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GENERALIZED MATRIX GRAPHS AND COMPLETELY INDEPENDENT CRITICAL CLIQUES IN ANY DIMENSION

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Abstract

For natural numbers k and n, where $2 \leq k \leq n$, the vertices of a graph are labeled using the elements of the k-fold Cartesian product $I_n \times I_n \times \cdots \times I_n$. Two particular graph constructions will be given and the graphs so constructed are called generalized matrix graphs. Properties of generalized matrix graphs are determined and their application to completely independent critical cliques is investigated. It is shown that there exists a vertex critical graph which admits a family of k completely independent critical cliques for any k, where $k \geq 2$. Some attention is given to this application and its relationship with the double-critical conjecture that the only vertex double-critical graph is the complete graph.

Keywords: matrix graph, chromatic number, critical clique, completely independent critical cliques, double-critical conjecture.

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1. Introduction and Notation

In the late 1940's, G.A. Dirac defined critical graphs for the purpose of simplifying the central problems in the theory of graph coloring. Several results on critical graphs containing few edges have been established; e.g., [4], [5], [10], and [11]. Likewise, several results on critical graphs containing many edges have been established; e.g., [15] and [7]. However, determining bounds on the number of edges in critical graphs is not the objective of this paper. Rather, relations between particular sets of critical vertices will be investigated. This paper is a continuation and generalization of the results obtained in [12].

Most of the notation and terminology follows that found in [2] and [3]. The graphs considered in this paper are finite, undirected, and simple. For a given graph G, the vertex set of G and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. The order of G is the cardinality of $V(G)$ and is denoted by $|V(G)|$. The complete graph having order r shall be denoted by K_r. An r-clique of G is a subgraph K of G isomorphic to K_r. A subset I of $V(G)$ is said to be independent whenever no two distinct vertices in I are adjacent. The maximum cardinality of an independent subset of $V(G)$ is denoted by $\alpha(G)$. For a subset X of $V(G)$, the subgraph of G induced by X is denoted by $G[X]$. All vertex colorings considered will be proper; i.e., a partition of $V(G)$ into independent subsets of $V(G)$ called color classes. The minimum cardinality of a partition of $V(G)$ admitted by a proper vertex coloring of G is called the chromatic number of G and is denoted by $\chi(G)$. Lastly, the graph G is k-chromatic whenever $\chi(G) = k$.

The set $\mathbb{N} = \{1, 2, 3, \ldots\}$ will denote the set of natural numbers. For $n \in \mathbb{N}$, the set S_n is the set of all permutations on a set of n letters. The general element $\sigma \in S_n$ will be written as $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$. When convenient, σ may also be written as a formal string, i.e., $\sigma = \sigma_1\sigma_2\cdots\sigma_n$. If $r, n \in \mathbb{N}$, with $1 \leq r \leq n$, then by an r-permutation, we mean any permutation on a set of r out of n letters. An n-permutation is usually referred to simply as a permutation. Let $\mathcal{S} = \{S_\omega : \omega \in \Omega\}$ be an indexed family of sets. The generalized union of this indexed family of sets, denoted by $\cup\mathcal{S}$, is given by $\cup\mathcal{S} = \bigcup_{\omega\in\Omega} S_\omega$.

For $n \in \mathbb{N}$, an arbitrary Latin square of order n will be denoted by L_n. The ith row of a general Latin square L_n, where $1 \leq i \leq n$, will be denoted by $\Lambda_i = \lambda_{i1}\lambda_{i2}\cdots\lambda_{in}$. The Kronecker delta function, δ_{ij}, is given by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j\,. \end{cases}$$

Lastly, let $M = \{b_1 \cdot x_1, b_2 \cdot x_2, \ldots, b_m \cdot x_m\}$ be a (finite) multiset having m distinct elements $x_1, x_2, \ldots, x_m$. Here, the natural numbers $b_1, b_2, \ldots, b_m$ are called the repetition numbers and denote the number of times the corresponding element appears in the multiset M. Hence, it is clear that $|M| = \sum_{i=1}^m b_i$. A submultiset of M is a multiset $T = \{a_1 \cdot x_1, a_2 \cdot x_2, \ldots, a_m \cdot x_m\}$ satisfying $0 \leq a_i \leq b_i$ for all i with $1 \leq i \leq m$. An r-submultiset (or r-combination) is an unordered selection of r of the objects of M and satisfies $|T| = \sum_{i=1}^m a_i = r$.

2. GENERALIZED MATRIX GRAPHS

The concept of a matrix graph arose out of the idea of placing the vertices of a graph into a rectangular array and by defining the adjacencies in terms of submatrices. Two similar families of graphs resulted from this notion. For lack of better terminology, these graphs will be called either a Type I or Type II matrix graph. Although we give both constructions, particular emphasis shall be placed on Type I matrix graphs.

2.1. Indices of generalized k-dimensional n-square matrices

For $n \in \mathbb{N}$, let I_n represent the nth segment of the natural numbers, that is, $I_n = \{1, 2, \ldots, n\}$. For $m, n \in \mathbb{N}$, the general element of a standard $m \times n$ matrix A is typically denoted by a_{ij}, where $i, j \in \mathbb{N}$ satisfy $1 \leq i \leq m$ and $1 \leq j \leq n$. Observe that the indices of an $m \times n$ matrix can be viewed as elements of the Cartesian product $I_m \times I_n$. Hence, the indices of a square $n \times n$ matrix can be viewed as elements of $I_n \times I_n$. Such a matrix could be called a 2-dimensional n-square matrix. More generally, for $k, n \in \mathbb{N}$, the indices of a k-dimensional n-square matrix can be viewed as the elements of the k-fold Cartesian product of I_n: $I_n^k = \prod_{i=1}^k I_n = I_n \times I_n \times \cdots \times I_n$.

The graphs constructed below will be called generalized matrix graphs, either Type I or Type II, because the vertices have been labeled using the elements of I_n^k; i.e., the indices of a generalized k-dimensional n-square matrix. It is helpful to view these graphs as "square"arrays, at least in two and three dimensions.

The general element of I_n^k will be denoted by $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. When convenient, α may also be viewed as a function, $\alpha : I_k \longrightarrow I_n$, defined by the rule $\alpha(i) = \alpha_i$, and written as the formal string $\alpha = \alpha_1\alpha_2 \cdots \alpha_k$. Moreover, we adopt the conventions that $I_n^0 = \{1\}$ and $I_n^1 = I_n$. For $T \subseteq \mathbb{N}$ and $c \in \mathbb{N}$, the scalar product cT is defined by $cT = \{ct : t \in T\}$. Certain subsets of I_n^k occur often enough to warrant special consideration. For $i \in I_k$ and $j \in I_n$ the set F_{ij} is given by

$$F_{ij} = \left\{\alpha \in I_n^k : \alpha_i = j\right\} = \prod_{t=1}^k j^{\delta_{it}} I_n^{1-\delta_{it}}.$$

Additionally, for $\alpha \in I_n^k$ and $i \in I_k$, we set $F_i(\alpha) = \left\{\beta \in I_n^k : \alpha_i = \beta_i\right\}$ and $\mathcal{F}_\alpha = \{F_i(\alpha) : i \in I_k\}$. It follows immediately that $F_i(\alpha) = F_{i,\alpha_i}$. Moreover, $F_i(\alpha) = F_i(\beta)$ if and only if $\alpha_i = \beta_i$. In a similar fashion, for $\alpha \in I_n^k$ and $i \in I_k$ the set $\eta_{F_i}^{\perp}(\alpha)$ is given by

$$\eta_{F_i}^{\perp}(\alpha) = \left\{\beta \in I_n^k : \beta_j = \alpha_j \text{ for all } j \neq i\right\}.$$

Lastly, for $\alpha \in I_n^k$, we set $\eta^{\perp}(\alpha) = \left\{\eta_{F_i}^{\perp}(\alpha) : i \in I_k\right\}$. A relation exists between F_{ij} and $\eta_{F_i}^{\perp}(\alpha)$, namely,

$$\eta_{F_i}^{\perp}(\alpha) = \bigcap_{t \in I_k \setminus \{i\}} F_t(\alpha).$$

2.2. Constructions of generalized matrix graphs

Here, two matrix graph constructions are given. These graphs are not symmetric in the literal sense of the word but they do possess a type of symmetry by construction. This symmetric nature of matrix graphs is exploited and, in fact, essential to the proof of Theorem 5 in Section 3.2.

