

## THE $i$ -CHORDS OF CYCLES AND PATHS

TERRY A. MCKEE

*Department of Mathematics and Statistics*  
*Wright State University*  
*Dayton, Ohio 45435 USA*

**e-mail:** terry.mckee@wright.edu

### Abstract

An  $i$ -chord of a cycle or path is an edge whose endpoints are a distance  $i \geq 2$  apart along the cycle or path. Motivated by many standard graph classes being describable by the existence of chords, we investigate what happens when  $i$ -chords are required for specific values of  $i$ . Results include the following: A graph is strongly chordal if and only if, for  $i \in \{4, 6\}$ , every cycle  $C$  with  $|V(C)| \geq i$  has an  $(i/2)$ -chord. A graph is a threshold graph if and only if, for  $i \in \{4, 5\}$ , every path  $P$  with  $|V(P)| \geq i$  has an  $(i-2)$ -chord.

**Keywords:** chord, chordal graph, strongly chordal graph, ptolemaic graph, trivially perfect graph, threshold graph.

**2010 Mathematics Subject Classification:** 05C75, 05C38.

### 1. INTRODUCTION

A *chord* of a cycle  $C$  or path  $P$  is an edge  $vw$  between two nonconsecutive vertices  $v$  and  $w$  of  $C$  or  $P$ , and  $vw$  is an  $i$ -chord if the distance between  $v$  and  $w$  is  $i$  within  $C$  or  $P$ . Chords  $vw$  and  $xy$  are *crossing chords* of  $C$  if the four vertices  $v, x, w, y$  come in that order around  $C$ .

Many graph classes have been characterized by chords existing in long-enough cycles (or, less often, paths). Using  $i$ -chords for specific  $i$  allows finer distinctions to be made. Section 2 will discuss several graph classes in terms of  $i$ -chords of cycles, with similar—yet only somewhat similar—results in Section 3 for  $i$ -chords of paths. Sections 4 and 5 will discuss some of the corresponding results for bipartite graphs.

## 2. CHORDS OF CYCLES IN GRAPHS

As in [2, 9], a graph is *chordal* if every cycle  $C$  with  $|V(C)| \geq 4$  has a chord (equivalently, every cycle long enough to have a chord does have a chord). Theorem 1 is a very simple characterization of being chordal.

**Theorem 1.** *A graph is chordal if and only if every cycle  $C$  with  $|V(C)| \geq 4$  has a 2-chord.*

**Proof.** The ‘if direction’ is immediate. The ‘only if direction’ follows immediately from the well-known result that every induced subgraph of a chordal graph has a *simplicial vertex*, meaning a vertex whose open neighborhood induces a complete subgraph [2, 9]. ■

For comparison with Theorems 2 and 5, note that Theorem 1 could be rephrased as follows: *A graph is chordal if and only if, for  $i \in \{4\}$ , every cycle  $C$  with  $|V(C)| \geq i$  has an  $(i/2)$ -chord.*

Theorem 2 characterizes *strongly chordal graphs*—the chordal graphs in which every cycle of even length at least 6 has an  $i$ -chord where  $i$  is odd [2, 4, 8, 9] (equivalently, for each  $i \in \{2, 3\}$ , every cycle long enough to have an  $i$ -chord does have an  $i$ -chord).

**Theorem 2.** *A graph is strongly chordal if and only if, for  $i \in \{4, 6\}$ , every cycle  $C$  with  $|V(C)| \geq i$  has an  $(i/2)$ -chord.*

**Proof.** The ‘if direction’ is immediate. The ‘only if direction’ follows from the well-known result that every induced subgraph of a strongly chordal graph has a *simple vertex*, meaning a vertex  $v$  such that the closed neighborhoods of every two neighbors of  $v$  are comparable by inclusion [2, 9]. ■

The chordal graph formed from a 6-cycle by inserting three noncrossing 2-chords shows that cycles of chordal graphs with  $|V(C)| \geq 6$  might not have 3-chords. The strongly chordal graph shown in Figure 1 shows that cycles of strongly chordal graphs with  $|V(C)| \geq 8$  might not have 4-chords.

