GENERALIZED GRAPH CORDIALITY

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Abstract

Hovey introduced A-cordial labelings in [4] as a simultaneous generalization of cordial and harmonious labelings. If A is an abelian group, then a labeling $f : V(G) \to A$ of the vertices of some graph G induces an edge-labeling on G; the edge uv receives the label $f(u) + f(v)$. A graph G is *A-cordial* if there is a vertex-labeling such that (1) the vertex label classes differ in size by at most one and (2) the induced edge label classes differ in size by at most one.

Research on A-cordiality has focused on the case where A is cyclic. In this paper, we investigate V_4-cordiality of many families of graphs, namely complete bipartite graphs, paths, cycles, ladders, prisms, and hypercubes. We find that all complete bipartite graphs are V_4-cordial except $K_{m,n}$ where $m, n \equiv 2 \pmod 4$. All paths are V_4-cordial except P_4 and P_5. All cycles are V_4-cordial except C_4, C_5, and C_k, where $k \equiv 2 \pmod 4$. All ladders $P_2 \square P_k$ are V_4-cordial except C_4. All prisms are V_4-cordial except $P_2 \square C_k$, where $k \equiv 2 \pmod 4$. All hypercubes are V_4-cordial, except C_4.

Finally, we introduce a generalization of A-cordiality involving digraphs and quasigroups, and we show that there are infinitely many Q-cordial digraphs for every quasigroup Q.

Keywords: graph labeling, cordial graph, A-cordial, quasigroup.

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1. INTRODUCTION

Graph labelings of diverse types are the subject of much study. The state of the field is described in detail in Gallian's dynamic survey [2]. Results obtained so far, while numerous, are mainly piecemeal in nature and lack generality. In an

attempt to provide something of a framework for these results, Hovey introduced
A-cordial labelings in [4] as a common generalization of cordial labeling (intro-
duced by Cahit [1]) and harmonious labeling (introduced by Graham and Sloane
[3]).

If A is an additive abelian group, then a vertex-labeling $f \colon V(G) \to A$ of
the vertices of some graph G induces an edge-labeling on G as well by giving the
edge uv the label $f(u) + f(v)$.

Definition 1.1. Let A be an abelian group. We say that a graph G is A-cordial
if there is a vertex-labeling $f \colon V(G) \to A$ such that:

(1) the vertex label classes differ in size by at most one, and

(2) the induced edge label classes differ in size by at most one.

Such a labeling is *balanced*. If the sizes of the vertex label classes are exactly
equal, then that vertex labeling is *perfectly balanced*. Similarly, if the sizes of the
edge label classes are exactly equal, then that edge labeling is *perfectly balanced*.

Cordial graphs are simply the $\mathbb{Z}_2$-cordial graphs, while harmonious graphs are
simply the $\mathbb{Z}_{|E(G)|}$-cordial graphs. Both of these concepts have been much studied.
Almost all other works on A-cordiality have also focused on the case where A
is cyclic. This case is indeed very interesting, particularly in light of Hovey's
conjecture from [4] that all trees are A-cordial for all cyclic groups A (which he
proved for $|A| < 6$). The conjecture does not extend even to the smallest non-
cyclic group, V_4 (i.e. $\mathbb{Z}_2 \times \mathbb{Z}_2$); the paths P_4 and P_5 are easily seen to be not
V_4-cordial. Hence, it is natural to investigate V_4-cordiality to see how it differs
from A-cordiality when A is cyclic, as we do in Section 3.

Throughout this paper, all graphs are finite and simple, and all quasigroups
are finite. Section 2 considers some conditions guaranteeing that a graph G is
not A-cordial for certain A. Section 3 considers the case $A \cong V_4$. Finally, Section
4 introduces a generalization of A-cordiality involving digraphs and quasigroups,
showing that there are infinitely many Q-cordial digraphs for every quasigroup
Q.

2. Necessary Conditions for A-Cordiality

The following propositions will be used in the next section. The *exponent* of an
additive abelian group A is the least $n \in \mathbb{Z}^+$ such that $na = 0$ for all $a \in A$.