Type I Matrix Graph. Let k, $n \in \mathbb{N}$ with $2 \leq k \leq n$. Construct a Type I matrix graph, denoted by $G_1^{k,\,n}$, by setting $V\left(G_1^{k,\,n}\right) = I_n^k$ and by defining the edge set of $G_1^{k,\,n}$ according to the following rule: For α, $\beta \in I_n^k$,

$$\alpha\beta \in E\left(G_1^{k,\,n}\right) \text{ if and only if } \beta \notin \cup\mathcal{F}_\alpha. \tag{1}$$

Type II Matrix Graph. Let k, $n \in \mathbb{N}$ with $2 \leq k \leq n$. Construct a Type II matrix graph, denoted by $G_2^{k,\,n}$, by setting $V\left(G_2^{k,\,n}\right) = I_n^k$ and by defining the edge set of $G_2^{k,\,n}$ according to the following rule: For α, $\beta \in I_n^k$,

$$\alpha\beta \in E\left(G_2^{k,\,n}\right) \text{ if and only if } \beta \notin \cup\eta^\perp(\alpha).$$

Although the construction of matrix graphs is somewhat elementary, one of these constructions, namely a Type I matrix graph, will give rise to a more complex construction of a family of graphs, the members of which contain a set of cliques satisfying a certain property to be defined below. In what follows, various aspects of matrix graphs are investigated.

2.3. Subgraphs of generalized matrix graphs

Let k, $n \in \mathbb{N}$, where $k \geq 2$ and $n \geq k+1$. In the Type I matrix graph $G_1^{k+1,\,n}$, we would first like to establish the existence of a subset X of I_n^{k+1} such that the induced subgraph $G_1^{k+1,\,n}[X]$ is isomorphic to the Type I matrix $G_1^{k,\,n}$; i.e., $G_1^{k+1,\,n}[X] \cong G_1^{k,\,n}$. It will then be demonstrated that I_n^k can be partitioned into n subsets in such a way that each element of the partition determines an induced subgraph of $G_1^{k+1,\,n}$ that is isomorphic to $G_1^{k,\,n}$. These facts are established by the next two results.

Lemma 1. *Let k, $n \in \mathbb{N}$, where $k \geq 2$ and $n \geq k+1$. The graph $G_1^{k+1,\,n}$ contains an induced subgraph isomorphic to $G_1^{k,\,n}$.*

Proof. Let k, $n \in \mathbb{N}$, where $k \geq 2$ and $n \geq k+1$. Consider the Type I matrix graph $G_1^{k+1,\,n}$. Select an arbitrary permutation of I_n and an arbitrary 2-permutation of I_k, say $\lambda = \lambda_1\lambda_2\cdots\lambda_n$ and $\rho = \rho_1\rho_2$, respectively. The set

$$T(\lambda,\,\rho) = \left\{\prod_{i=1}^{k+1} j^{\delta_{i\rho_1}}\lambda_j^{\delta_{i\rho_2}} I_n^{1-\delta_{i\rho_1}-\delta_{i\rho_2}} : j \in I_n\right\}$$

consists of n subsets of I_n^{k+1} each one of which corresponds to a set $F_{\rho_1}(\alpha) \cap F_{\rho_2}(\alpha)$ for any α satisfying $\alpha_{\rho_1} = j$ and $\alpha_{\rho_2} = \lambda_j$. Since two out of the $k+1$ coordinate positions, namely ρ_1 and ρ_2, of elements in each member of $T(\lambda, \rho)$ are fixed, it is clear that each member of $T(\lambda, \rho)$ is a nonempty subset of I_n^{k+1} having cardinality n^{k-1}. Define $X = \cup T(\lambda, \rho)$. Then $|X| = n \cdot n^{k-1} = n^k$; and, moreover, $n \geq k+1 > k$. Now, observe that for all $\alpha, \beta \in X$, we have $\beta \notin (\cup \mathcal{F}_\alpha) \cap X$ if and only if $\beta \notin \cup \mathcal{F}_\alpha$. Hence, it follows that $G_1^{k+1,n}[X] \cong G_1^{k,n}$. ∎

Proposition 1. Let $k, n \in \mathbb{N}$, where $k \geq 2$ and $n \geq k+1$. For the matrix graph $G_1^{k+1,n}$, there exists a partition $\mathcal{P}$ of I_n^{k+1} such that $G_1^{k+1,n}[P] \cong G_1^{k,n}$ for each $P \in \mathcal{P}$.

Proof. By Lemma 1, there exists of a subset of I_n^{k+1} which admits an induced subgraph of $G_1^{k+1,n}$ that is isomorphic to $G_1^{k,n}$. It is now demonstrated that I_n^k can be partitioned in such a way that each element of the partition admits an induced subgraph of $G_1^{k+1,n}$ that is isomorphic to $G_1^{k,n}$. Select an arbitrary Latin square having order n, say

$$L_n = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{nn} \end{bmatrix} = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_n \end{bmatrix},$$

where $\Lambda_i = (\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in})$ is the ith row of the Latin square L_n. For a fixed 2-permutation of I_k, say $\rho = \rho_1\rho_2$, define

$$T(L_n, \rho) = \bigcup_{q=1}^{n} \{\cup T(\Lambda_q, \rho)\},$$

where, as in the proof of Lemma 1 above, the set $T(\Lambda_q, \rho)$ is defined as

$$T(\Lambda_q, \rho) = \left\{ \prod_{i=1}^{k+1} j^{\delta_{i\rho_1}} \lambda_{qj}^{\delta_{i\rho_2}} I_n^{1-\delta_{i\rho_1}-\delta_{i\rho_2}} : j \in I_n \right\}.$$