A graph  $G$  is *distance-hereditary* if the distance between vertices in connected induced subgraphs of  $G$  always equals the distance between them in  $G$ . A graph is *ptolemaic* if it is both chordal and distance-hereditary. Reference [2] contains many other characterizations of these concepts. In particular, from [5], a graph is ptolemaic if and only if it is chordal with no induced subgraph isomorphic to a *gem*—the graph obtained from a 5-cycle by inserting two noncrossing 2-chords. Also, a graph is ptolemaic if and only if it is chordal and every cycle of length at least 5 has crossing chords. Every ptolemaic graph is strongly chordal [2] (but the strongly chordal graph in Figure 1 is not ptolemaic). Theorem 5 will show that, for each  $i \in \{2, 3, 4\}$ , every cycle of a ptolemaic graph that is long enough to have an  $i$ -chord in fact does have an  $i$ -chord.

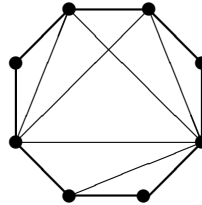


Figure 1. A strongly chordal graph spanned by an 8-cycle that has a 2-chord and a 3-chord, but no 4-chord.

**Lemma 3** (Howorka [5]). *Every 4-cycle, 5-cycle, and 6-cycle in a ptolemaic graph will have, respectively, at least 1, 3, or 4 chords (and so, respectively, at most 1, 2, or 5 nonadjacent pairs of vertices).*

**Proof.** This follows from the characterization of ptolemaic graphs in [5] by every  $k$ -cycle having at least  $\lfloor 3(k-3)/2 \rfloor$  chords (and so having at most  $k(k-3)/2 - \lfloor 3(k-3)/2 \rfloor$  nonadjacent pairs of vertices). ■

Let  $v \sim w$  and  $v \not\sim w$  denote that vertices  $v$  and  $w$  are, respectively, adjacent or nonadjacent.

**Lemma 4.** *If  $a, b, c, d, e, f$  is a path (possibly a closed path with  $a = f$ ) in a ptolemaic graph with  $b \not\sim d$  and  $c \not\sim e$ , then  $b \not\sim e$ . If also  $a \neq f$ , then  $a \not\sim e$  and  $b \not\sim f$ .*

**Proof.** Inserting an edge  $be$  (or  $ae$  or  $bf$  if  $a \neq f$ ) would violate Lemma 3 by creating a cycle with too few chords. ■

**Theorem 5.** *In a ptolemaic graph, if  $i \in \{4, 6, 8\}$ , then every cycle  $C$  with  $|V(C)| \geq i$  has an  $(i/2)$ -chord.*

**Proof.** Suppose  $G$  is ptolemaic (and so is strongly chordal). By Theorem 2, every cycle  $C$  with  $|V(C)| \geq 4$  has a 2-chord and every  $C$  with  $|V(C)| \geq 6$  has a 3-chord. Suppose  $C = v_1, v_2, \dots, v_k, v_1$  is a  $k$ -cycle in  $G$ . Argue by induction on  $k \geq 8$  that  $C$  has a 4-chord.

For the basis step, suppose  $k = 8$ , but  $C$  has no 4-chord (arguing by contradiction); thus  $v_1 \not\sim v_5$ ,  $v_2 \not\sim v_6$ ,  $v_3 \not\sim v_7$ , and  $v_4 \not\sim v_8$ . Since  $G$  is strongly chordal,  $C$  has a 3-chord; without loss of generality, say  $v_3v_6$  is a chord. Lemma 4 on the path  $v_1, v_2, v_3, v_6, v_7, v_8$  implies  $v_1 \not\sim v_7 \not\sim v_2 \not\sim v_8$ . Therefore, the 6-cycle  $C^- = v_1, v_2, v_3, v_6, v_7, v_8, v_1$  has five nonadjacent pairs of vertices and so, by Lemma 3,  $C^-$  has all the other four possible chords  $v_1v_3$ ,  $v_1v_6$ ,  $v_3v_8$  and  $v_6v_8$ . Lemma 4 on the path  $v_4, v_5, v_6, v_1, v_2, v_3$  now implies  $v_3 \not\sim v_5$ . Similarly, the path  $v_5, v_4, v_3, v_8, v_7, v_6$  implies  $v_4 \not\sim v_6$ . But the chordless 4-cycle  $v_3, v_4, v_5, v_6, v_3$  would now contradict Lemma 3.