Lemma 2.1. *If A is an abelian group of exponent 2, then $|A|$ is even. If further
$|A| > 2$, then*

$$\sum_{a \in A} a = 0.$$

Proof. By the Fundamental Theorem of Finitely Generated Abelian Groups, an abelian group of exponent 2 is a direct product of copies of $\mathbb{Z}_2$. The lemma follows. ∎

Proposition 2.2. *Let A be an abelian group of exponent 2 and order $N > 2$. If G is an Eulerian graph with $m = |E(G)| \equiv \pm 2 (\bmod\ N)$, then G is not A-cordial.*

Proof. Take an Eulerian circuit through G, and label the vertices along it $g_1, \ldots, g_m$ in order. For all i, let $h_i = g_i + g_{i+1}$ (taking the indices modulo m); these are precisely the labels assigned to corresponding edges. In particular, $\sum_{i=1}^{m} h_i$ is the sum of all the edge labels. Clearly,

$$\sum_{i=1}^{\frac{m}{2}} h_{2i-1} = \sum_{i=1}^{m} g_i = \sum_{i=1}^{\frac{m}{2}} h_{2i}.$$

Since any element added to itself equals 0, we conclude that

$$\sum_{i=1}^{m} h_i = \sum_{i=1}^{\frac{m}{2}} h_{2i-1} + \sum_{i=1}^{\frac{m}{2}} h_{2i} = 2 \sum_{i=1}^{m} g_i = 0.$$

If the edge label classes were balanced, all but two edge labels would appear an equal number of times. By Lemma 2.1, the sum of all the elements of A is 0. Canceling sets of N distinct summands implies that there are two distinct elements of A that sum to 0, which is impossible, since every element of A is its own inverse. Hence, the edge label classes cannot be balanced and G is not A-cordial. ∎

Definition 2.3. A graph G is 1-*factorable* if the edges of G can be partitioned into disjoint perfect matchings.

Proposition 2.4. *Let A be an abelian group of exponent 2 and order $N > 2$. Let G be a 1-factorable graph with kN vertices and $\ell N \pm 2$ edges, where $k, \ell \in \mathbb{N}$. Then G is not A-cordial.*

Proof. In an A-cordial labeling of G, the vertices must be perfectly balanced, since the number of vertices is divisible by N. Partition the edges of G into edge-disjoint perfect matchings. In each perfect matching, the sum of the vertex labels must be equal to the sum of the edge labels. Thus by Lemma 2.1, the sum of the labels on the edges in each of these matchings must be 0. Thus, the sum of all the edge labels of G is 0. But G has $\ell N \pm 2$ edges, and we have assumed that the edge labeling is balanced. Canceling sets of N edges with distinct labels implies that there are two distinct elements of A that sum to 0, which is impossible, since every element of A is its own inverse. Thus, G is not A-cordial. ∎

3. V_4-CORDIALITY FOR SOME FAMILIES OF GRAPHS

We denote the elements of V_4 by $0, a, b, c$; the sum of any two of $\{a, b, c\}$ is the third, and $g + g = 0$ for any $g \in V_4$.

The study of V_4-cordiality was initiated by Riskin [6], who claimed the following results.

Claim 3.1 (Riskin, [6])**.** *The complete graph K_n is V_4-cordial if and only if $n < 4$.*

Claim 3.2 (Riskin, [6])**.** *All complete bipartite graphs $K_{m,n}$ are V_4-cordial except $K_{2,2}$.*

Riskin's proof of Claim 3.1 is essentially correct, except for some arithmetical errors. However, Claim 3.2 is not true.[1] We provide a corrected version of it.

Theorem 3.3. *The complete bipartite graph $K_{m,n}$ is V_4-cordial if and only if m and n are not both congruent to $2(\bmod\ 4)$.*

Proof. Let X and Y be the partite sets, with $|X| = m$ and $|Y| = n$. Suppose that $\max\{m, n\} \geq 4$ and suppose that we have a V_4-cordial labeling of $K_{m,n}$. We note that in V_4, for distinct s, t, u, w, we have $s + t = u + w$. We claim that one of the partite sets has four vertices with distinct labels. If not, then some label u appears only in X and some other label w appears only in Y. This implies that the number of edges joining u-vertices to w-vertices is at least $((m + n)/4 - 1)^2 = (m + n)^2/16 - (m + n)/2 + 1$. We will derive a contradiction by showing that there are more than $\lceil mn/4 \rceil$ $(u + w)$-edges.

