# ON SUPER $(a, d)$-EDGE ANTIMAGIC TOTAL LABELING OF CERTAIN FAMILIES OF GRAPHS 

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#### Abstract

A $(p, q)$-graph $G$ is $(a, d)$-edge antimagic total if there exists a bijection $f: V(G) \cup E(G) \rightarrow\{1,2, \ldots, p+q\}$ such that the edge weights $\Lambda(u v)=$ $f(u)+f(u v)+f(v), u v \in E(G)$ form an arithmetic progression with first term $a$ and common difference $d$. It is said to be a super ( $a, d$ )-edge antimagic total if the vertex labels are $\{1,2, \ldots, p\}$ and the edge labels are $\{p+1, p+$ $2, \ldots, p+q\}$. In this paper, we study the super ( $a, d$ )-edge antimagic total labeling of special classes of graphs derived from copies of generalized ladder, fan, generalized prism and web graph.


Keywords: edge weight, magic labeling, antimagic labeling, ladder, fan graph, prism and web graph.
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## 1. Introduction

By a graph $G$ we mean a finite, undirected, connected graph without any loops or multiple edges. Let $V(G)$ and $E(G)$ be the set of vertices and edges of a graph $G$, respectively. The order and size of a graph $G$ is denoted as $p=|V(G)|$ and $q=|E(G)|$ respectively. For general graph theoretic notions we refer Harrary [6].

By a labeling we mean a one-to-one mapping that carries the set of graph elements onto a set of numbers (usually positive or non-negative integers), called labels. There are several types of labelings and a detailed survey of many of them can be found in the dynamic survey of graph labeling by Gallian [5].

Kotzig and Rosa [9] introduced the concept of magic labeling. They define an edge magic total labeling of a $(p, q)$-graph $G$ as a bijection $f$ from $V(G) \cup E(G)$ to the set $\{1,2, \ldots, p+q\}$ such that for each edge $u v \in E(G)$, the edge weight $f(u)+f(u v)+f(v)$ is a constant.

Enomoto et al. [3] defined a super edge magic labeling as an edge magic total labeling such that the vertex labels are $\{1,2, \ldots, p\}$ and edge labels are $\{p+1, p+2, \ldots, p+q\}$. They have proved that if a graph with $p$ vertices and $q$ edges is super edge magic then, $q \leq 2 p-3$. They also conjectured that every tree is super edge magic.

As a natural extension of the notion of edge magic total labeling, Hartsfield and Ringel [7] introduced the concept of an antimagic labeling and they defined an antimagic labeling of a $(p, q)$-graph $G$ as a bijection $f$ from $E(G)$ to the set $\{1,2, \ldots, q\}$ such that the sums of label of the edges incident with each vertex $v \in V(G)$ are distinct.

Simanjuntak et al. [10] defined an (a,d)-edge antimagic total labeling as a one to one mapping $f$ from $V(G) \cup E(G)$ to $\{1,2, \ldots, p+q\}$ such that the set of edge weight $\{f(u)+f(u v)+f(v): u v \in E(G)\}$ is equal to $\{a, a+d, a+2 d, \ldots, a+$ $(q-1) d\}$ for any two integers $a>0$ and $d \geq 0$.

An $(a, d)$-edge antimagic total labeling of a $(p, q)$-graph $G$ is said to be super $(a, d)$-edge antimagic total if the vertex labeles are $\{1,2, \ldots, p\}$ and the edge labeles are $\{p+1, p+2, \ldots, p+q\}$. The super ( $a, 0$ )-edge antimagic total labeling is usually called as super edge magic in the literature (see $[3,4]$ ).

An $(a, d)$-edge antimagic vertex labeling of a $(p, q)$-graph $G$ is defined as a one to one mapping $f$ from $V(G)$ to the set $\{1,2, \ldots, p\}$ such that the set of edge weight $\{f(u)+f(v): u v \in E(G)\}$ is equal to $\{a, a+d, a+2 d, \ldots, a+(q-1) d\}$ for any two integers $a>0$ and $d \geq 0$.

In [2] Bača et al. proved that if a $(p, q)$-graph $G$ has an $(a, d)$-edge antimagic vertex labeling then $d(q-1) \leq 2 p-1-a \leq 2 p-4$.