Therefore suppose  $k \geq 9$  and every  $k'$ -cycle with  $8 \leq k' < k$  has a 4-chord, but also suppose that  $C$  has no 4-chord (arguing by contradiction); thus  $v_1 \not\sim v_{k-3}$ ,  $v_2 \not\sim v_{k-2}$ ,  $v_3 \not\sim v_{k-1}$ ,  $v_4 \not\sim v_k$ ,  $v_1 \not\sim v_5$ ,  $v_2 \not\sim v_6$ , and  $v_3 \not\sim v_7$ . Since  $G$  is chordal,  $C$  has a 2-chord; without loss of generality, say  $v_1 v_{k-1}$  is a chord. Let  $C'$  be the cycle with edge set  $E(C) - \{v_{k-1} v_k, v_1 v_k\} \cup \{v_1 v_{k-1}\}$  and length  $k - 1 \geq 8$ . The inductive hypothesis implies that  $C'$  has a 4-chord (that is not a 4-chord of  $C$ ), and so  $k \geq 10$  and either  $v_1 \sim v_{k-4}$  or  $v_2 \sim v_{k-3}$  or  $v_3 \sim v_{k-2}$  or  $v_4 \sim v_{k-1}$ . Observe that  $v_2 \not\sim v_{k-3}$ ; otherwise Lemma 4 on the path  $v_k, v_1, v_2, v_{k-3}, v_{k-2}, v_{k-1}$  would contradict  $v_1 \sim v_{k-1}$ . Similarly,  $v_3 \not\sim v_{k-2}$ . Therefore, either  $v_1 \sim v_{k-4}$  or  $v_4 \sim v_{k-1}$ ; without loss of generality, suppose  $v_4 \sim v_{k-1}$ .

Lemma 4 on the path  $v_2, v_3, v_4, v_{k-1}, v_k, v_1$  implies  $v_1 \not\sim v_3 \not\sim v_k \not\sim v_2$ . Lemma 3 on the 5-cycle  $v_1, v_2, v_3, v_4, v_{k-1}, v_1$  with the two nonadjacent pairs  $\{v_1, v_3\}$  and  $\{v_3, v_{k-1}\}$  implies  $v_1 \sim v_4 \sim v_2 \sim v_{k-1}$ . Lemma 4 on the path  $v_{k-3}, v_{k-2}, v_{k-1}, v_2, v_3, v_4$  then implies  $v_4 \not\sim v_{k-2}$ , and also, now on the path  $v_{k-1}, v_k, v_1, v_4, v_5, v_6$ , implies  $v_{k-1} \not\sim v_5 \not\sim v_k \not\sim v_6$ . Finally, Lemma 4 on the path  $v_{k-3}, v_{k-2}, v_{k-1}, v_4, v_5, v_6$  implies  $v_5 \not\sim v_{k-2}$ .

Let  $C''$  be the cycle with edge set  $E(C) - \{v_1 v_2, v_2 v_3, v_3 v_4\} \cup \{v_1 v_4\}$  and length  $k - 2 \geq 8$ . The inductive hypothesis implies that  $C''$  has a 4-chord (that is not a 4-chord of  $C$ ), and so  $k \geq 11$  and either  $v_4 \sim v_{k-2}$  or  $v_5 \sim v_{k-1}$  or  $v_6 \sim v_k$  or  $v_1 \sim v_7$ . Since we have proved that  $v_4 \not\sim v_{k-2}$  and  $v_5 \not\sim v_{k-1}$  and  $v_6 \not\sim v_k$ , it follows that  $v_1 \sim v_7$ . Observe that  $v_1 \not\sim v_6$ , since otherwise Lemma 4 on the path  $v_4, v_5, v_6, v_1, v_2, v_3$  would contradict  $v_2 \sim v_4$ . Lemma 3 on the 5-cycle  $v_1, v_4, v_5, v_6, v_7, v_1$  with the two nonadjacent pairs  $\{v_1, v_5\}$  and  $\{v_1, v_6\}$  then implies  $v_5 \sim v_7 \sim v_4 \sim v_6$ . Note that  $v_2 \not\sim v_5$ , since otherwise the 5-cycle  $v_1, v_2, v_5, v_6, v_7, v_1$  with the three nonadjacent pairs  $\{v_1, v_5\}$ ,  $\{v_1, v_6\}$  and  $\{v_2, v_6\}$  would contradict Lemma 3. Furthermore, Lemma 3 on the 5-cycle  $v_1, v_2, v_4, v_5, v_7, v_1$  with the two nonadjacent pairs  $\{v_1, v_5\}$  and  $\{v_2, v_5\}$  implies  $v_2 \sim v_7$ . But then Lemma 4 on the path  $v_5, v_6, v_7, v_2, v_3, v_4$  would contradict  $v_4 \sim v_6$ . ■