By the inequality of arithmetic and geometric means, $(m + n)^2/16 \geq mn/4$. It remains to show that there are more than $(m + n)/2 - 1$ other $(u + w)$-edges, which we do by counting those joining s-vertices to t-vertices. Let s_X, t_X be the number of vertices in X labeled s, t respectively. Then ignoring rounding we see that the number of such edges is $(s_X + t_X)(m + n)/4 - 2s_X t_X$. Since there must be at least $\lfloor mn/4 \rfloor$ edges labeled 0, again ignoring rounding we have that $(s_X + t_X)(m + n)/4 - s_X^2 - t_X^2 \geq mn/4$. Therefore $(s_X + t_X)(m + n)/4 - 2s_X t_X \geq (m + n)/2 - 1$, except in very small cases, which may be verified by hand. Hence the number of edges labeled $u + w$ is strictly greater than $\lceil mn/4 \rceil$, a contradiction. Thus one of the partite sets has four vertices with distinct labels.

Deleting these four vertices yields a V_4-cordial labeling of $K_{m-4,n}$ or $K_{m,n-4}$. Thus it suffices to consider $m, n < 4$. In this family, case analysis shows that $K_{m,n}$ is V_4-cordial if and only if m and n are not both equal to 2. ∎

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[1]An anonymous reviewer has informed us that some of these mistakes were also identified in an unpublished undergraduate thesis [5]. This thesis may also anticipate some of our other results. We were unable to obtain a copy.

Let P_n denote the n-vertex path. As noted above, the paths P_4 and P_5 are not V_4-cordial. However they are exceptional in this regard.

Theorem 3.4. *The path P_n is V_4-cordial unless $n \in \{4,5\}$.*

Proof. If $n < 4$, the path P_n is obviously V_4-cordial.

The path P_6 has a V_4-cordial labeling with vertices labeled $(c, c, 0, b, 0, a)$ in order. The path P_8 has a V_4-cordial with vertices labeled $(a, c, a, b, b, c, 0, 0)$ in order. The path P_{12} has a V_4-cordial labeling with vertices labeled $(a, 0, b, 0, c, c, c, a, b, b, a, 0)$ in order.

The following two claims complete the proof by induction.

Claim 1. *If P_n is V_4-cordial and $n \not\equiv 3 (\bmod 4)$, then P_{n+1} is V_4-cordial.*

Claim 2. *For all $n \in \mathbb{N}$, if P_n is V_4-cordial, then P_{n+8} is V_4-cordial.*

We begin by proving Claim 1. Given a V_4-cordial labeling of P_n, we append a vertex v to one end and extend the labeling to v, while maintaining V_4-cordiality. We consider three cases for n modulo 4. Let w be the neighbor of v.

When $n = 4k$, there are exactly k vertices with each label, so the vertex label classes will be balanced in P_{n+1} regardless of how we label v. One edge label appears $k - 1$ times, the others k times. Label v so that the edge vw receives the label that was deficient.

When $n = 4k + 1$, there are exactly k edges with each label, so the edge label classes will be balanced in P_{n+1} regardless of how we label v. Label v so that the vertex label classes remain balanced.

When $n = 4k + 2$, there are two labels we could use on v to keep the vertex label classes balanced. Only one label on vw would cause an imbalance in the edge label classes, so at least one of the two potential labels for v avoids this label on vw.

We now prove Claim 2. If P_n has a V_4-cordial labeling with an endvertex labeled 0, extend by eight edges at that vertex and label the new vertices $a, c, a, b, b, c, 0, 0$ in order.

Otherwise, without loss of generality, P_n has an endvertex labeled a. In this case, extend by eight edges at that vertex and label the new vertices $0, 0, c, b, b, a, c, a$ in order. ∎

We now determine which cycles C_n are V_4-cordial. Obviously, C_3 is V_4-cordial and the square C_4 is not. By an easy but somewhat tedious consideration of cases, it can also be seen that C_5 is not V_4-cordial.

Theorem 3.5. *The cycle C_n is V_4-cordial if and only if $n \notin \{4, 5\}$ and $n \not\equiv 2 (\bmod 4)$.*

Proof. It follows from Proposition 2.2 that C_n is not V_4-cordial when $n \equiv 2$ (mod 4), since V_4 has exponent 2 and order 4.

We now prove that C_n is V_4-cordial whenever n is a nontrivial multiple of 4. We proceed by induction with base cases C_8 and C_{12}. The vertex labels $(a, c, a, b, b, c, 0, 0)$ in order show C_8 is V_4-cordial. The vertex labels $(0, a, b, b, a, c,$ $c, c, 0, b, 0, a)$ in order show C_{12} is V_4-cordial.

