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DECOMPOSITIONS OF A COMPLETE MULTIDIGRAPH INTO ALMOST ARBITRARY PATHS [1]

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Abstract

For $n \geq 4$, the complete n-vertex multidigraph with arc multiplicity λ is proved to have a decomposition into directed paths of arbitrarily prescribed lengths $\leq n-1$ and different from $n-2$, unless $n = 5$, $\lambda = 1$, and all lengths are to be $n-1 = 4$. For $\lambda = 1$, a more general decomposition exists; namely, up to five paths of length $n-2$ can also be prescribed.

Keywords: complete digraph, multidigraph, tour girth, arbitrary path decomposition.

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1. Introduction

We use standard notation and terminology of graph theory [1, 3, 4] unless otherwise stated. Multigraphs and multidigraphs may have multiple edges and multiple arcs, respectively, loops are forbidden. For a multigraph G, let $\mathcal{D}G$ denote a multidigraph obtained from G by replacing each edge with two opposite arcs connecting endvertices of the edge.

Given a positive integer λ, the symbol ${}^{\lambda}\mathcal{D}K_n$ stands for the *complete λ-multidigraph* on n vertices, obtained by replacing each arc of $\mathcal{D}K_n$ by λ arcs (with the same endvertices).

By a *decomposition* of a multidigraph G we mean a family of arc-disjoint submultidigraphs of G which include all arcs of G.

In [6] we have stated the following general conjecture.

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Conjecture. *The complete n-vertex multidigraph ${}^{\lambda}\mathcal{D}K_n$ is decomposable into paths of arbitrarily prescribed lengths ($\leq n-1$) provided that the lengths sum up to the size $\lambda\, n\,(n-1)$ of ${}^{\lambda}\mathcal{D}K_n$, unless all paths are hamiltonian and either $n = 3$ and λ is odd or $n = 5$ and $\lambda = 1$.*

The known supporting results are summarized in three theorems.

Theorem A (Bosák [3, Corollary 11.9A])**.** *The multidigraph ${}^{\lambda}\mathcal{D}K_n$ is decomposable into hamiltonian paths if and only if neither $n = 3$ and λ is odd nor $n = 5$ and $\lambda = 1$.*

In case $\lambda = 1$ the assertion in Theorem A was noted by Bermond and Faber [2] for even n and completed by Tillson [9] for odd $n \geq 7$. The assertion answers a question which (according to Mendelsohn [5]) was posed by E.G. Strauss. Bosák settled the cases $n = 3, 5$ by extending (to any λ) former contributions in the case $\lambda = 1$, see [2] for contributions in general.

Theorem B (Meszka and Skupień [6])**.** *For $n \geq 3$, the complete n-vertex multidigraph ${}^{\lambda}\mathcal{D}K_n$ is decomposable into nonhamiltonian paths of arbitrarily prescribed lengths ($\leq n-2$) provided that the lengths sum up to the size $\lambda\, n\,(n-1)$ of ${}^{\lambda}\mathcal{D}K_n$.*

The following observation can easily be checked.

Theorem C. *Conjecture is true for $n \leq 4$ and $\lambda = 1$.*

In this paper we contribute to the results mentioned above by showing that the conjecture holds true in case when only the length $n-2$ is excluded. In the following theorem, which is the first of our main results, up to five paths of length $n-2$ are allowed.

Theorem 1. *For any integer $n \geq 4$, the complete n-vertex digraph $\mathcal{D}K_n$ has a decomposition into paths of arbitrarily prescribed lengths provided that the number of paths of length $n-2$ is not greater than 5 and lengths of paths sum up to the size $n\,(n-1)$ of $\mathcal{D}K_n$, unless $n = 5$ and all paths are to be hamiltonian.*

Corollary 2. *For any positive integer $n \geq 4$, the complete n-vertex digraph $\mathcal{D}K_n$ has an anti-1-defective path decomposition if the arbitrarily prescribed lengths of paths ($\neq n-2$) sum up to the size $n\,(n-1)$ of $\mathcal{D}K_n$, unless $n = 5$ and all paths are to be hamiltonian.*

Next we shall give a short proof, an adaptation of the related proof in [6], of the following extension from digraphs to the case of multidigraphs. The proof involves partitioning of the decomposition problem for a complete multidigraph ${}^{\lambda}\mathcal{D}K_n$ into λ problems each for the complete digraph $\mathcal{D}K_n$.

