $v_{4}, v_{5}, v_{6}, v_{1}, v_{2}, v_{3}$ would contradict $v_{2} \sim v_{4}$. Lemma 3 on the 5 -cycle $v_{1}, v_{4}, v_{5}, v_{6}, v_{7}, v_{1}$ with the two nonadjacent pairs $\left\{v_{1}, v_{5}\right\}$ and $\left\{v_{1}, v_{6}\right\}$ then implies $v_{5} \sim v_{7} \sim v_{4} \sim v_{6}$. Note that $v_{2} \nsim v_{5}$, since otherwise the 5 -cycle $v_{1}, v_{2}, v_{5}, v_{6}, v_{7}, v_{1}$ with the three nonadjacent pairs $\left\{v_{1}, v_{5}\right\},\left\{v_{1}, v_{6}\right\}$ and $\left\{v_{2}, v_{6}\right\}$ would contradict Lemma 3 . Furthermore, Lemma 3 on the 5 -cycle $v_{1}, v_{2}, v_{4}, v_{5}, v_{7}, v_{1}$ with the two nonadjacent pairs $\left\{v_{1}, v_{5}\right\}$ and $\left\{v_{2}, v_{5}\right\}$ implies $v_{2} \sim v_{7}$. But then Lemma 4 on the path $v_{5}, v_{6}, v_{7}, v_{2}, v_{3}, v_{4}$ would contradict $v_{4} \sim v_{6}$.


Figure 2. A ptolemaic graph spanned by a 14 -cycle that has a 2 -chord, a 3 -chord, and a 4 -chord (and a 6 -chord and a 7 -chord), but no 5 -chord.

The gem graph shows that the converse of Theorem 5 fails. The graph in Figure 2 is ptolemaic - Corollary 6 of [1] is an easy way to verify this - and shows that cycles of ptolemaic graphs with $|V(C)| \geq 10$ might not have 5 -chords. (This is a minimum-order counterexample: cycles with $10 \leq|V(C)| \leq 13$ in ptolemaic graphs turn out to always have 5 -chords.)

## 3. Chords of Paths in Graphs

Let $C_{n}$ and $P_{n}$ denote, respectively, a cycle and path on $n$ vertices (so $C_{n}$ has length $n$ and $P_{n}$ has length $n-1$ ). For any graph $H$, a graph $G$ is $H$-free if $G$ contains no induced subgraph isomorphic to $H$.

Theorem 6. For every $i \geq 3$, a graph is both $P_{i}$-free and chordal if and only if every path $P$ with $|V(P)| \geq i$ has a 2-chord.

Proof. First suppose $G$ is $P_{i}$-free and chordal with a path $P$ where $|V(P)| \geq$ $i \geq 3$. Consider the minimum $j$ such that $P$ has a $j$-chord. If $j \geq 3$, then that $j$-chord would combine with $P$ to form a chordless $(j+1)$-cycle where $j+1 \geq 4$, contradicting that $G$ is chordal. Therefore, $j=2$.

Conversely, suppose every path $P$ of $G$ with $|V(P)| \geq i \geq 3$ has a 2-chord. Therefore, $G$ contains no induced subgraph isomorphic to any such $P_{i}$ or to any $C_{n}$ with $n \geq 4$, and so $G$ is $P_{i}$-free chordal.

The trivially perfect graphs have many names and characterizations [2, 9], one of which is that they are precisely the $P_{4}$-free chordal graphs. Corollary 7 is the path analog of Theorem 1.

Corollary 7. A graph has complete components if and only if every path $P$ with $|V(P)| \geq 3$ has a 2-chord. A graph is trivially perfect if and only if every path $P$ with $|V(P)| \geq 4$ has a 2-chord.

Proof. These are the $i=3$ and $i=4$ cases of Theorem 6.
The threshold graphs also have many characterizations [2, 7, 9], one of which is that they are precisely the $2 K_{2}$-free trivially perfect graphs ( $2 K_{2}$ is the complement of $C_{4}$ ). For a connected graph, this is equivalent-see Theorem 1.2.4 of [7] for the history - to being constructible from a single vertex by recursively appending either an isolated vertex or a dominating vertex (often called a universal vertex, meaning a vertex adjacent to all the previously-existing vertices). Thus, a graph is a threshold graph if and only if every induced subgraph has either an isolated vertex or a dominating vertex. Theorem 8 is a path analog of Theorem 2.

Theorem 8. The following are equivalent for all graphs $G$ :
(8.1) $G$ is a threshold graph.
(8.2) For all $i \geq 4$, every path $P$ with $|V(P)| \geq i$ has an $(i-2)$-chord.
(8.3) For $i \in\{4,5\}$, every path $P$ with $|V(P)| \geq i$ has an $(i-2)$-chord.