Also in [1] Bača and Barrientos proved the following: if a graph with $q$ edges and $q+1$ vertices has an $\alpha$-labeling, then it has an $(a, 1)$-edge antimagic vertex labeling. A tree has $(3,2)$-edge antimagic vertex labeling if and only if it has an $\alpha$-labeling and the number of vertices in its two partite set differ by at most 1 . If a tree with at least two vertices has a super $(a, d)$-edge antimagic total labeling, then $d$ is at most 3 . If a graph has an ( $a, 1$ )-edge antimagic vertex labeling, then it also has a super $\left(a_{1}, 0\right)$-edge antimagic total labeling and a super $\left(a_{2}, 2\right)$-edge antimagic total labeling.

In [12] Sugeng et al. studied the super ( $a, d$ )-edge antimagic total properties
of ladders, generalized prisms and antiprisms.
We make use of the following lemmas for our further discussion.
Lemma 1. If $a(p, q)$-graph $G$ is super ( $a, d$ )-edge antimagic total, then $d \leq$ $\frac{2 p+q-5}{q-1}$.

Lemma 2. If a $(p, q)$-graph $G$ has an (a,1)-edge antimagic vertex labeling and odd number of edges, then it has a super ( $\left.a^{\prime}, 1\right)$-edge antimagic total labeling, where $a^{\prime}=a+p+\frac{q+1}{2}$.

Lemma 3. If a $(p, q)$-graph $G$ has an ( $a, d$ )-edge antimagic vertex labeling, then $G$ has a super $\left(a^{\prime}, d^{\prime}\right)$-edge antimagic total labeling, where $a^{\prime}=a+p+1$ and $d^{\prime}=d+1$ or $a^{\prime}=a+p+q$ and $d^{\prime}=d-1$.

Lemma 2 appeared in [11] and Lemma 3 appeared in [2].
In this paper, we study the super $(a, d)$-edge antimagic total labeling of special classes of graphs derived from copies of generalized ladder, fan, generalized prism and web graph.

## 2. A Graph Derived from Copies of Generalized Ladder

Let $\left(u_{i, 1}, u_{i, 2}, \ldots, u_{i, n}, v_{i, 1}, v_{i, 2}, \ldots, v_{i, n}\right), 1 \leq i \leq t$, be a collection of $t$ disjoint copies of the generalized ladder $£_{n}, n \geq 2$, such that $u_{i, j}$ is adjacent to $u_{i, j+1}$, $v_{i, j+1}$ and $v_{i, j}$ is adjacent to $v_{i, j+1}$ for $1 \leq j \leq n-1$ and $u_{i, j}$ is adjacent to $v_{i, j}$ for $1 \leq j \leq n$. We denote the graph obtained by joining $u_{i, n}$ to $u_{i+1,1}, u_{i+1,2}, v_{i+1,1}$, $1 \leq i \leq t-1$, as $£_{n}^{(t)}$. Clearly, the vertex set $V$ and the edge set $E$ of the graph $£_{n}^{(t)}$ are given by
$V\left(£_{n}^{(t)}\right)=\left\{u_{i, j}, v_{i, j}: 1 \leq i \leq t, 1 \leq j \leq n\right\}$ and $E\left(£_{n}^{(t)}\right)=E_{1} \cup E_{2} \cup E_{3}$ where
$E_{1}=\left\{u_{i, j} u_{i, j+1}, v_{i, j} v_{i, j+1}, u_{i, j} v_{i, j+1}: 1 \leq i \leq t, 1 \leq j \leq n-1\right\}$,
$E_{2}=\left\{u_{i, j} v_{i, j}: 1 \leq i \leq t, 1 \leq j \leq n\right\}$,
$E_{3}=\left\{u_{i, n} u_{i+1,1}, u_{i, n} u_{i+1,2}, u_{i, n} v_{i+1,1}: 1 \leq i \leq t-1\right\}$.
It is easy to see that for $£_{n}^{(t)}$, we have $p=2 n t$ and $q=4 n t-3$.
Lemma 4. The graph $£_{n}^{(t)}, n, t \geq 2$ has an (a, 1)-edge antimagic vertex labeling.
Proof. Let us define a bijection $f_{1}: V\left(£_{n}^{(t)}\right) \rightarrow\{1,2, \ldots, 2 n t\}$ as follows:

$$
\begin{array}{ll}
f_{1}\left(u_{i, j}\right)=2(i-1) n+2 j-1 & \text { if } 1 \leq i \leq t \text { and } 1 \leq j \leq n, \\
f_{1}\left(v_{i, j}\right)=2(i-1) n+2 j & \text { if } 1 \leq i \leq t \text { and } 1 \leq j \leq n .
\end{array}
$$

By direct computation, we observe that the edge weights of all the edges of $£_{n}^{(t)}$, constitute an arithmetic sequence $\{3,4, \ldots, 4 n t-1\}$. Thus $f_{1}$ is an (3,1)-edge antimagic vertex labeling of $£_{n}^{(t)}$.