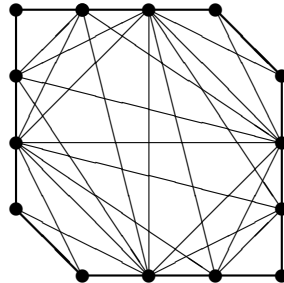


Figure 2. A ptolemaic graph spanned by a 14-cycle that has a 2-chord, a 3-chord, and a 4-chord (and a 6-chord and a 7-chord), but no 5-chord.

The gem graph shows that the converse of Theorem 5 fails. The graph in Figure 2 is ptolemaic—Corollary 6 of [1] is an easy way to verify this—and shows that cycles of ptolemaic graphs with  $|V(C)| \geq 10$  might not have 5-chords. (This is a minimum-order counterexample: cycles with  $10 \leq |V(C)| \leq 13$  in ptolemaic graphs turn out to always have 5-chords.)

### 3. CHORDS OF PATHS IN GRAPHS

Let  $C_n$  and  $P_n$  denote, respectively, a cycle and path on  $n$  vertices (so  $C_n$  has length  $n$  and  $P_n$  has length  $n - 1$ ). For any graph  $H$ , a graph  $G$  is  $H$ -free if  $G$  contains no induced subgraph isomorphic to  $H$ .

**Theorem 6.** *For every  $i \geq 3$ , a graph is both  $P_i$ -free and chordal if and only if every path  $P$  with  $|V(P)| \geq i$  has a 2-chord.*

**Proof.** First suppose  $G$  is  $P_i$ -free and chordal with a path  $P$  where  $|V(P)| \geq i \geq 3$ . Consider the minimum  $j$  such that  $P$  has a  $j$ -chord. If  $j \geq 3$ , then that  $j$ -chord would combine with  $P$  to form a chordless  $(j + 1)$ -cycle where  $j + 1 \geq 4$ , contradicting that  $G$  is chordal. Therefore,  $j = 2$ .

Conversely, suppose every path  $P$  of  $G$  with  $|V(P)| \geq i \geq 3$  has a 2-chord. Therefore,  $G$  contains no induced subgraph isomorphic to any such  $P_i$  or to any  $C_n$  with  $n \geq 4$ , and so  $G$  is  $P_i$ -free chordal. ■

The *trivially perfect graphs* have many names and characterizations [2, 9], one of which is that they are precisely the  $P_4$ -free chordal graphs. Corollary 7 is the path analog of Theorem 1.

**Corollary 7.** *A graph has complete components if and only if every path  $P$  with  $|V(P)| \geq 3$  has a 2-chord. A graph is trivially perfect if and only if every path  $P$  with  $|V(P)| \geq 4$  has a 2-chord.*

**Proof.** These are the  $i = 3$  and  $i = 4$  cases of Theorem 6. ■

The *threshold graphs* also have many characterizations [2, 7, 9], one of which is that they are precisely the  $2K_2$ -free trivially perfect graphs ( $2K_2$  is the complement of  $C_4$ ). For a connected graph, this is equivalent—see Theorem 1.2.4 of [7] for the history—to being constructible from a single vertex by recursively appending either an isolated vertex or a *dominating vertex* (often called a *universal vertex*, meaning a vertex adjacent to all the previously-existing vertices). Thus, a graph is a threshold graph if and only if every induced subgraph has either an isolated vertex or a dominating vertex. Theorem 8 is a path analog of Theorem 2.