To see that the set $T(L_n, \rho)$ is a pairwise disjoint collection of subsets of I_n^{k+1}, suppose to the contrary that $\alpha \in [\cup T(\Lambda_{q_1}, \rho)] \cap [\cup T(\Lambda_{q_2}, \rho)]$, where $q_1, q_2 \in I_n$ with $q_1 \neq q_2$. By the definition of $T(\Lambda_q, \rho)$, the $(k+1)$-tuple α would be expressible in two ways as

$$\alpha = j_1^{\delta_{i\rho_1}} \lambda_{q_1 j_1}^{\delta_{i\rho_2}} I_n^{1-\delta_{i\rho_1}-\delta_{i\rho_2}}$$

and

$$\alpha = j_2^{\delta_{i\rho_1}} \lambda_{q_2 j_2}^{\delta_{i\rho_2}} I_n^{1-\delta_{i\rho_1}-\delta_{i\rho_2}},$$

for some $j_1, j_2 \in I_n$. Necessarily, $j_1 = j_2$ and $\lambda_{q_1 j_1} = \lambda_{q_2 j_2}$. But since L_n is a Latin square, it would have to be that $q_1 = q_2$ contrary to the assumption that $q_1 \neq q_2$. Indeed, $T(L_n, \rho)$ is a pairwise disjoint collection of nonempty subsets of I_n^{k+1}. Moreover, $|\cup T(L_n, \rho)| = n \cdot n^k = n^{k+1}$ so that $\cup T(L_n, \rho) = I_n^{k+1}$. Finally, for $q \in I_n$, define $P_q = \cup T(\Lambda_q, \rho)$. By Lemma 1, each set P_q satisfies $G_1^{k+1,n}[P_q] \cong G_1^{k,n}$. It follows that the collection $\mathcal{P} = \{P_q : q \in I_n\}$ is such a partition. ∎

Proposition 2. Let $X = \{\alpha^1, \alpha^2, \ldots, \alpha^t\}$ be a set of t vertices in $G_1^{k,n}$, where $t \in \mathbb{N}$ and $1 \leq t \leq n$. For $i = 1, 2, \ldots, k$, define $Z_i = \left\{\alpha_i^j : j \in I_t\right\}$. If $|Z_i| = t$ for all $i \in I_k$, then $G_1^{k,n}[X] \cong K_t$.

Proof. The condition $|Z_i| = t$ for all $i \in I_k$ implies that there do not exist indices $j_1, j_2 \in I_t$ and an index $i_0 \in I_k$ for which $\alpha_{i_0}^{j_1} = \alpha_{i_0}^{j_2}$. Hence, $\alpha^{j_1} \notin \cup\mathcal{F}_{\alpha^{j_2}}$. Therefore, by (1), $\alpha^{j_1}\alpha^{j_2} \in E\left(G_1^{k,n}\right)$ and it follows that $G_1^{k,n}[X] \cong K_t$. ■

An immediate consequence of Proposition 2 is given in the next corollary. The main diagonal of I_n^k, denoted by Δ, is the set

$$\Delta = \left\{\alpha \in I_n^k : \alpha = c\,(1, 1, \ldots, 1) \text{ for some } c \in I_n\right\}.$$

Corollary 1. The induced subgraph $G_1^{k,n}[\Delta]$ is isomorphic to K_n.

2.4. The chromatic number of generalized matrix graphs

In this section, we determine the chromatic number of a Type I matrix graph. Similar results can be obtained for Type II matrix graphs. First, certain independent subsets of I_n^k in a Type I matrix graph are identified.

Proposition 3. Let $k, n \in \mathbb{N}$ with $2 \leq k \leq n$. Also, let $i \in I_k$ and $\alpha \in I_n^k$. In a Type I matrix graph, the set $F_i(\alpha)$ is an independent subset of I_n^k.

Proof. Let $k, n \in \mathbb{N}$ with $2 \leq k \leq n$. Also, let $i \in I_k$ and $\alpha \in I_n^k$. Consider the set $F_i(\alpha)$ in the Type I matrix graph $G_1^{k,n}$. If $\beta \in I_n^k$ and $\beta \in F_i(\alpha)$, then $\beta \in \cup\mathcal{F}_\alpha$. Therefore, $\alpha\beta \notin E\left(G_1^{k,n}\right)$, which follows immediately from the definition of the edge set of $G_1^{k,n}$ in (1). Hence, the set $F_i(\alpha)$ is an independent subset of I_n^k. ■

Corollary 2. Let $k, n \in \mathbb{N}$ with $2 \leq k \leq n$. Also, let $i \in I_k$ and $\alpha \in I_n^k$. In a Type I matrix graph, the set $\eta_{F_i}^{\perp}(\alpha)$ is an independent subset of I_n^k.

Lemma 2. *For $n \geq 2$, the independence number of $G_1^{2,n}$ is equal to n.*

Proof. Let $n \geq 2$ and consider an arbitrary independent subset J of I_n^2. Because sets that contain a single vertex are independent and since $n \geq 2$, it can be assumed that $|J| \geq 2$. By Proposition 3, the set $F_i(\alpha)$ is an independent subset of I_n^2. Since $k = 2$, we see that $|F_i(\alpha)| = n$. Thus, it is clear that $\alpha\left(G_1^{2,n}\right) \geq n$. Moreover, it now suffices to prove that $J \subseteq F_i(\alpha)$ for some $F_i(\alpha)$. To this end, suppose that α and β are distinct vertices in J, say $\alpha = (i_1, j_1)$ and $\beta = (i_2, j_2)$. Observe that either $i_1 = i_2$ or $j_1 = j_2$, but not both. This is because, otherwise, either $\alpha\beta \in E\left(G_1^{2,n}\right)$ or $\alpha = \beta$. Hence, without loss of generality, we may assume that $i_1 = i_2 = i$ so that $\alpha = (i, j_1)$ and $\beta = (i, j_2)$, where $j_1 \neq j_2$. Now consider an arbitrary $\gamma = (k, j) \in J$ and suppose to the contrary that $k \neq i$. Now, either $j \neq j_1$ or $j \neq j_2$. Consequently, either $\alpha\gamma \in E\left(G_1^{2,n}\right)$ or $\beta\gamma \in E\left(G_1^{2,n}\right)$, which contradicts that fact that $\gamma \in J$. It follows that $J \subseteq F_i(\alpha)$. Therefore, $\alpha\left(G_1^{2,n}\right) = n$. ■

Proposition 4. Let $k, n \in \mathbb{N}$ with $2 \leq k \leq n$. The independence number of $G_1^{k,n}$ is equal to n^{k-1}.

Proof. The proof proceeds by induction on k. By Lemma 2, the result holds for $k = 2$. Inductively assume the result holds for $k = r$, where $r \geq 2$, that is to say, $\alpha\left(G_1^{r,n}\right) = n^{r-1}$. It is demonstrated that the result holds for $k = r + 1$. Consider the graph $G_1^{r+1,n}$ and an arbitrary independent subset J of I_n^{r+1}. It must be shown that $|J| \leq n^r$. By Proposition 1, there exists a partition $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ of I_n^{r+1} such that $G_1^{r+1,n}[P_i] \cong G_1^{r,n}$ for each $i \in I_n$. Observe that $J = \bigcup_{i=1}^n (J \cap P_i)$. By the inductive hypothesis, $\alpha\left(G_1^{r,n}\right) = n^{r-1}$ so that $|J \cap P_i| \leq n^{r-1}$ for each $i \in I_n$. Therefore,

$$|J| = \textstyle\sum_{i=1}^n |J \cap P_i| \leq \sum_{i=1}^n n^{r-1} = n^r.$$

As the sets $F_i(\alpha)$ of I_n^{r+1} are independent subsets having n^r elements, it follows that $\alpha\left(G_1^{r+1,n}\right) = n^r$. ■

Corollary 3. The sets $F_i(\alpha)$ of a Type I matrix graph are independent subsets of maximum cardinality.

We are now in the position to determine the chromatic number of the Type I matrix graph $G_1^{k,n}$.