Theorem 3. *For $n \geq 4$, the complete n-vertex multidigraph ${}^{\lambda}\mathcal{D}K_n$ has a decomposition into paths of arbitrarily prescribed lengths different from $n-2$, provided that the lengths of paths sum up to the size $\lambda\, n\,(n-1)$ of ${}^{\lambda}\mathcal{D}K_n$, unless $n = 5$, $\lambda = 1$, and all paths are to be hamiltonian.*

The corresponding decompositions of a complete multigraph into arbitrary paths was originated by Tarsi [8], see [6] for some subsequent results.

2. Preliminaries

The symbol $v_1 \to v_2$ denotes the arc which goes from the *tail* v_1 to the *head* v_2, whilst the symbol $v_1 \rightsquigarrow v_2$ is used to denote a path with the initial vertex v_1 and the terminal one v_2. Given a multidigraph, the names *walk*, *trail* and *path* stand for alternating sequences of vertices and (consistently oriented) arcs where each arc a is preceded by the tail of a and is followed by the head of a. Recall that arcs are not repeated in trails. Vertices (and arcs) are not repeated in open paths. Closed trails and closed paths are named *tours* and *cycles*, respectively.

Note that names *path* and *cycle* can stand also for digraphs $\vec{P}_n$, $\vec{C}_n$, respectively, where the subscript n denotes the number of vertices; $n \geq 1$ and $n \geq 2$, respectively.

2.1. Useful tours

Let W_0 be a sequence of (possibly repeated) vertices of the digraph $\mathcal{D}K_n$, say $W_0 = \langle x_1, x_2, \ldots, x_k\rangle$ where denotation involves angle brackets. In what follows we use the convention that the phrase 'walk W_0' refers to the walk whose subsequence of vertices is W_0. If applicable, the word 'walk' in the phrase is replaced by 'trail', 'path', 'tour', or 'cycle'. Moreover, the symbol $\langle W_0\rangle$ stands for the digraph induced by the arc set of the walk W_0.

Definition 1. Assume that $n \geq 5$. For odd and even n separately, the vertex sequence denoted by $W_0(n)$ or W_0 is defined as follows:
(i) For odd $n \geq 5$, the vertices are denoted by $\infty, 0, 1, \ldots, n-2$ and $W_0 = \langle \infty, 0, 1, \ldots, \frac{n-3}{2}, \frac{n+1}{2}, \infty\rangle$, which represents a cycle $\vec{C}_{n-1}$ in $\mathcal{D}K_n$. It is assumed that the walk W_0 avoids the vertex $\frac{n-1}{2}$ but includes the initial path $\infty \to 0 \to 1 \to n-2$ together with the following arcs:

$$\begin{array}{rcll} n-k & \to & k, & 2 \leq k \leq \frac{n-3}{2}, \\ & & k \;\to\; n-k-1, & 2 \leq k \leq \frac{n-3}{2}, \end{array}$$

and the terminal arc $\frac{n+1}{2} \to \infty$, see Figure 1, wherein $n = 9$. Thus the walk W_0 is indeed a cycle.

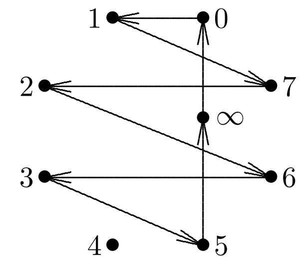

Figure 1. $n = 9$

(ii) For even $n \geq 6$, the vertices are denoted by $\infty, \overline{\infty}, 0, 1, \ldots, n-3$ and $W_0 = \langle \infty, 0, \overline{\infty}, 1, n-3, \ldots, \frac{n-4}{2}, \frac{n}{2}, \infty \rangle$, which represents a cycle $\vec{C}_{n-1}$ in $\mathcal{D}K_n$. We assume that the walk W_0 avoids the vertex $\frac{n-2}{2}$ and comprises the initial path $\infty \to 0 \to \overline{\infty} \to 1 \to n-3$ as well as the following arcs:

$$\begin{array}{rcll} n-k-1 & \to & k, & 2 \leq k \leq \frac{n-4}{2}, \\ k & \to & n-k-2, & 2 \leq k \leq \frac{n-4}{2}, \end{array}$$

and the terminal arc $\frac{n}{2} \to \infty$, see Figure 2, for $n = 10$.