Consider a V_4-cordial labeling of C_n, where $n \neq 3$. There is an edge labeled 0; its endpoints have the same label. Without loss of generality, assume the endpoints are either both labeled 0 or both labeled a. In either case, insert eight vertices into the cycle between the two endpoints and label them $(a, c, a, b, b, c, 0, 0)$ in order to obtain a V_4-cordial labeling of C_{n+8}.

Finally, we show that if C_n is V_4-cordial and n is a multiple of 4, then C_{n-1} and C_{n+1} are also V_4-cordial. Let $n = 4k$. In a V_4-cordial labeling of C_{4k}, there are exactly k vertices with each label and exactly k edges with each label. In particular, there is an edge labeled 0, the endpoints of which must share the same label, say g. Contracting this edge or subdividing it by a new vertex with label g yields V_4-cordial labelings of C_{4k-1} and C_{4k+1}, respectively.   ∎

We next determine which ladders $P_2 \square P_n$ are V_4-cordial. The copies of P_2 that appear in each ladder will be referred to as *rungs*. A rung whose vertices are labeled g and h will be called a (g, h)-*rung*.

Theorem 3.6. *All ladders $P_2 \square P_k$ are V_4-cordial, except $P_2 \square P_2$.*

Proof. We first note that the ladders $P_2 \square P_3$, $P_2 \square P_4$, $P_2 \square P_5$, and $P_2 \square P_6$ are V_4-cordial, as shown in Figure 1. In particular, there is a V_4-cordial labeling of these ladders such that one of the end rungs is a $(0, 0)$-rung.
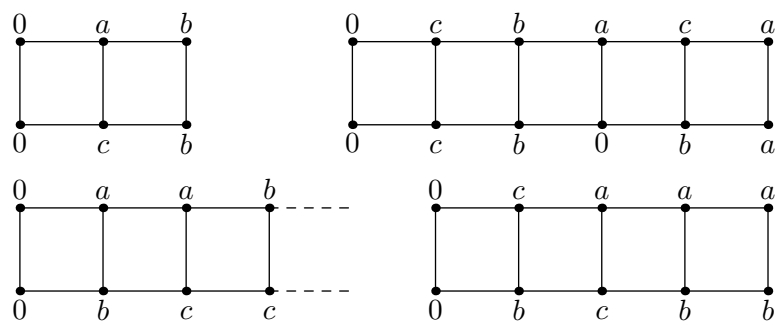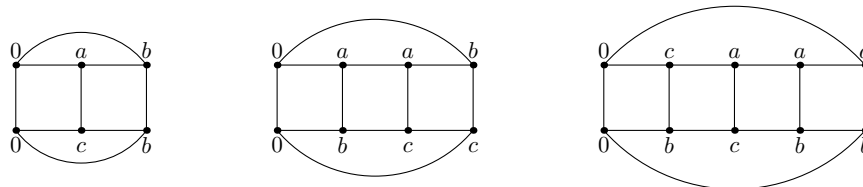

Figure 1. V_4-cordial labelings of the ladders $P_2 \square P_3$, $P_2 \square P_4$, $P_2 \square P_5$, and $P_2 \square P_6$.

If the (b, c)-rung of the 4-ladder $P_2 \square P_4$ shown in Figure 1 is made adjacent to an end $(0, 0)$-rung of any labeled ladder (as suggested in Figure 1), then the added vertices and edges are both perfectly balanced. Using this process, we construct

a V_4-cordial $P_2 \square P_{k+4}$ with an end $(0,0)$-rung from a V_4-cordial $P_2 \square P_k$ with an end $(0,0)$-rung. With the base cases, we construct V_4-cordial labelings for all ladders except $P_2 \square P_2$. ∎

We next determine which prisms $P_2 \square C_n$ are V_4-cordial, using "rungs" as above.

Theorem 3.7. *The prism $P_2 \square C_k$ is V_4-cordial if and only if $k \not\equiv 2(\bmod 4)$.*

Proof. We first note that the prisms $P_2 \square C_3$, $P_2 \square C_4$, and $P_2 \square C_5$ are V_4-cordial, as shown in Figure 2. In particular, there is a V_4-cordial labeling of these prisms such that one of the rungs is a $(0,0)$-rung.

Figure 2. V_4-cordial labelings of the prisms $P_2 \square C_3$, $P_2 \square C_4$, and $P_2 \square C_5$.