Theorem 3. *Let $k, n \in \mathbb{N}$ with $2 \leq k \leq n$. The graph $G_1^{k,n}$ is n-chromatic.*

Proof. By Corollary 1, $\chi\left(G_1^{k,n}\right) \geq n$. Now, the sets F_{ij} are independent subsets of I_n^k. Therefore, an n-coloring of $G_1^{k,n}$ can be exhibited by coloring each face $F_{i_0 j}$ with color c_j for $j = 1, 2, \ldots, n$, where $i_0 \in I_k$ is an arbitrary but fixed coordinate position. Hence, $G_1^{k,n}$ is n-chromatic. ■

Remark 1. Theorem 3 also could have been obtained as a consequence of Proposition 4. Indeed,

$$\chi\left(G_1^{k,n}\right) \geq \frac{\left|V\left(G_1^{k,n}\right)\right|}{\alpha\left(G_1^{k,n}\right)} = \frac{n^k}{n^{k-1}} = n$$

and by coloring the faces as in the proof of Theorem 3, the desired result is obtained.

3. On a Generalization of Completely Independent Critical Cliques

The results of this section provide a complete generalization of the results obtained in [12]. Because the notation and terminology are not standard, two definitions will be given in Section 3.1 below.

3.1. Preliminary terminology

Recall that a vertex $v \in V(G)$ is a critical vertex of G provided that the chromatic number of G decreases upon the removal of v. In fact, the chromatic number of G decreases by exactly one whenever v is a critical vertex; i.e., $\chi(G-v) = \chi(G) - 1$. A vertex critical graph is a graph in which every vertex is critical. Also, a vertex k-critical graph is a k-chromatic vertex critical graph. A graph G is called vertex double-critical provided that $\chi(G-u-v) = \chi(G) - 2$ for every pair u, v of adjacent vertices.

Clearly, the induced subgraph $G[\{v\}]$ satisfies $G[\{v\}] \cong K_1$. When v is a critical vertex, there is a natural generalization of this concept.

Definition 1. Let K be an r-clique of G. Then K is a critical r-clique of G, written K_r^c, provided that $\chi(G-K) = \chi(G) - r$.

First note that in general, $\chi(G-K) \geq \chi(G) - r$ for an arbitrary r-clique. Thus equality in Definition 1 imposes additional structure in G. It is straightforward to prove that any subgraph K of order r satisfying the equation $\chi(G-K) = \chi(G) - r$ is necessarily an r-clique. Recall from Section 1 that a set U of vertices is independent provided that no two vertices in U are adjacent. Equivalently, U is an independent subset of $V(G)$ whenever the induced subgraph $G[U]$ is isomorphic to an empty graph. The next definition can be thought of as a generalization of an independent set containing two critical vertices, provided that each vertex in the independent set is viewed as an induced subgraph of G isomorphic to K_1.

Definition 2. Let $K_r^{c(1)}$ and $K_s^{c(2)}$ be two critical cliques having orders r and s, respectively. Then $K_r^{c(1)}$ and $K_s^{c(2)}$ are completely independent provided $N(v) \cap V\left(K_s^{c(2)}\right) = \emptyset$ for every vertex $v \in V\left(K_r^{c(1)}\right)$.

In the Definition 2, $K_r^{c(1)}$ and $K_s^{c(2)}$ are completely independent provided that $\chi\left(G - K_r^{c(1)}\right) = \chi(G) - r$ and $\chi\left(G - K_s^{c(2)}\right) = \chi(G) - s$, and, moreover, no vertex of $K_r^{c(1)}$ is adjacent to a vertex of $K_s^{c(2)}$. The motivation for this definition arises out of its connection with a conjecture of Lovász in [6] that the only vertex double-critical graph is the complete graph. The double-critical conjecture has been proven in the affirmative by Stiebitz in [14] only in the case of a 5-chromatic double-critical graph. A more general statement that includes the conjecture of Lovász as a special case is the Erdős-Lovász Tihany conjecture. This more general conjecture and a brief history of some of the known results can be found in [8]. Related results for quasi-line graphs are given in [1]. The edge analogue of this conjecture has been resolved in the affirmative and can be found in [9] and [13].

It seems reasonable that for a single critical clique K_r^c, there would be many edges from vertices in $V(K_r^c)$ to vertices in $V(G - K_r^c)$. Thus, it might seem just as reasonable that for a family of critical cliques, there would be many edges from the vertices in one critical clique to vertices in the other critical cliques. The main result of this paper declares this not to be the case! The indexed family $\mathcal{K} = \left\{K_{r_\alpha}^{c(\alpha)} : \alpha \in \Omega\right\}$ of critical r_α-cliques is said to be a family of completely independent critical cliques provided that the elements of $\mathcal{K}$ are pairwise completely independent critical cliques. Note that a family of completely independent critical cliques is a generalization of an independent set of critical vertices. In [12], the existence of a completely independent family $\mathcal{K}$ for the case $|\mathcal{K}| = 2$ was addressed and it was demonstrated that there exists a vertex k-critical graph admitting two completely independent critical cliques having orders r and s for any r and s, with r, $s \geq 1$. This result was a 2-dimensional version of our current result. A k-dimensional generalization is now provided by establishing the existence of an infinite family of vertex critical graphs each admitting k completely independent critical cliques. This will confirm the existence of a completely independent family $\mathcal{K}$ for $|\mathcal{K}| = k$, where $k \in \mathbb{N}$ with $k \geq 2$.

3.2. The main construction

If $\tau \in S_n$ with $\tau = \tau_1\tau_2\cdots\tau_n$, then $\widehat{\tau}(i)$ denotes the formal substring of τ determined by deleting the ith character of τ. More precisely,

$$\widehat{\tau}(i) = \begin{cases} \tau_2\tau_3\cdots\tau_n & \text{if } i = 1, \\ \tau_1\cdots\tau_{i-1}\tau_{i+1}\cdots\tau_n & \text{if } 1 < i < n, \\ \tau_1\tau_2\cdots\tau_{n-1} & \text{if } i = n\,. \end{cases}$$

The jth character of the formal string $\widehat{\tau}(i)$, for $1 \leq j \leq n-1$, will be denoted by $\widehat{\tau}_j(i)$ and is given by

$$\widehat{\tau}_j(i) = \begin{cases} \tau_j & \text{if } j < i, \\ \tau_{j+1} & \text{if } j \geq i\,. \end{cases}$$

To begin the main construction, suppose that k, $n \in \mathbb{N}$ with $2 \leq k \leq n$ and consider the Type I matrix graph $G_1^{k,n}$. In what follows, the graph $G_1^{k,n}$ will be referred to as the planet. Adjoin to the planet $G_1^{k,n}$ exactly k complete graphs of order $n-k+1$, say $K^{j}_{n-k+1}(s_j)$, where

$$V\left(K^{j}_{n-k+1}(s_j)\right)=\left\{x^{j,1}_{\widehat{\sigma}_1(s_j)},\, x^{j,2}_{\widehat{\sigma}_2(s_j)},\, \ldots,\, x^{j,\,n-k+1}_{\widehat{\sigma}_{n-k+1}(s_j)}\right\},$$

for $j=1,2,\ldots,k$. Here, an arbitrary k-submultiset (or k-combination) of I_{n-k+2} has been selected, say $\{s_1, s_2, \ldots, s_k\}$, as well as the fixed permutation $\sigma = 1\,2\,\cdots\,(n-k+2)$ of the elements of I_{n-k+2}. These k complete graphs adjoined to the planet will be referred to as the satellites. The reason why such a peculiar notation for these satellites is needed will become clear below. This family of satellites will be written as $\mathcal{K}=\left\{K^{j}_{n-k+1}(s_j) : j=1,2,\ldots,k\right\}$. For notational convenience, let

$$\mathcal{S}=\bigcup\nolimits_{j\in I_k} V(K^{j}_{n-k+1}(s_j))$$

denote the set of all satellite vertices. Now, construct a graph $G=G^{k,\,n}_{1}(\mathcal{K})$ by defining $V(G)$ to be the set $V(G)=I^k_n\cup\mathcal{S}$ and by defining $E(G)$ according to the following two prescriptions.