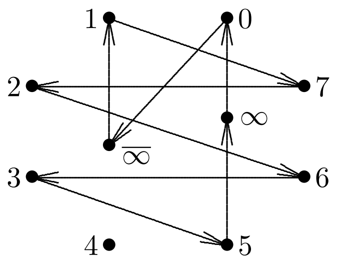

Figure 2. $n = 10$

Note that vertex labels in $\mathcal{D}K_n$ which are finite (not ∞ or $\overline{\infty}$) range over all integers modulo $\tilde{n}$ where

$$\tilde{n} := \begin{cases} n-1 & \text{for odd } n, \\ n-2 & \text{for even } n. \end{cases} \tag{1}$$

Definition 2. Given any positive integer x, let W_x stand for the sequence $W_0 + x$ obtained from the sequence W_0 by adding x to each term of W_0, the addition being modulo $\tilde{n}$, with $\infty + x = \infty$, $\overline{\infty} + x = \overline{\infty}$. Therefore the symbol W_x stands for a walk obtained from the walk W_0 by x-fold rotation γ^x around either the fixed vertex ∞ if n is odd or the two fixed vertices ∞ and $\overline{\infty}$ if n is even, that is, $W_x = \gamma^x[W_0]$ with convention that $\gamma[\cdot]$ is the extension of γ to sequences, γ^x is the iterate of γ, and γ is the permutation

$$(2) \qquad \gamma := \begin{cases} (\infty)\,(0,\,1,\,2,\,\ldots,\,n-2) & \text{for odd } n, \\ (\infty)(\overline{\infty})\,(0,\,1,\,2,\,\ldots,\,n-3) & \text{for even } n. \end{cases}$$

Definition 3. Using the abbreviation $\tilde{n}$, define W, $W = W_0\,W_1 \ldots W_{\tilde{n}-1}$, to be the unification of the $\tilde{n}$ sequences W_x such that the neighboring symbols ∞ are glued together to the single ∞. Arcs of $\mathcal{D}K_n$ which are not represented in W constitute either the $(n-1)$-cycle

$$C = \langle 0,\, n-2,\, n-1,\, \ldots, 0\rangle \qquad \text{if } n \text{ is odd}$$

or otherwise the union of three cycles of which one, $C'' := \langle \infty,\, \overline{\infty},\, \infty\rangle$, is of length 2 but $C := \langle 0,\, n-3,\, n-4,\, \ldots, 0\rangle$ and $C' := \langle 0,\, 1,\, \ldots, n-3,\, 0\rangle$ are both of length $n-2$.

Note that W represents a closed walk of $\mathcal{D}K_n$. In fact, the walk is a tour because arcs do not repeat for the following reasons:

- The initial tour W_0 does not include any arc joining vertices which are fixed under γ.
- The indegree and outdegree of any fixed vertex (∞ or $\overline{\infty}$) are (at most) one in W_0. Hence any arc incident to a fixed vertex does not repeat in W.
- Arc lengths along the $\tilde{n}$-cycle of γ for all remaining arcs in the initial tour W_0 are mutually distinct. Recall that the length of the arc $u \to v$, defined to be $v - u \bmod \tilde{n}$, is an invariant under γ.

2.2. Useful conventions

We assume that the names of vertices as well as the related subscript x which refers to x-fold rotation γ^x both read modulo $\tilde{n}$. Given a term v of the sequence $W_x =< t_0, t_1, \ldots, t_{n-1} >$, an integer j is called a *position* of v in W_x whenever $v = t_j$, $0 \le j < n-1$. Hence the position j of v in W_x is uniquely determined. In particular, 0 is defined to be the position of ∞ in each W_x. However, if $v = \infty$, we use the symbols ∞_x and ∞'_x to denote the first and second appearance of the vertex ∞ in W_x; in fact, $\infty'_x = \infty_{x+1}$. Note that, for even n, $j = 2$ is the position of $\overline{\infty}$ in any W_x.

"*u, v encoding*". Letters v and u stand for vertices only. Then given a vertex w with $w = u$ or v, any subscript at w is assumed to refer to a rotation so that w_x denotes the image of w under the x-fold rotation γ^x. Then $w_x = (w + x) \bmod \tilde{n}$ and therefore $w = w_0$ for each vertex w which is not a fixed point of γ, otherwise $\infty_x = \infty$ and $\overline{\infty}_x = \overline{\infty}$ for each subscript x. Hence, if $w \neq \infty$ then the 'situation' of w_x on W_x, that is, either the position of w_x in W_x or the fact that W_x avoids w_x, is the same as that of w $(= w_0)$ on W_0.