Proof. Suppose $G$ is any connected graph.
$(8.1) \Rightarrow(8.2)$ : Suppose $G$ is a threshold graph (and so is trivially perfect) and $i \geq 4$. Corollary 7 implies that every path $P$ with $|V(P)| \geq 4$ has a 2-chord. Therefore assume path $P=v_{1}, \ldots, v_{p}$ has $p=|V(P)| \geq i>4$ with subpath $Q=v_{1}, \ldots, v_{i}$. Suppose $P$ has no $(i-2)$-chord (arguing by contradiction), and so $v_{1} \nsim v_{i-1}$ and $v_{2} \nsim v_{i}$. But then the subgraph induced by $\left\{v_{1}, v_{2}, v_{i-1}, v_{i}\right\}$ would be isomorphic to $2 K_{2}$ or $P_{4}$ or $C_{4}$ (contradicting that $G$ is a threshold graph).
$(8.2) \Rightarrow(8.3):$ This implication is immediate.
$(8.3) \Rightarrow(8.1)$ : Suppose every path with $|V(P)| \geq 4$ has a 2-chord and every path with $|V(P)| \geq 5$ has a 3-chord, yet $G$ is not a threshold graph (arguing by contradiction). By Corollary 7, $G$ is trivially perfect. Since $G$ is not a threshold graph, $G$ must contain two edges $v w$ and $v^{\prime} w^{\prime}$ in an induced $2 K_{2}$. Suppose $P$ is a minimum-length path that contains both $v w$ and $v^{\prime} w^{\prime}$. But $v \nsim v^{\prime} \nsim w$ and $v \nsim w^{\prime} \nsim w$ imply $V(P) \geq 5$. Therefore, $P$ would have a 3 -chord (contradicting the minimality of $P$ ).

## 4. Chords of Cycles in Bipartite Graphs

As in $[2,9]$, a graph $G$ is chordal bipartite if $G$ is bipartite and every cycle $C$ with $|V(C)| \geq 6$ has a chord (equivalently, every cycle long enough to have a chord does have a chord).

Theorem 9. A graph is chordal bipartite if and only if every cycle $C$ with $|V(C)| \geq 6$ has a 3-chord.

Proof. The 'if direction' is immediate. The 'only if direction' follows from the well-known result that every induced subgraph of a chordal bipartite graph has a simplicial edge, meaning an edge $v w$ such that the union of the neighborhoods of $v$ and $w$ induce a complete subgraph $[2,9]$.

A graph is bipartite distance-hereditary if it is both bipartite and distance-hereditary; see section 6 of [1]. This is equivalent to being chordal bipartite with no induced subgraph isomorphic to a domino - the graph obtained from a 6 -cycle by inserting one 3 -chord. Theorem 10 is a bipartite analog of Theorems 2 and 5 simultaneously.

Theorem 10. In a bipartite distance-hereditary graph, if $i \in\{6,10\}$, then every cycle $C$ with $|V(C)| \geq i$ has an (i/2)-chord.

Proof. Suppose $G$ is a bipartite distance-hereditary graph. By [1], $G$ is chordal bipartite with no induced subgraph isomorphic to a domino. By Theorem 9, every cycle $C$ with $|V(C)| \geq 6$ has a 3 -chord. Suppose $C=v_{1}, v_{2}, \ldots, v_{k}, v_{1}$ is a $k$-cycle in $G$ ( $k$ is even, of course). Argue by induction on even $k \geq 10$ that $C$ has a 5 -chord.

For the basis step, suppose $k=10$, but $C$ has no 5 -chord (arguing by contradiction); thus $v_{1} \not \nsim v_{6}, v_{2} \not \nsim v_{7}, v_{3} \not \nsim v_{8}, v_{4} \nsim v_{9}$, and $v_{5} \nsim v_{10}$. Since $G$ is chordal bipartite, $C$ has a 3 -chord; without loss of generality, say $v_{3} v_{6}$ is a chord. Thus $v_{1} \not \nsim v_{8}$ (to avoid $v_{1}, v_{2}, v_{3}, v_{6}, v_{7}, v_{8}, v_{1}$ being a 6 -cycle with no chords), $v_{2} \nsim v_{9}$ (to avoid $v_{2}, v_{3}, v_{6}, v_{7}, v_{8}, v_{9}, v_{2}$ either being a 6 -cycle with no chords or spanning an induced domino), and $v_{7} \not \nsim v_{10}$ (similarly). Therefore, the 8 -cycle $v_{1}, v_{2}, v_{3}, v_{6}, v_{7}, v_{8}, v_{9}, v_{10}, v_{1}$ must have both of the only possible 3 chords $v_{3} v_{10}$ and $v_{6} v_{9}$ (both of them, to avoid inducing a domino). But then $v_{1}, v_{2}, v_{3}, v_{6}, v_{9}, v_{10}, v_{1}$ would span an induced domino, a contradiction.