**Theorem 8.** *The following are equivalent for all graphs  $G$ :*

(8.1)  *$G$  is a threshold graph.*

(8.2) *For all  $i \geq 4$ , every path  $P$  with  $|V(P)| \geq i$  has an  $(i - 2)$ -chord.*

(8.3) *For  $i \in \{4, 5\}$ , every path  $P$  with  $|V(P)| \geq i$  has an  $(i - 2)$ -chord.*

**Proof.** Suppose  $G$  is any connected graph.

(8.1)  $\Rightarrow$  (8.2): Suppose  $G$  is a threshold graph (and so is trivially perfect) and  $i \geq 4$ . Corollary 7 implies that every path  $P$  with  $|V(P)| \geq 4$  has a 2-chord. Therefore assume path  $P = v_1, \dots, v_p$  has  $p = |V(P)| \geq i > 4$  with subpath  $Q = v_1, \dots, v_i$ . Suppose  $P$  has no  $(i - 2)$ -chord (arguing by contradiction), and so  $v_1 \not\sim v_{i-1}$  and  $v_2 \not\sim v_i$ . But then the subgraph induced by  $\{v_1, v_2, v_{i-1}, v_i\}$  would be isomorphic to  $2K_2$  or  $P_4$  or  $C_4$  (contradicting that  $G$  is a threshold graph).

(8.2)  $\Rightarrow$  (8.3): This implication is immediate.

(8.3)  $\Rightarrow$  (8.1): Suppose every path with  $|V(P)| \geq 4$  has a 2-chord and every path with  $|V(P)| \geq 5$  has a 3-chord, yet  $G$  is not a threshold graph (arguing by contradiction). By Corollary 7,  $G$  is trivially perfect. Since  $G$  is not a threshold graph,  $G$  must contain two edges  $vw$  and  $v'w'$  in an induced  $2K_2$ . Suppose  $P$  is a minimum-length path that contains both  $vw$  and  $v'w'$ . But  $v \not\sim v' \not\sim w$  and  $v \not\sim w' \not\sim w$  imply  $|V(P)| \geq 5$ . Therefore,  $P$  would have a 3-chord (contradicting the minimality of  $P$ ). ■

#### 4. CHORDS OF CYCLES IN BIPARTITE GRAPHS

As in [2, 9], a graph  $G$  is *chordal bipartite* if  $G$  is bipartite and every cycle  $C$  with  $|V(C)| \geq 6$  has a chord (equivalently, every cycle long enough to have a chord does have a chord).

**Theorem 9.** *A graph is chordal bipartite if and only if every cycle  $C$  with  $|V(C)| \geq 6$  has a 3-chord.*

**Proof.** The ‘if direction’ is immediate. The ‘only if direction’ follows from the well-known result that every induced subgraph of a chordal bipartite graph has a *simplicial edge*, meaning an edge  $vw$  such that the union of the neighborhoods of  $v$  and  $w$  induce a complete subgraph [2, 9]. ■

A graph is *bipartite distance-hereditary* if it is both bipartite and distance-hereditary; see section 6 of [1]. This is equivalent to being chordal bipartite with no induced subgraph isomorphic to a *domino*—the graph obtained from a 6-cycle by inserting one 3-chord. Theorem 10 is a bipartite analog of Theorems 2 and 5 simultaneously.

**Theorem 10.** *In a bipartite distance-hereditary graph, if  $i \in \{6, 10\}$ , then every cycle  $C$  with  $|V(C)| \geq i$  has an  $(i/2)$ -chord.*

**Proof.** Suppose  $G$  is a bipartite distance-hereditary graph. By [1],  $G$  is chordal bipartite with no induced subgraph isomorphic to a domino. By Theorem 9, every cycle  $C$  with  $|V(C)| \geq 6$  has a 3-chord. Suppose  $C = v_1, v_2, \dots, v_k, v_1$  is a  $k$ -cycle in  $G$  ( $k$  is even, of course). Argue by induction on even  $k \geq 10$  that  $C$  has a 5-chord.