From a V_4-cordial labeling of $P_2 \square C_k$ with a $(0,0)$-rung, we will construct a V_4-cordially-labeled prism $P_2 \square C_{k+4}$ with a $(0,0)$-rung. Take a V_4-cordially-labeled prism $P_2 \square C_k$ with a $(0,0)$-rung and cut it into a ladder by removing two edges, so that the $(0,0)$-rung becomes an end rung. Now make the (b,c)-rung of the ladder $P_2 \square P_4$ from Figure 1 adjacent to this $(0,0)$-rung and add two edges to turn the resulting ladder into a prism. This operation has not changed the balance of the labelings. By induction, all prisms $P_2 \square C_n$ with $n \not\equiv 2(\bmod 4)$ are V_4-cordial.

Proposition 2.4 shows that $P_2 \square C_{4k+2}$ is not V_4-cordial. ∎

We next determine which hypercubes Q_d are V_4-cordial. As we saw previously, the square Q_2 is not V_4-cordial.

**Theorem 3.8.** *The d-dimensional hypercube Q_d is V_4-cordial, unless $d = 2$.*

Proof. We prove a stronger statement by induction. We show that if $d > 2$, then Q_d not only has a V_4-cordial labeling, but it has such a labeling with the property that we can cut Q_d into a pair of $(d-1)$-dimensional subcubes by removing a perfectly balanced set of 2^{d-1} edges.

A V_4-cordial-labeling of the cube Q_3 is shown in Figure 3. This labeling has the property that the inside square is cut from the outside square by removing a perfectly balanced set of four edges.

Now suppose that Q_d has a V_4-cordial labeling as specified. Let F_1 and F_2 be the two $(d-1)$-dimensional subcubes obtained by deleting a balanced cut of size 2^{d-1}. We construct a V_4-cordial-labeling of Q_{d+1} by joining two copies of each of
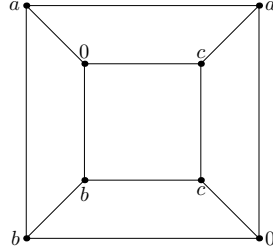
Figure 3. A V_4-cordial labeling of the cube Q_3.

F_1 and F_2 as shown in Figure 4. The hypercubes F_1 and F_2 are labeled as in a V_4-cordial-labeling of Q_d. Each of the four sets of 2^{d-1} edges between F_1 and F_2 is perfectly balanced. Furthermore this labeling of Q_{d+1} has the property that it may be cut into two d-dimensional subcubes by removing a perfectly balanced set of 2^d edges.   ∎
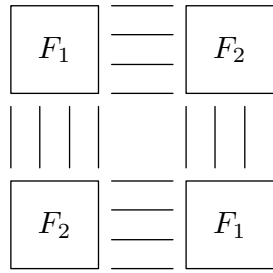
Figure 4. A V_4-cordial labeling of the hypercube Q_{d+1}.

Further research on V_4-cordiality could address which grids $P_h \Box P_k$ are V_4-cordial. Our results on ladders resolve the case $h = 2$. Additionally, it is not hard to see that the Petersen graph is V_4-cordial. By Proposition 2.2, the Kneser graph $K(n, k)$ is not V_4-cordial, if

$$\binom{n-k}{k} \text{ is even and } \frac{\binom{n-k}{k}\binom{n}{k}}{2} \equiv 2(\!\!\mod 4).$$

For example, $K(7,3)$ is not V_4-cordial. Further research could address which other generalized Petersen graphs or Kneser graphs are V_4-cordial.

4. BEYOND ABELIAN GROUPS

We now generalize the idea of A-cordial graphs to labelings from quasigroups. A *quasigroup* Q is a set with a binary operation $\cdot$ such that for all $a, b \in Q$,

there exist unique $c, d \in Q$ such that $a \cdot c = b$ and $d \cdot a = b$. In particular, (non-abelian) groups are quasigroups. Lack of commutativity suggests labeling digraphs. We do not delve deeply here into the study of Q-cordial graphs where Q is a quasigroup; our goal is merely to motivate the definition by demonstrating that, for each Q, there is an interesting theory of Q-cordial digraphs.