Theorem 5. The graph $£_{n}^{(t)}, n, t \geq 2$, has a super $(a, d)$-edge antimagic total labeling if and only if $d \in\{0,1,2\}$.

Proof. If the graph $£_{n}^{(t)}, n, t \geq 2$, is super ( $a, d$ )-edge antimagic total, then by Lemma 1, we get $d \leq 2$.

Conversely, by Lemma 4 and Lemma 3, we see that the graph $£_{n}^{(t)}, n, t \geq 2$ has a super $(6 n t, 0)$-edge antimagic total labeling and a super $(2 n t+4,2)$-edge antimagic total labeling.

Also by Lemma 2, we conclude that the graph $£_{n}^{(t)}, n, t \geq 2$, has a super $(4 n t+2,1)$-edge antimagic total labeling, since $q=4 n t-3$, which is odd for all $n$ and $t$.


Figure 1. (a, 1)-edge antimagic vertex labeling of $£_{4}^{(3)}$.

## 3. A Graph Derived from Copies of Fan Graph

Let $\left(u_{i}, w_{i}, v_{i, 1}, v_{i, 2}, \ldots, v_{i, m}\right), 1 \leq i \leq t$, be a collection of $t$ disjoint copies of the fan graph $\mathcal{F}_{m, 2}, m \geq 2$, such that $u_{i}$ is adjacent to $w_{i}$ and $v_{i, j}$ is adjacent to both $u_{i}$ and $w_{i}$ for $1 \leq j \leq m$. We denote the graph [8] obtained by joining $v_{i, m}$ to $u_{i+1}, v_{i+1,1}, v_{i+1,2}, 1 \leq i \leq t-1$, as $\mathcal{F}_{m, 2}^{(t)}$. Clearly, the vertex set $V$ and the edge set $E$ of the graph $\mathcal{F}_{m, 2}^{(t)}$ are given by

$$
\begin{aligned}
V\left(\mathcal{F}_{m, 2}^{(t)}\right) & =\left\{u_{i}, w_{i}, v_{i, j}: 1 \leq i \leq t, 1 \leq j \leq m\right\} \text { and } \\
E\left(\mathcal{F}_{m, 2}^{(t)}\right) & =\left\{u_{i} w_{i}, u_{i} v_{i, j}, w_{i} v_{i, j}: 1 \leq i \leq t, 1 \leq j \leq m\right\} \\
& \cup\left\{v_{i, m} u_{i+1}, v_{i, m} v_{i+1,1}, v_{i, m} v_{i+1,2}: 1 \leq i \leq t-1\right\}
\end{aligned}
$$

It is easy to see that for $\mathcal{F}_{m, 2}^{(t)}$, we have $p=(m+2) t$ and $q=(m+2) 2 t-3$.
Lemma 6. The graph $\mathcal{F}_{m, 2}^{(t)}, m, t \geq 2$, has an $(a, 1)$-edge antimagic vertex labeling.
Proof. Let us define a bijection $f_{2}: V\left(\mathcal{F}_{m, 2}^{(t)}\right) \rightarrow\{1,2, \ldots,(m+2) t\}$ as follows:

$$
\begin{array}{ll}
f_{2}\left(u_{i}\right)=(i-1)(m+2)+1 & \text { if } 1 \leq i \leq t \\
f_{2}\left(w_{i}\right)=(m+2) i & \text { if } 1 \leq i \leq t \\
f_{2}\left(v_{i, j}\right)=f_{2}\left(u_{i}\right)+j & \text { if } 1 \leq i \leq t \text { and } 1 \leq j \leq m
\end{array}
$$

By direct computation, we observe that the edge weights of all the edges of $\mathcal{F}_{m, 2}^{(t)}$ constitute an arithmetic sequence $\{3,4, \ldots, 2 t(m+2)-1\}$. Thus $f_{2}$ is an $(3,1)$ edge antimagic vertex labeling of $\mathcal{F}_{m, 2}^{(t)}$.

Theorem 7. The graph $\mathcal{F}_{m, 2}^{(t)}, m, t \geq 2$, has a super (a,d)-edge antimagic total labeling if and only if $d \in\{0,1,2\}$.