For the basis step, suppose  $k = 10$ , but  $C$  has no 5-chord (arguing by contradiction); thus  $v_1 \not\sim v_6$ ,  $v_2 \not\sim v_7$ ,  $v_3 \not\sim v_8$ ,  $v_4 \not\sim v_9$ , and  $v_5 \not\sim v_{10}$ . Since  $G$  is chordal bipartite,  $C$  has a 3-chord; without loss of generality, say  $v_3v_6$  is a chord. Thus  $v_1 \not\sim v_8$  (to avoid  $v_1, v_2, v_3, v_6, v_7, v_8, v_1$  being a 6-cycle with no chords),  $v_2 \not\sim v_9$  (to avoid  $v_2, v_3, v_6, v_7, v_8, v_9, v_2$  either being a 6-cycle with no chords or spanning an induced domino), and  $v_7 \not\sim v_{10}$  (similarly). Therefore, the 8-cycle  $v_1, v_2, v_3, v_6, v_7, v_8, v_9, v_{10}, v_1$  must have both of the only possible 3-chords  $v_3v_{10}$  and  $v_6v_9$  (both of them, to avoid inducing a domino). But then  $v_1, v_2, v_3, v_6, v_9, v_{10}, v_1$  would span an induced domino, a contradiction.

Therefore suppose  $k \geq 12$  and every  $k'$ -cycle with  $10 \leq k' < k$  has a 5-chord, and again suppose that  $C$  has no 5-chord arguing by contradiction. By Theorem 9,  $C$  has a 3-chord; without loss of generality, say  $v_3v_6$  is a chord. Let  $C'$  be the cycle with edge set  $E(C) - \{v_3v_4, v_4v_5, v_5v_6\} \cup \{v_3v_6\}$  and length  $k - 2 \geq 10$ . The inductive hypothesis implies that  $C'$  has a 5-chord (that is not a 5-chord of  $C$ ), and so either  $v_1 \sim v_8$  or  $v_2 \sim v_9$  or  $v_3 \sim v_{10}$  or  $v_6 \sim v_{k-1}$  or  $v_7 \sim v_k$ . Thus  $k \geq 14$  (since those five edges would be 5-chords of  $C$  if  $k = 12$ ). By the same argument used in the basis step,  $v_1 \not\sim v_8$  and  $v_2 \not\sim v_9$  and  $v_7 \not\sim v_k$ . Therefore either  $v_3 \sim v_{10}$  or  $v_6 \sim v_{k-1}$ ; without loss of generality, suppose  $v_3 \sim v_{10}$ . Since  $v_3 \not\sim v_8$  (because  $C$  has no 5-chords) and  $v_3, v_6, v_7, v_8, v_9, v_{10}, v_3$  must have a 3-chord without spanning an induced domino, it follows that  $v_6 \sim v_9$  and  $v_7 \sim v_{10}$ . But then the cycle  $v_3, v_4, v_5, v_6, v_9, v_{10}, v_3$  would span an induced domino [a contradiction]. ■

The domino graph shows that the converse of Theorem 10 fails. The graph in Figure 3 is bipartite distance-hereditary—Corollary 3 of [1] is an easy way to verify this—and shows that cycles of bipartite distance-hereditary graphs with  $|V(C)| \geq 14$  might not have 7-chords.

## 5. CHORDS OF PATHS IN BIPARTITE GRAPHS

Theorem 11 is the bipartite analog of Theorem 6.

**Theorem 11.** *For every  $i \geq 4$ , a bipartite graph is both  $P_i$ -free and chordal bipartite if and only if every path  $P$  with  $|V(P)| \geq i$  has a 3-chord.*

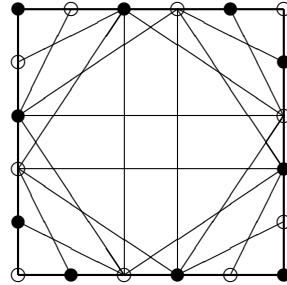


Figure 3. A bipartite distance-hereditary graph spanned by a 20-cycle that has a 3-chord and a 5-chord (and a 9-chord), but no 7-chord.

**Proof.** This is proved in the same way as Theorem 6 (with  $i \geq 4$  and  $j \geq 4$  in the first paragraph and with  $n \geq 5$  in the second). ■

The bipartite graphs that are  $P_5$ -free chordal bipartite graphs have been characterized—see Corollary 3.2 of [3] or Theorem 4 of [6]—by every connected induced subgraph of  $G$  having either a dominating vertex or a *dominating edge* (meaning an edge  $vw$  such that every vertex is adjacent to  $v$  or  $w$ ). Thus, a bipartite graph is  $P_5$ -free chordal bipartite if and only if every induced subgraph has an isolated vertex or a dominating edge.