$$\text{(2)}\qquad \begin{array}{rl} \text{(I)} & \text{For all } \alpha,\,\beta\in I^k_n,\\ & \alpha\beta\in E(G) \text{ if and only if } \beta\notin\cup\mathcal{F}_\alpha.\\ \text{(II)} & \text{For all } x^{j,\,t}_m\in\mathcal{S} \text{ and } \alpha\in I^k_n,\\ & x^{j,\,t}_m\alpha\in E(G) \text{ if and only if } \alpha_p\neq m \text{ when } p\neq j. \end{array}$$

Example 1. As an example to illustrate this labeling scheme, consider the graph $G^{3,\,5}_{1}(\mathcal{K})$. Suppose that $\{1,3,4\}$ is chosen as the 3-submultiset of I_4. Also, let $\sigma=1\,2\,3\,4$ so that $\widehat{\sigma}(2)=1\,3\,4$. Then $\widehat{\sigma}_1(2)=1$, $\widehat{\sigma}_2(2)=3$, and $\widehat{\sigma}_3(2)=4$. In this case, the vertices of the three satellites would be labeled as follows:

$$V(K^1_3(1))=\{x^{1,1}_2,\,x^{1,2}_3,\,x^{1,3}_4\},\qquad V(K^2_3(3))=\{x^{2,1}_1,\,x^{2,2}_2,\,x^{2,3}_4\},$$

and

$$V\left(K^3_3(4)\right)=\left\{x^{3,1}_1,\,x^{3,2}_2,\,x^{3,3}_3\right\}.$$

Remark 2. In (II) of (2), it is somewhat cumbersome to visualize how the vertices in the planet are adjacent to the vertices in the satellites. An equivalent formulation of the second prescription is the following:

$$x^{j,\,t}_m\alpha\in E(G) \text{ if and only if } \alpha\in I^k_n\backslash\left(\bigcup\nolimits_{i\neq j}F_{im}\right).$$

In other words, connect the vertex $x^{j,\,t}_m$ from the satellite $K^{j}_{n-k+1}(s_j)$ to all vertices that remain in the planet $G^{k,\,n}_1$ after removing all of the vertices from each of the sets F_{im} except for when $i=j$.

Observe now that the choice for the notation in the labeling of the vertices of each satellite is necessary as it provides a convenient method for describing exactly how the satellites are attached to the planet. The first objective is to determine the exact value of the chromatic number of $G_1^{k,n}(\mathcal{K})$.

1-cliques	C_1	C_2
$K_1^1(2)$	$x_1^{1,1}$	
$K_1^2(2)$	$x_1^{2,1}$	
$\vdots$	$\vdots$	
$K_1^k(2)$	$x_1^{k,1}$	

Table 1. Pattern I—$\{k \cdot 2\}$.

1-cliques	C_1	C_2
$K_1^1(2)$	$x_1^{1,1}$	
$K_1^2(2)$	$x_1^{2,1}$	
$\vdots$	$\vdots$	
$K_1^{k-q}(2)$	$x_1^{k-q,1}$	
$K_1^{k-q+1}(1)$		$x_2^{k-q+1,1}$
$K_1^{k-q+2}(1)$		$x_2^{k-q+2,1}$
$\vdots$		$\vdots$
$K_1^k(1)$		$x_2^{k,1}$

Table 2. Pattern II—$\{q \cdot 1, (k-q) \cdot 2\}$.

Theorem 4. *For every $k, n \in \mathbb{N}$ with $2 \le k \le n$, the graph $G_1^{k,n}(\mathcal{K})$ is $(2n-k+1)$-chromatic.*

Proof. From Theorem 3, it follows that $\chi\left(G_1^{k,n}\right) = n$. Clearly, $K_{n-k+1}^j(s_j)$ is $(n-k+1)$-chromatic. When the satellites are adjoined to form the graph $G_1^{k,n}(\mathcal{K})$, we assert that $\chi\left(G_1^{k,n}(\mathcal{K})\right) = n + (n-k+1) = 2n-k+1$.

The proof proceeds by induction on n. For the base case of $n = k$, consider the graph $G_1^{k,k}(\mathcal{K})$. It must be shown that $G_1^{k,k}(\mathcal{K})$ is $(k+1)$-chromatic. The planet $G_1^{k,k}$ is k-chromatic by Theorem 3. Because each satellite is a 1-clique and because there do not exist edges between distinct satellites, by the definition of the edge set of $G_1^{k,k}(\mathcal{K})$ in (2), it is clear that $\chi\left(G_1^{k,k}(\mathcal{K})\right) \le k+1$. It remains to show that $\chi\left(G_1^{k,k}(\mathcal{K})\right) \ge k+1$. To this end, consider an arbitrary partition $\mathcal{P}$ of $I_k^k \cup \mathcal{S}$ into a minimum number of independent subsets. Every $X \in \mathcal{P}$ has the form $X = X_S \cup X_P$, where X_S is an independent subset of $\mathcal{S}$, the set of all satellite vertices; and, X_P is an independent subset of I_k^k, the set of all planet vertices. There are, up to automorphism, exactly two distinct patterns as to how the vertices in the satellites can be adjacent to vertices in the planet. They are distinguished by the repetition numbers of the associated k-submultiset of I_2. The k-submultiset that determines the pattern is given in parentheses. These two patterns are illustrated in Table 1 and Table 2. In each pattern, the vertices of the satellites are represented in the rows of the table and the vertices in the columns (either C_1 or C_2 in Table 1 and Table 2) will be grouped in various ways to represent how the satellite vertices are distributed among the independent subsets of $\mathcal{P}$.

Now for each of these two patterns, there are two ways in which vertices in the columns can be colored; i.e., distributed among independent subsets of $\mathcal{P}$. These ways are indicated in (A) and (B) below.

(A) For each column, all vertices in the column are in a single color class.

(B) There exists a column for which not all of the vertices in the column are contained in a single color class.

For instance, in Pattern II, if the first type of coloration as in (A) is illustrated, then we would have

$$\begin{aligned} X &= X_S \cup X_P = \left\{x_1^{1,1},\, x_1^{2,1},\, \ldots,\, x_1^{k-q,1}\right\} \cup X_P, \\ Y &= Y_S \cup Y_P = \left\{x_2^{k-q+1,1},\, x_2^{k-q+2,1},\, \ldots,\, x_2^{k,1}\right\} \cup Y_P \end{aligned}$$

as two representative elements from $\mathcal{P}$. Observe, as is taken into consideration below, that it is possible for all of the satellite vertices in Table 2 to be contained in a single color calss. Also, in Pattern I, if the second type of coloration as in (B) is illustrated, then we would have

$$\begin{aligned} X &= X_S \cup X_P = \left\{x_1^{1,1}\right\} \cup X_P, \\ Y &= Y_S \cup Y_P = \left\{x_1^{2,1},\, x_1^{3,1},\, \ldots,\, x_1^{k,1}\right\} \cup Y_P \end{aligned}$$

as two representative elements from $\mathcal{P}$. Observe that all vertices in the first column of Pattern I are shared (distributed) among two distinct independent subsets contained in $\mathcal{P}$.