Proof. If the graph $\mathcal{F}_{m, 2}^{(t)}, m, t \geq 2$, is super ( $a, d$ )-edge antimagic total, then by Lemma 1 , we get $d \leq 2$.

Conversely, by Lemmas 3 and 6 , we see that the graph $\mathcal{F}_{m, 2}^{(t)}, m, t \geq 2$, has a super $((m+2) 3 t, 0)$-edge antimagic total labeling and a super $((m+2) t+4,2)$ edge antimagic total labeling.

Also by Lemma 2, we conclude that the graph $\mathcal{F}_{m, 2}^{(t)}, m, t \geq 2$, has a super $((m+2) 2 t+2,1)$-edge antimagic total labeling, since $q=(m+2) 2 t-3$, which is odd for all $m$ and $t$.


Figure 2. ( $a, 1$ )-edge antimagic vertex labeling of $\mathcal{F}_{3,2}^{(3)}$.

## 4. A Graph Derived from Copies of Generalized Prism

Let $\left(v_{i, j}^{(k)}, 1 \leq i \leq m, 1 \leq j \leq n\right), 1 \leq k \leq t$, be a collection of $t$ disjoint copies of the generalized prism $C_{m} \times P_{n}, m \geq 3, n \geq 2$, such that $v_{i, j}^{(k)}$ is adjacent to $v_{i+1, j}^{(k)}$ for $1 \leq i \leq m-1,1 \leq j \leq n, v_{m, j}^{(k)}$ is adjacent to $v_{1, j}^{(k)}$ for $1 \leq j \leq n$ and $v_{i, j}^{(k)}$ is adjacent to $v_{i, j+1}^{(k)}$ for $1 \leq i \leq m, 1 \leq j \leq n-1$. We denote the graph obtained by joining $v_{m, n}^{(k)}$ to $v_{i, 1}^{(k+1)}$ if $n$ is odd or $v_{1, n}^{(k)}$ to $v_{i, 1}^{(k+1)}$ if $n$ is even for $1 \leq i \leq m$, $1 \leq k \leq t-1$ as $\left(C_{m} \times P_{n}\right)^{(t)}$. Clearly, the vertex set $V$ and the edge set $E$ of the graph $\left(C_{m} \times P_{n}\right)^{(t)}$ are given by $V\left(\left(C_{m} \times P_{n}\right)^{(t)}\right)=\left\{v_{i, j}^{(k)}: 1 \leq k \leq t, 1 \leq i \leq\right.$
$m, 1 \leq j \leq n\}$ and $E\left(\left(C_{m} \times P_{n}\right)^{(t)}\right)=E_{1} \cup E_{2} \cup E_{3}$ where

$$
\begin{aligned}
E_{1} & =\left\{v_{i, j}^{(k)} v_{i+1, j}^{(k)}: 1 \leq k \leq t, 1 \leq i \leq m-1,1 \leq j \leq n\right\} \\
& \cup\left\{v_{m, j}^{(k)} v_{1, j}^{(k)}: 1 \leq k \leq t, 1 \leq j \leq n\right\}, \\
E_{2} & =\left\{v_{i, j}^{(k)} v_{i, j+1}^{(k)}: 1 \leq k \leq t, 1 \leq i \leq m, 1 \leq j \leq n-1\right\}, \\
E_{3} & =\left\{v_{m, n}^{(k)} v_{i, 1}^{(k+1)}: \text { if } n \text { is odd and } 1 \leq k \leq t-1,1 \leq i \leq m\right\} \\
& \cup\left\{v_{1, n}^{(k)} v_{i, 1}^{(k+1)}: \text { if } n \text { is even and } 1 \leq k \leq t-1,1 \leq i \leq m\right\} .
\end{aligned}
$$

It is easy to see that for $\left(C_{m} \times P_{n}\right)^{(t)}$, we have $p=m n t$ and $q=m(2 n t-1)$.
Lemma 8. For odd $m, m \geq 3$ and $n, t \geq 2$, the graph $\left(C_{m} \times P_{n}\right)^{(t)}$ has an ( $a, 1$ )-edge antimagic vertex labeling.
Proof. Let us define a bijection $f_{3}: V\left(\left(C_{m} \times P_{n}\right)^{(t)}\right) \rightarrow\{1,2, \ldots, m n t\}$ as follows.
If $j$ is odd and $1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq t$, then

$$
f_{3}\left(v_{i, j}^{(k)}\right)= \begin{cases}(k-1) m n+(j-1) m+\frac{i+1}{2} & \text { if } i \text { is odd, } \\ (k-1) m n+(j-1) m+\frac{m+i+1}{2} & \text { if } i \text { is even. }\end{cases}
$$