Therefore suppose $k \geq 12$ and every $k^{\prime}$-cycle with $10 \leq k^{\prime}<k$ has a 5chord, and again suppose that $C$ has no 5 -chord arguing by contradiction. By Theorem $9, C$ has a 3 -chord; without loss of generality, say $v_{3} v_{6}$ is a chord. Let $C^{\prime}$ be the cycle with edge set $E(C)-\left\{v_{3} v_{4}, v_{4} v_{5}, v_{5} v_{6}\right\} \cup\left\{v_{3} v_{6}\right\}$ and length $k-2 \geq 10$. The inductive hypothesis implies that $C^{\prime}$ has a 5 -chord (that is not a 5 -chord of $C$ ), and so either $v_{1} \sim v_{8}$ or $v_{2} \sim v_{9}$ or $v_{3} \sim v_{10}$ or $v_{6} \sim v_{k-1}$ or $v_{7} \sim v_{k}$. Thus $k \geq 14$ (since those five edges would be 5 -chords of $C$ if $k=12$ ). By the same argument used in the basis step, $v_{1} \nsim v_{8}$ and $v_{2} \nsim v_{9}$ and $v_{7} \nsim$ $v_{k}$. Therefore either $v_{3} \sim v_{10}$ or $v_{6} \sim v_{k-1}$; without loss of generality, suppose $v_{3} \sim v_{10}$. Since $v_{3} \nsim v_{8}$ (because $C$ has no 5 -chords) and $v_{3}, v_{6}, v_{7}, v_{8}, v_{9}, v_{10}, v_{3}$ must have a 3 -chord without spanning an induced domino, it follows that $v_{6} \sim v_{9}$ and $v_{7} \sim v_{10}$. But then the cycle $v_{3}, v_{4}, v_{5}, v_{6}, v_{9}, v_{10}, v_{3}$ would span an induced domino [a contradiction].

The domino graph shows that the converse of Theorem 10 fails. The graph in Figure 3 is bipartite distance-hereditary-Corollary 3 of [1] is an easy way to verify this-and shows that cycles of bipartite distance-hereditary graphs with $|V(C)| \geq 14$ might not have 7 -chords.

## 5. Chords of Paths in Bipartite Graphs

Theorem 11 is the bipartite analog of Theorem 6.
Theorem 11. For every $i \geq 4$, a bipartite graph is both $P_{i}$-free and chordal bipartite if and only if every path $P$ with $|V(P)| \geq i$ has a 3 -chord.


Figure 3. A bipartite distance-hereditary graph spanned by a 20 -cycle that has a 3 -chord and a 5 -chord (and a 9 -chord), but no 7 -chord.

Proof. This is proved in the same way as Theorem 6 (with $i \geq 4$ and $j \geq 4$ in the first paragraph and with $n \geq 5$ in the second).

The bipartite graphs that are $P_{5}$-free chordal bipartite graphs have been charac-terized-see Corollary 3.2 of [3] or Theorem 4 of [6]-by every connected induced subgraph of $G$ having either a dominating vertex or a dominating edge (meaning an edge $v w$ such that every vertex is adjacent to $v$ or $w$ ). Thus, a bipartite graph is $P_{5}$-free chordal bipartite if and only if every induced subgraph has an isolated vertex or a dominating edge.

Corollary 12. Every connected bipartite graph is complete bipartite if and only if every path $P$ with $|V(P)| \geq 4$ has a 3-chord. Every connected induced subgraph of a bipartite graph has a dominating vertex or edge if and only if every path $P$ with $|V(P)| \geq 5$ has a 3 -chord.

Proof. These are the $i=4$ and $i=5$ cases of Theorem 11.
Comparing Corollary 12 to Corollary 7, reference [10] shows that a graph is trivially perfect if and only if every connected induced subgraph of $G$ has a dominating vertex.

The difference graphs - these are close relatives of threshold graphs and are also called chain graphs, see $[7,9]$-are the $P_{5}$-free bipartite graphs. The chordal bipartite difference graphs are the $2 K_{2}$-free (and so $P_{5}$-free) chordal bipartite graphs. Theorem 13 is a bipartite analog of Theorem 8 . When comparing it to Corollary 12 , note that the second condition in the second part of Corollary 12 could be rephrased as for $i \in\{5\}$, every path $P$ with $|V(P)| \geq i$ has an $(i-2)$ chord.

Theorem 13. The following are equivalent for all bipartite graphs $G$ :
(13.1) $G$ is a chordal bipartite difference graph.
(13.2) For all odd $i \geq 5$, every path $P$ with $|V(P)| \geq i$ has an $(i-2)$-chord.
(13.3) For $i \in\{5,7\}$, every path $P$ with $|V(P)| \geq i$ has an $(i-2)$-chord.

Proof. This is proved in the same way as Theorem 8 (for $(13.1) \Rightarrow(13.2)$, the subgraph induced by $\left\{v_{1}, v_{2}, v_{i-1}, v_{i}\right\}$ would be isomorphic to $2 K_{2}$ ).

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