Consider the first type of coloration given by (A) for each pattern separately. For Pattern I, the arbitrary partition $\mathcal{P}$ contains an element X, say

$$X = \left\{x_1^{1,1},\, x_1^{2,1},\, \ldots,\, x_1^{k,1}\right\} \cup X_P.$$

To prove there are at least k elements of $\mathcal{P} \setminus \{X\}$, it suffices to prove that $G_1^{k,k}(\mathcal{K}) - X$ contains a k-clique. To this end, construct a $k \times k$ matrix B having elements of I_k in the following manner. Set $b_{ii} = 1$ for all $i \in I_k$ and then randomly select the non-diagonal entries, b_{ij} with $i \neq j$, in such a way that each column B_j consists of k distinct elements, $b_{1j}, b_{2j}, \ldots, b_{kj}$, of I_k. By the definition of the edge set of $G_1^{k,k}(\mathcal{K})$ in (2) and Proposition 2, the rows of this matrix determine a set of vertices of $G_1^{k,k}$ in which each vertex is adjacent to at least one vertex of X and which admits an induced subgraph of $G_1^{k,k}$ isomorphic to K_k. It follows that $\chi\left(G_1^{k,k}(\mathcal{K})\right) \geq k+1$ and hence $G_1^{k,k}(\mathcal{K})$ is $(k+1)$-chromatic for Pattern I provided the type of coloration indicated by (A) is implemented.

For Pattern II, assume, without loss of generality, that $k, q \in \mathbb{N}$ satisfy $k-q=2$. Still under the assumption that the first type of coloration given by (A) is being implemented, suppose it is the case that there exists an element X of $\mathcal{P}$ such that

$$X = \left\{x_1^{1,1},\, x_1^{2,1},\, x_2^{3,1},\, x_2^{4,1},\, \ldots,\, x_2^{k,1}\right\}.$$

In other words, all of the satellite vertices are contained in a single color class. Then, upon the removal of the color class X from $\mathcal{P}$, it is clear that $G_1^{k,k}[\Delta]$ is a subgraph of $G_1^{k,k}(\mathcal{K}) - X$ so that $\chi\left(G_1^{k,k}(\mathcal{K})\right) \geq k+1$. Hence, $\chi\left(G_1^{k,k}(\mathcal{K})\right) = k+1$ in this event. Otherwise, it must be the case that the arbitrary partition $\mathcal{P}$ contains elements X and Y having the form:

$$\begin{aligned} X &= \left\{x_1^{1,1},\, x_1^{2,1}\right\} \cup X_P, \\ Y &= \left\{x_2^{3,1},\, x_2^{4,1},\, \ldots,\, x_2^{k,1}\right\} \cup Y_P. \end{aligned}$$

Select any satellite vertex from each set, say, without loss of generality, $x_1^{1,1} \in X$ and $x_2^{3,1} \in Y$. Then form the new (ordered) list L of satellite vertices by appending the remaining satellite vertices of X and Y in order:

$$L = \left(x_1^{1,1},\, x_2^{3,1},\, x_1^{2,1},\, x_2^{4,1},\, x_2^{5,1},\, \ldots,\, x_2^{k,1}\right).$$

The first superscript from each element of L determines a vertex in $\alpha \in I_k^k$, say

$$\alpha = (1,\, 3,\, 2,\, 4,\, 5,\, \ldots,\, k)\,.$$

For each of the last $k-2$ vertices in L, associate a vertex $\beta^{(j)}$ by the rule

$$x_m^{j,t} \longrightarrow \beta^{(j)} = \left(\beta_1^{(j)},\, \beta_2^{(j)},\, \ldots,\, \beta_k^{(j)}\right),$$

where

$$\beta_s^{(j)} = \begin{cases} m & \text{if } s = j, \\ \alpha_j & \text{if } s \neq j\,. \end{cases}$$

Then by Proposition 2, the set $W = \left\{\alpha,\, \beta^{(j_1)},\, \beta^{(j_2)},\, \ldots,\, \beta^{(j_{k-2})}\right\}$ determines an induced subgraph of $G_1^{k,k}$ that is isomorphic to K_{k-1}. This shows that $\chi\left(G_1^{k,k}(\mathcal{K})\right) \geq k+1$ and hence $G_1^{k,k}(\mathcal{K})$ is $(k+1)$-chromatic for Pattern II provided the type of coloration indicated by (A) is implemented. Therefore, in either pattern, we conclude that $G_1^{k,\,k}\,(\mathcal{K})$ is $(k+1)$-chromatic for the coloration indicated by (A).

Suppose now that the latter type of coloration in (B) is implemented. Then it is no longer necessary to consider separately the two patterns of adjacencies described above by Pattern I and Pattern II. Assume there exists a column, say C_m, in which there are two vertices in distinct satellites that are in distinct color classes. Call these vertices $x_m^{j_1,\,1}$ and $x_m^{j_2,\,1}$ and suppose they belong to the color classes X and Y, respectively. Upon the removal of the color classes X and Y from the partition $\mathcal{P}$, it is clear that the vertices in the set $Z = \left\{(i,\, i,\, \ldots,\, i) \in I_k^k : i \neq m\right\}$ determine a $(k-1)$-clique in $G_1^{k,\,k}\,(\mathcal{K}) - X - Y$. Consequently, we find that if the coloration in (B) is implemented, then $G_1^{k,\,k}\,(\mathcal{K})$ is $(k+1)$-chromatic. The base case when $n = k$ is henceforth established.

$(r-k+2)$-cliques	C_1	C_2	$\cdots$	C_{r-k+2}	C_{r-k+3}
$K_{r-k+2}^1\,(r-k+3)$	$x_1^{1,\,1}$	$x_2^{1,\,2}$	$\cdots$	$x_{r-k+2}^{1,\,r-k+2}$	
$K_{r-k+2}^2\,(r-k+3)$	$x_1^{2,\,1}$	$x_2^{2,\,2}$	$\cdots$	$x_{r-k+2}^{2,\,r-k+2}$	
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	
$K_{r-k+2}^k\,(r-k+3)$	$x_1^{k,\,1}$	$x_2^{k,\,2}$	$\cdots$	$x_{r-k+2}^{k,\,r-k+2}$	

Table 3. Pattern I—$\{k \cdot (r-k+3)\}$.

Inductively assume that $G_1^{k,\,r}\,(\mathcal{K})$ is $(2r-k+1)$-chromatic for $r \geq k$ and consider the graph $G_1^{k,\,r+1}\,(\mathcal{K})$. Let $\mathcal{P}$ be an arbitrary partition of $I_{r+1}^k \cup \mathcal{S}$ into a minimum number of independent subsets. For any fixed $i_0 \in I_k$, color the vertices in the set $F_{i_0 j}$ with color c_j for $j \in I_{r+1}$. Necessarily, the satellite $K_{r-k+2}^{i_0}\,(s_i)$ would require $r-k+2$ additional colors, say $c_{r+2},\, c_{r+3},\, \ldots,\, c_{2r-k+3}$, while the remaining satellites can be colored from among the colors $c_1,\, c_2,\, \ldots,\, c_{r+1}$. This coloration shows that $\chi\left(G_1^{k,\,r+1}\,(\mathcal{K})\right) \leq 2r-k+3$. Thus, it suffices to show that $\chi\left(G_1^{k,\,r+1}\,(\mathcal{K})\right) \geq 2r-k+3$. As in the base case, there are exactly two distinct patterns, up to automorphism, for how vertices in the satellites can be adjacent to vertices in the planet. These patterns are indicated in Table 3 and Table 4. Pattern I represents a k-multiset having exactly one repetition number being equal to k, and, Pattern II represents a k-multiset having at least two nonzero repetition numbers.