If $j$ is even and $1 \leq i \leq m, 2 \leq j \leq n, 1 \leq k \leq t$, then

$$
f_{3}\left(v_{i, j}^{(k)}\right)= \begin{cases}(k-1) m n+(j-1) m+\frac{m+i}{2} & \text { if } i \text { is odd } \\ (k-1) m n+(j-1) m+\frac{i}{2} & \text { if } i \text { is even. }\end{cases}
$$

By direct computation, we observe that the edge weights of all the edges of ( $C_{m} \times$ $\left.P_{n}\right)^{(t)}$ constitute an arithmetic sequence $\left\{\frac{m+3}{2}, \frac{m+5}{2}, \ldots, \frac{m+4 m n t-3}{2}\right\}$. Clearly $\frac{m+3}{2}$ is an integer only when $m$ is odd. Thus $f_{3}$ is an $\left(\frac{m+3}{2}, 1\right)$-edge antimagic vertex labeling of $\left(C_{m} \times P_{n}\right)^{(t)}$, for odd $m$.

Theorem 9. For odd $m, m \geq 3$ and $n, t \geq 2$, the graph $\left(C_{m} \times P_{n}\right)^{(t)}$ has a super (a,d)-edge antimagic total labeling if and only if $d \in\{0,1,2\}$.
Proof. If the graph $\left(C_{m} \times P_{n}\right)^{(t)}, m \geq 3$ and $n, t \geq 2$, is super ( $a, d$ )-edge antimagic total, then by Lemma 1 we get

$$
d \leq \frac{2 p+q-5}{q-1}=\frac{2 m n t+m(2 n t-1)-5}{m(2 n t-1)-1}=2+\frac{m-3}{2 m n t-m-1} .
$$

Since $2 m n t-m-1>0$, for $m \geq 3, n, t \geq 2$, it follows that $\frac{m-3}{2 m n t-m-1}<1$ and hence $d<3$.

Conversely, by Lemma 8 and Lemma 3, we obtain that for odd $m$, the graph $\left(C_{m} \times P_{n}\right)^{(t)}, m \geq 3, n, t \geq 2$, is both super $\left(\frac{m+3}{2}+p+q, 0\right)$-edge antimagic total and super $\left(\frac{m+3}{2}+p+1,2\right)$-edge antimagic total.

Also by Lemma 2 , we conclude that the graph $\left(C_{m} \times P_{n}\right)^{(t)}, m \geq 3, n, t \geq 2$, has a super $\left(\frac{m+3}{2}+p+\frac{q+1}{2}, 1\right)$-edge antimagic total labeling, since $q=m(2 n t-$ 1 ), which is odd for odd $m$.


Figure 3. $(a, 1)$-edge antimagic vertex labeling of $\left(C_{3} \times P_{2}\right)^{(2)}$.

## 5. A Graph Derived from Copies of Generalized Web Graph

Let $\left(v_{i, j}^{(k)}, 1 \leq i \leq m, 1 \leq j \leq n+1\right), 1 \leq k \leq t$, be a collection of $t$ disjoint copies of the generalized web graph $W(m, n), m \geq 3, n \geq 2$, such that $v_{i, j}^{(k)}$ is adjacent to $v_{i+1, j}^{(k)}$ for $1 \leq i \leq m-1,1 \leq j \leq n, v_{m, j}^{(k)}$ is adjacent to $v_{1, j}^{(k)}$ for $1 \leq j \leq n$ and $v_{i, j}^{(k)}$ is adjacent to $v_{i, j+1}^{(k)}$ for $1 \leq i \leq m, 1 \leq j \leq n$. We denote the graph obtained by joining $v_{1, n}^{(k)}$ to $v_{i, 1}^{(k+1)}$ and $v_{i, 2}^{(k+1)}$ for $1 \leq i \leq m, 1 \leq k \leq t-1$ as $(W(m, n))^{(t)}$. Clearly, the vertex set $V$ and the edge set $E$ of the graph $(W(m, n))^{(t)}$ are given by $V\left((W(m, n))^{(t)}\right)=\left\{v_{i, j}^{(k)}: 1 \leq k \leq t, 1 \leq i \leq m, 1 \leq j \leq n+1\right\}$ and $E\left((W(m, n))^{(t)}\right)=E_{1} \cup E_{2} \cup E_{3}$ where