**Corollary 12.** *Every connected bipartite graph is complete bipartite if and only if every path  $P$  with  $|V(P)| \geq 4$  has a 3-chord. Every connected induced subgraph of a bipartite graph has a dominating vertex or edge if and only if every path  $P$  with  $|V(P)| \geq 5$  has a 3-chord.*

**Proof.** These are the  $i = 4$  and  $i = 5$  cases of Theorem 11. ■

Comparing Corollary 12 to Corollary 7, reference [10] shows that a graph is trivially perfect if and only if every connected induced subgraph of  $G$  has a dominating vertex.

The *difference graphs*—these are close relatives of threshold graphs and are also called *chain graphs*, see [7, 9]—are the  $P_5$ -free bipartite graphs. The *chordal bipartite difference graphs* are the  $2K_2$ -free (and so  $P_5$ -free) chordal bipartite graphs. Theorem 13 is a bipartite analog of Theorem 8. When comparing it to Corollary 12, note that the second condition in the second part of Corollary 12 could be rephrased as for  $i \in \{5\}$ , every path  $P$  with  $|V(P)| \geq i$  has an  $(i - 2)$ -chord.

**Theorem 13.** *The following are equivalent for all bipartite graphs  $G$ :*

(13.1)  *$G$  is a chordal bipartite difference graph.*

(13.2) *For all odd  $i \geq 5$ , every path  $P$  with  $|V(P)| \geq i$  has an  $(i - 2)$ -chord.*



(13.3) For  $i \in \{5, 7\}$ , every path  $P$  with  $|V(P)| \geq i$  has an  $(i - 2)$ -chord.

**Proof.** This is proved in the same way as Theorem 8 (for (13.1)  $\Rightarrow$  (13.2), the subgraph induced by  $\{v_1, v_2, v_{i-1}, v_i\}$  would be isomorphic to  $2K_2$ ). ■

## REFERENCES

- [1] H.-J. Bandelt and H.M. Mulder, *Distance-hereditary graphs*, J. Combin. Theory (B) **41** (1986) 182–208.  
doi:10.1016/0095-8956(86)90043-2
- [2] A. Brandstädt, V.B. Le and J.P. Spinrad, *Graph Classes: A Survey* (Society for Industrial and Applied Mathematics, Philadelphia, 1999).  
doi:10.1137/1.9780898719796
- [3] M.B. Cozzens and L.L. Kelleher, *Dominating cliques in graphs*, Discrete Math. **86** (1990) 101–116.  
doi:10.1016/0012-365X(90)90353-J
- [4] M. Farber, *Characterizations of strongly chordal graphs*, Discrete Math. **43** (1983) 173–189.  
doi:10.1016/0012-365X(83)90154-1
- [5] E. Howorka, *A characterization of ptolemaic graphs*, J. Graph Theory **5** (1981) 323–331.  
doi:10.1002/jgt.3190050314
- [6] J. Liu and H.S. Zhou, *Dominating subgraphs in graphs with some forbidden structures*, Discrete Math. **135** (1994) 163–168.  
doi:10.1016/0012-365X(93)E0111-G
- [7] N.V.R. Mahadev and U.N. Peled, *Threshold Graphs and Related Topics* (North-Holland, Amsterdam, 1995).
- [8] A. McKee, *Constrained chords in strongly chordal and distance-hereditary graphs*, Utilitas Math. **87** (2012) 3–12.
- [9] T.A. McKee and F.R. McMorris, *Topics in Intersection Graph Theory* (Society for Industrial and Applied Mathematics, Philadelphia, 1999).  
doi:10.1137/1.9780898719802
- [10] E.S. Wolk, *The comparability graph of a tree*, Proc. Amer. Math. Soc. **13** (1962) 789–795.  
doi:10.1090/S0002-9939-1962-0172273-0

Received 29 July 2011

Revised 4 November 2011

Accepted 4 November 2011