Now for each of these two patterns, there are two ways in which vertices in the columns can be colored; i.e., distributed among independent subsets of

$I^k_{r+1} \cup \mathcal{S}$. These ways are exactly as they were in the base case above. They are restated here for convenience.

(A) For each column, all vertices in the column are in a single color class.

(B) There exists a column for which not all of the vertices in the column are contained in a single color class.

$(r-k+2)$-cliques	C_1	C_2	C_3	C_4	$\cdots$	C_{r-k+2}	C_{r-k+3}
$K^1_{r-k+2}\,(2)$	$x_1^{1,\,1}$		$x_3^{1,\,2}$	$x_4^{1,\,3}$	$\cdots$	$x_{r-k+2}^{1,\,r-k+1}$	$x_{r-k+3}^{1,\,r-k+2}$
$K^2_{r-k+2}\,(1)$		$x_2^{2,\,1}$	$x_3^{2,\,2}$	$x_4^{2,\,3}$	$\cdots$	$x_{r-k+2}^{2,\,r-k+1}$	$x_{r-k+3}^{2,\,r-k+2}$
$K^3_{r-k+2}\,(r-k+3)$	$x_1^{3,\,1}$	$x_2^{3,\,2}$	$x_3^{3,\,3}$	$x_4^{3,\,4}$	$\cdots$	$x_{r-k+2}^{3,\,r-k+2}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$K^{k-1}_{r-k+2}\,(2)$	$x_1^{k-1,\,1}$		$x_3^{k-1,\,2}$	$x_4^{k-1,\,3}$	$\cdots$	$x_{r-k+2}^{k-1,\,r-k+1}$	$x_{r-k+3}^{k-1,r-k+2}$
$K^k_{r-k+2}\,(1)$		$x_2^{k,\,1}$	$x_3^{k,\,2}$	$x_4^{k,\,3}$	$\cdots$	$x_{r-k+2}^{k,\,r-k+1}$	$x_{r-k+3}^{k,\,r-k+2}$

Table 4. Pattern II—A k-multiset with at least two nonzero repetition numbers.

Let us assume the former coloration in (A) is implemented. For Pattern I, after $r-k+2$ elements of the partition $\mathcal{P}$ are removed, the vertices

$$\begin{aligned}
\gamma^{(1)} &= (1,\,2,\,3,\ldots,\,k-1,\,k)\,,\\
\gamma^{(2)} &= (2,\,3,\,4,\ldots,\,k,\,k+1)\,,\\
&\;\;\vdots\\
\gamma^{(r-k+2)} &= (r-k+2,\,r-k+3,\,r-k+4,\ldots,\,r,\,r+1)\,,\\
\gamma^{(r-k+3)} &= (r-k+3,\,r-k+4,\,r-k+5,\ldots,\,r+1,\,1)\,,\\
&\;\;\vdots\\
\gamma^{(r+1)} &= (r+1,\;\;1,\,2,\,\ldots,\,k-2,\,k-1)
\end{aligned}$$

determine, by Proposition 2, an $(r+1)$-clique in the resulting subgraph of $G_1^{k,\,r+1}$. As a result, there must be at least $r+1$ elements of $\mathcal{P}$ that remain upon the removal of the $r-k+2$ color classes indicated by columns C_1 through C_{r-k+2} in Pattern I of Table 3. Thus, it follows that $G_1^{k,\,r+1}$ is $(2r-k+3)$-chromatic for Pattern I provided the type of coloration indicated by (A) is implemented.

For Pattern II, after removing $r-k+3$ elements from the partition $\mathcal{P}$, it is shown that there are at least r elements remaining in $\mathcal{P}$. To this end, it suffices to prove the existence of an r-clique in the resulting subgraph. The following method will be used to construct a set of r vertices that determines such an r-clique. First, construct the $k \times r$ matrix Q such that $q_{ij} = j$ for each pair $(i,\, j)$, where $1 \leq i \leq k$ and $1 \leq j \leq r$. Thus,

$$Q = \begin{bmatrix} 1 & 2 & \cdots & r-1 & r \\ 1 & 2 & \cdots & r-1 & r \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & \cdots & r-1 & r \end{bmatrix}.$$

Second, for $j = 1,\, 2,\, \ldots,\, k$, redefine $q_{j,\, s_j}$ as $q_{j,\, s_j} = r+1$, except for when $s_j = r+1$. Third, and lastly, construct the set W of vertices, say

$$W = \left\{\alpha^{(1)},\, \alpha^{(2)},\, \ldots,\, \alpha^{(r)}\right\},$$

where the coordinates of $\alpha^{(i)}$, for $i = 1,\, 2,\, \ldots,\, r$, are determined by certain entries of the matrix Q^* formed by appending to Q, as redefined, the first $r-1$ columns of Q on the right. More precisely, define $\alpha_j^{(i)}$ as

$$\alpha_j^{(i)} = Q^*_{j,\, i+j-1}.$$

In other words, by using the same randomly selected k-combination as in Table 2, i.e., $\{2,\, 1,\, r-k+3,\, \ldots,\, 1\}$, we obtain

$$Q^* = \begin{bmatrix} 1 & r+1 & \cdots & r-k+3 & \cdots & r-1 & r & 1 & r+1 & \cdots & r-k+3 & \cdots & r-1 \\ r+1 & 2 & \cdots & r-k+3 & \cdots & r-1 & r & r+1 & 2 & \cdots & r-k+3 & \cdots & r-1 \\ 1 & 2 & \cdots & r+1 & \cdots & r-1 & r & 1 & 2 & \cdots & r+1 & \cdots & r-1 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ r+1 & 2 & \cdots & r-k+3 & \cdots & r-1 & r & r+1 & 2 & \cdots & r-k+3 & \cdots & r-1 \end{bmatrix}.$$

The set W then consists of the following elements of I_n^k:

$$\begin{aligned} \alpha^{(1)} &= (1,\, 2,\, 3,\, \ldots,\, r-k+3,\, \ldots,\, k)\,, \\ \alpha^{(2)} &= (r+1,\, 3,\, 4,\, \ldots,\, r-k+3,\, \ldots,\, k+1)\,, \\ \alpha^{(3)} &= (3,\, 4,\, 5,\, \ldots,\, r+1,\, \ldots,\, k+2)\,, \\ &\vdots \\ \alpha^{(r)} &= (r,\, r+1,\, 2,\, \ldots,\, r-k+3,\, \ldots,\, k-1)\,. \end{aligned}$$

By Proposition 2, the subgraph of $G_1^{k,\, r+1}$ induced by W is isomorphic to K_r. Thus, it follows that $G_1^{k,\, r+1}(\mathcal{K})$ is $(2r-k+3)$-chromatic for Pattern II provided the type of coloration indicated by (A) is implemented. Therefore, in either pattern, we conclude that $G_1^{k,\, r+1}(\mathcal{K})$ is $(2r-k+3)$-chromatic for the coloration indicated by (A).