Definition 4.1. Let Q be a quasigroup. A labeling $f \colon V(G) \to Q$ of the vertices of a digraph G induces a labeling of the edges of G in the following way. If (a, b) is a directed edge with head b, then $f(a, b) = f(a) \cdot f(b)$. If there is a balanced vertex labeling of G that induces a balanced edge labeling of G, then we say that G is Q-cordial.
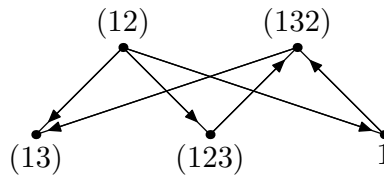

Figure 5. An S_3-cordial labeling of an orientation of $K_{2,3}$, with the convention that $\sigma\tau$ means apply σ then τ.

Theorem 4.2. *Let Q be an n-element quasigroup. If n is even, then for every positive integer m, there are orientations of C_{mn^2} and P_{mn^2} that are Q-cordial. If n is odd, then for every positive integer m, there are orientations of C_{2mn^2} and P_{2mn^2} that are Q-cordial.*

Proof. Enumerate the elements of Q as $q_1, \ldots, q_n$. Consider the graph $H = C_n \square C_n$, where we name the vertices by elements of $\{1, \ldots, n\} \times \{1, \ldots, n\}$ in the canonical way. We call an edge *horizontal* if its endpoints differ in their first coordinate. Edges that are not horizontal are *vertical*.

When n is even, it is easy to find a Hamiltonian cycle through H that alternates horizontal and vertical edges. Fix a direction along such a cycle. Label the vertex (i, j) with the quasigroup element q_i if we leave (i, j) by a vertical edge and with q_j if we leave by a horizontal edge. This gives a balanced labeling of the vertices of C_{n^2}. Orient each vertical edge of C_{n^2} in the direction that it is traversed and orient each horizontal edge in the opposite direction to how it is traversed. As there is now one edge labeled with each entry of the multiplication table for Q, this gives a balanced labeling of the edges of C_{n^2}, so this orientation of C_{n^2} is Q-cordial.

When n is odd, we may modify the construction by finding an Eulerian circuit through H that alternates vertical and horizontal moves, so that every vertex is

visited twice, producing a circuit of length $2n^2$. We label C_{2n^2} following the labeling of this circuit.

For $m > 1$, splice together m copies of the appropriate labeled and oriented cycle.

Deleting any edge, in a label class of maximal size, from any of the labeled and oriented cycles constructed above gives a Q-cordially-labeled oriented path. ∎

For abelian groups, the orientation of edges is irrelevant, so Theorem 4.2 gives results for undirected graphs. In particular, we identify the following easy but important consequence.

Corollary 4.3. *For every abelian group A, there are infinitely many A-cordial cycles and infinitely many A-cordial paths.*

In the case where A was V_4, we obtained much stronger results. Indeed by Theorem 3.4, all paths with six or more vertices are V_4-cordial. For any particular abelian group A, Corollary 4.3 is fairly weak. However, it suggests that, for each abelian group A, the class of A-cordial graphs will be an interesting object. It would be of interest to study how the structure of the abelian group A relates to the sequence of natural numbers n for which the path P_n is A-cordial. For example, V_4 has the special property that all sufficiently long paths are V_4-cordial. We can ask the following question:

Question 4.4. *Is it true that, for each abelian group A, there exists N such that P_n is A-cordial whenever $n > N$?*

If the answer is no, then a characterization of the groups that have this property would be very interesting. The only groups known to have this property are the cyclic groups (Theorem 2 in [4]) and V_4 (Theorem 3.4).

Acknowledgments

References

[1] I. Cahit, *Cordial graphs: a weaker version of graceful and harmonious graphs*, Ars Combin. **23** (1987) 201–207.

[2] J.A. Gallian, *A dynamic survey of graph labeling*, Electron. J. Combin. **18** (2011).

[3] R.L. Graham and N.J.A. Sloane, *On additive bases and harmonious graphs*, SIAM J. Algebraic Discrete Methods **1** (1980) 382–404.
doi:10.1137/0601045

[4] M. Hovey, *A-cordial graphs*, Discrete Math. **93** (1991) 183–194.
doi:10.1016/0012-365X(91)90254-Y

[5] G. McAlexander, Undergraduate thesis, (Mary Baldwin College, c.2007).

[6] A. Riskin, $\mathbb{Z}_2^2$-*cordiality of complete and complete bipartite graphs*, (http:// arxiv.org/abs/0709.0290∨1), September 2007.