$$
\begin{aligned}
E_{1} & =\left\{v_{i, j}^{(k)} v_{i+1, j}^{(k)}: 1 \leq k \leq t, 1 \leq i \leq m-1,1 \leq j \leq n\right\} \\
& \cup\left\{v_{m, j}^{(k)} v_{1, j}^{(k)}: 1 \leq k \leq t, 1 \leq j \leq n\right\}, \\
E_{2} & =\left\{v_{i, j}^{(k)} v_{i, j}^{(k)}: 1 \leq k \leq t, 1 \leq i \leq m, 1 \leq j \leq n\right\}, \\
E_{3} & =\left\{v_{1, n}^{(k)} v_{i, 1}^{(k+1)}, v_{1, n}^{(k)} v_{i, 2}^{(k+1)}: 1 \leq k \leq t-1,1 \leq i \leq m\right\} .
\end{aligned}
$$

It is easy to see that for $(W(m, n))^{(t)}$, we have $p=m t(n+1)$ and $q=2 m(n t+t-1)$.
Lemma 10. For odd $m, m \geq 3, n, t \geq 2$, the graph $(W(m, n))^{(t)}$ has an $(a, 1)$ edge antimagic vertex labeling.

Proof. Let us define a bijection $\left.f_{4}: V(W(m, n))^{(t)}\right) \rightarrow\{1,2, \ldots, m t(n+1)\}$ as follows:

Case (i): $n$ is even.
If $j$ is odd and $1 \leq i \leq m, 1 \leq j \leq n+1,1 \leq k \leq t$, then

$$
f_{4}\left(v_{i, j}^{(k)}\right)= \begin{cases}(k-1)(m n+m)+(j-1) m+\frac{i+1}{2} & \text { if } i \text { is odd, } \\ (k-1)(m n+m)+(j-1) m+\frac{m+i+1}{2} & \text { if } i \text { is even. }\end{cases}
$$

If $j$ is even and $1 \leq i \leq m, 2 \leq j \leq n, 1 \leq k \leq t$, then


Figure 4. $(a, 1)$-edge antimagic vertex labeling of $(W(3,3))^{(2)}$.

$$
f_{4}\left(v_{i, j}^{(k)}\right)= \begin{cases}(k-1)(m n+m)+(j-1) m+\frac{m+i}{2} & \text { if } i \text { is odd } \\ (k-1)(m n+m)+(j-1) m+\frac{i}{2} & \text { if } i \text { is even }\end{cases}
$$

Case (ii): $n$ is odd.
If $j$ is odd and $1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq t$, then

$$
f_{4}\left(v_{i, j}^{(k)}\right)= \begin{cases}(k-1)(m n+m)+(j-1) m+\frac{m+i}{2} & \text { if } i \text { is odd } \\ (k-1)(m n+m)+(j-1) m+\frac{i}{2} & \text { if } i \text { is even }\end{cases}
$$

If $j$ is even and $1 \leq i \leq m, 2 \leq j \leq n+1,1 \leq k \leq t$, then

$$
f_{4}\left(v_{i, j}^{(k)}\right)= \begin{cases}(k-1)(m n+m)+(j-1) m+\frac{i+1}{2} & \text { if } i \text { is odd } \\ (k-1)(m n+m)+(j-1) m+\frac{m+i+1}{2} & \text { if } i \text { is even }\end{cases}
$$

In both the cases, we observe that under the bijection $f_{4}$, the edge weights of all the edges of $(W(m, n))^{(t)}$ constitute an arithmetic sequence $\left\{\frac{m+3}{2}, \frac{m+5}{2}, \ldots\right.$, $\left.\frac{m+4 m n t+4 m(t-1)+1}{2}\right\}$. Clearly $\frac{m+3}{2}$ is an integer only when $m$ is odd. Hence the vertex labeling $f_{4}$ is an $\left(\frac{m+3}{2}, 1\right)$-edge antimagic vertex labeling of $(W(m, n))^{(t)}$, for odd $m$.

Theorem 11. For odd $m, m \geq 3, n, t \geq 2$ and $d \in\{0,2\}$, the $\operatorname{graph}(W(m, n))^{(t)}$, has a super (a,d)-edge antimagic total labeling.

Proof. By Lemmas 3 and 10, we see that for odd $m$, the graph $(W(m, n))^{(t)}$, $m \geq 3, n, t \geq 2$ has a super $\left(\frac{m+3}{2}+p+q, 0\right)$-edge antimagic total labeling and a super $\left(\frac{m+3}{2}+p+1,2\right)$-edge antimagic total labeling.

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