Suppose now that the latter type of coloration (B) is implemented. In this event, it no longer becomes necessary to consider separately the two patterns of adjacencies described in Table 3 and Table 4. Assume that there exists a column, say C_m, in which there are two vertices in distinct satellites that are in distinct color classes. Call these vertices $x_m^{j_1,\, t_1}$ and $x_m^{j_2,\, t_2}$. Moreover, suppose these two vertices belong to the color classes X and Y, respectively. Upon the removal of the color classes X and Y from the partition $\mathcal{P}$, we assert that there exists a subgraph H of $G_1^{k,\, r+1}(\mathcal{K})$, in the resulting subgraph, that is isomorphic to $G_1^{k,\, r}(\mathcal{K}^*)$, where $\mathcal{K}^*$ is some appropriate collection of satellites. To see this, we consider the following two cases.

Case 1 (Coloration B). When the color classes X and Y are removed from $\mathcal{P}$, at most one vertex from each satellite is removed. In this case, $r-k+1$ vertices remain in each clique when X and Y are removed. Moreover, the subgraph of $G_1^{k,\, r+1}$ induced by the set $Z = \left\{(i,\, i,\, \ldots,\, i) \in I_{r+1}^k : i \neq m\right\}$ is isomorphic to $G_1^{k,\, r}$. Therefore, the subgraph of $G_1^{k,\, r+1}$ induced by the set consisting of these remaining satellite vertices together with the vertices of the aforementioned induced planet is such a subgraph H of $G_1^{k,\, r+1}$ that is isomorphic to $G_1^{k,\, r}(\mathcal{K}^*)$.

Case 2 (Coloration B). When the color classes X and Y are removed from $\mathcal{P}$, there exists a satellite for which two vertices have been removed. Note that there cannot be more than two vertices removed from any satellite when X and Y are removed from $\mathcal{P}$. For each such satellite that has two vertices removed, we claim that there is a vertex that can be taken from the planet and sent into orbit to serve as a new satellite vertex. If this can be established for each satellite that had two vertices removed, then the previous case applies, in which each satellite would have $r-k+1$ vertices. Suppose that $K_{r-k+2}^j(s_j)$ is a satellite that had two vertices removed. Recall that

$$V\left(K_{r-k+2}^j(s_j)\right) = \left\{x_{\widehat{\sigma}_1(s_j)}^{j,\, 1},\, x_{\widehat{\sigma}_2(s_j)}^{j,\, 2},\, \ldots,\, x_{\widehat{\sigma}_{r-k+2}(s_j)}^{j,\, r-k+2}\right\}.$$

Further assume that vertices $x_{\widehat{\sigma}_{n_1}(s_j)}^{j,\, n_1}$ and $x_{\widehat{\sigma}_{n_2}(s_j)}^{j,\, n_2}$ have been removed upon the removal of X and Y. Define $\beta^{(j)} \in I_n^k$ by the rule

$$\beta_i^{(j)} = \begin{cases} m & \text{if } i = j, \\ s_j & \text{if } i \neq j\,. \end{cases}$$

By the definitions of the satellite $K_{r-k+2}^j(s_j)$ and the edge set of $G_1^{k,\, r+1}(\mathcal{K})$, the vertex $\beta^{(j)}$ is adjacent to every vertex in $K_{r-k+2}^j(s_j)$. Therefore, $\beta^{(j)}$ remains upon the removal of the color classes X and Y. Thus, a new satellite is determined as the subgraph induced by the vertices in the set

$$\left(V\left(K_{r-k+2}^j(s_j)\right) \cup \left\{\beta^{(j)}\right\}\right) \setminus (X \cup Y)\,.$$

Moreover, this new satellite has order $r-k+1$. We remark that it is not required that $\beta^{(j)}$ be independent from the other satellites. This is because we are only interested in finding a subgraph H that is isomorphic to $G_1^{k,\,r}(\mathcal{K}^*)$. The subgraph of $G_1^{k,\,r+1}$ induced by these new satellite vertices together with the same induced planet as in the previous case determine such a subgraph H of $G_1^{k,\,r+1}$ that is isomorphic to $G_1^{k,\,r}(\mathcal{K}^*)$.

Now, in the case of coloration (B), we have shown that in the subgraph $G_1^{k,\,r+1}(\mathcal{K})$
$-X-Y$, there exists a subgraph H of satisfying $H \cong G_1^{k,\,r}(\mathcal{K}^*)$. Therefore, by the inductive hypothesis,

$$\chi\left(G_1^{k,\,r+1}(\mathcal{K}) - (X \cup Y)\right) \geq \chi\left(G_1^{k,\,r}(\mathcal{K}^*)\right) = 2r-k+1.$$

Consequently, $\chi\left(G_1^{k,\,r+1}(\mathcal{K})\right) \geq (2r-k+1)+2 = 2r-k+3$. Now, by coloring the planet $G_1^{k,\,r+1}$ with $r+1$ colors and by coloring the satellites with an additional $r-k+2$ colors, we find that the graph $G_1^{k,\,r+1}(\mathcal{K})$ is $(2r-k+3)$-chromatic if the coloration in (B) is implemented. By the principle of mathematical induction, we conclude that $G_1^{k,\,n}(\mathcal{K})$ is $(2n-k+1)$-chromatic for all $k,\, n \in \mathbb{N}$, with $2 \leq k \leq n$. ■

Theorem 5. *The set* $\mathcal{G} = \left\{G_1^{k,\,n}(\mathcal{K}) : k,\, n \in \mathbb{N} \text{ with } 2 \leq k \leq n\right\}$ *is a family of graphs such that:*

(1) $G_1^{k,\,n}(\mathcal{K})$ *is* $(2n-k+1)$*-chromatic;*

(2) $\mathcal{K}$ *is a family consisting of* k *completely independent critical cliques, each having order* $n-k+1$.

Proof. By Theorem 4, the graph $G_1^{k,\,n}(\mathcal{K})$ is $(2n-k+1)$-chromatic. For any fixed $i_0 \in I_k$, color the vertices in the set $F_{i_0 j}$ with color c_j for $j \in I_n$. Necessarily, the satellite $K_{n-k+1}^{i_0}(s_i)$ would require $n-k+1$ additional colors, say $c_{n+1},\, c_{n+2},\, \ldots,\, c_{2n-k+1}$, while the remaining satellites can be colored from among the colors $c_1,\, c_2,\, \ldots,\, c_n$. This coloration shows that the satellite $K_{n-k+1}^{i_0}(s_i)$ is a critical $(n-k+1)$-clique for each $i_0 \in I_k$. By the definition of the edge set of $G_1^{k,\,n}(\mathcal{K})$, the collection $\mathcal{K}$ is a family consisting of k completely independent critical cliques each having order $n-k+1$. ■

Corollary 4. For every $p_1,\, p_2,\, \ldots,\, p_k \in \mathbb{N}$, where $p_i \geq 1$ for each $i \in I_k$, there exists a vertex critical graph which admits a family $\mathcal{K}$ consisting of k completely independent critical cliques $K_{p_1^1},\, K_{p_2^2},\, \ldots,\, K_{p_k^k}$ having orders $p_1,\, p_2,\, \ldots,\, p_k$, respectively.

4. Concluding Remarks

It seems as though graphs admitting completely independent critical cliques are rare. It would be interesting to determine whether or not there are other families of graphs admitting completely independent critical cliques; or to be able to classify such graphs. Also, what condition(s) can be imposed on a graph to guarantee that a certain number of edges exist between critical cliques? It is believed that the answer to this question might lead to answering the double-critical conjecture of Lovász in the affirmative.

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