Given a vertex v taken as either a term of W_k or a vertex omitted by the tour W_k, we define the *preimage* of v to be ∞_0, the first vertex of W_0, if $v = \infty_x$ for any x. Otherwise, if $v \neq \infty_x$, the *preimage* of v is to be the vertex u, $u = u_0$ (possibly a term of W_0), such that $\gamma^k u_0 = v$, i.e., $u = (v - k) \bmod \tilde{n}$, see (1) for $\tilde{n}$.

2.3. Repetition distance, girth, and path structure

A trail is called *nonsimple* if a vertex is repeated. A *girth* of a trail (simple or nonsimple trail) is defined to be the least length among closed walks being sections of the trail. Thus the girth of a trail can be larger than the girth of the multidigraph induced by the arcs of the trail.

We intend to present a method of how to cut off all prescribed paths. To this end, we shall investigate the path structure of the tour W. Because of the rotational structure of W, it suffices to determine longest sections of W which are paths starting at any vertex v (which is not the last vertex) in the initial tour W_0, $v \neq \lfloor \frac{n-1}{2} \rfloor$ (the vertex omitted by W_0).

For each vertex v which appears both in W_0 and in W_1, let the *repetition distance* of v_0, denoted by $r(v_0)$, be the smallest nonzero length among $v_0 \rightsquigarrow v$ closed subwalks of the tour $W_0\, W_1$, where v_0 stands for the first appearance of v in W_0. Therefore $r(v)$ is not defined for $v = \lfloor \frac{n-1}{2} \rfloor$ $(\notin W_0)$ and for $v = \lfloor \frac{n+1}{2} \rfloor$ $(\notin W_1)$. Due to the rotational structure of W, the *girth of* W is the minimum value of the function $r(\cdot)$.

Lemma 4. *The girth of the tour W equals $n - 3$.*

Proof. It is enough to show that $\min r(\cdot) = n - 3$. For this purpose, note that $r(\infty) = r(\overline{\infty}) = n - 1$. For the remaining values of v_0, $r(v_0)$ is the length of the $v_0 \rightsquigarrow u_1$ tour where u_1 is a term of W_1 such that $u_1 = v$ and therefore the preimage u $(= u_0)$ of u_1 is equal to $u_0 = (v_0 - 1) \bmod \tilde{n}$. Note that the position of v, $v = u_1$, in W_1 is the same as that of u_0 in W_0. Therefore $r(v_0) = n - 1 + \rho$ where ρ is the value of the difference: position of the term u_0 in the sequence W_0 minus that of v_0. In other words if ℓ is the length of the subpath of $\langle W_0 \rangle \backslash \infty$ connecting u_0 and v_0 then $\rho = -\ell$ if u_0 precedes v_0 and $\rho = \ell$ if u_0 follows v_0 on the path. Applying Definition 1 we get the following.

For even $n \geq 6$:
$r(v) = n - 3$ for $v = 1, 2, \ldots, \frac{n-4}{2}$
because paths $u_0 = 0 \to \overline{\infty} \to 1 = v$ and
$u_0 = k - 1 \to n - k - 1 \to k = v$ (for $2 \leq k \leq \frac{n-4}{2}$) are in W_0;
$r(v) = n + 1$ for $v = \frac{n+2}{2}, \frac{n+4}{2}, \ldots, n - 3$
because paths $v = n - k - 1 \to k \to n - k - 2 = u_0$ (for $2 \leq k \leq \frac{n-4}{2}$) are in

W_0; moreover,
$r(0) = n + 2$ since the path $v = 0 \to \overline{\infty} \to 1 \to n - 3 = u_0$ is in W_0.
For odd $n \geq 5$:
$r(v) = n - 3$ for $v = 2, 3, \ldots, \frac{n-3}{2}$
because paths $u_0 = k - 1 \to n - k \to k = v$ (for $2 \leq k \leq \frac{n-3}{2}$) are in W_0;
$r(v) = n + 1$ for $v = \frac{n+3}{2}, \frac{n+5}{2}, \ldots, n - 2, 0$
because paths $v = n - k \to k \to n - k - 1 = u_0$ (for $2 \leq k \leq \frac{n-3}{2}$)
and $v = 0 \to 1 \to n - 2$ are in W_0; moreover,
$r(1) = n - 2$ since $u_0 = 0$ and the arc $0 \to 1$ is in W_0.
Hence $\min r(\cdot) = n - 3$. ■

Corollary 5. *The tour W can be cut freely into paths of lengths not greater than $n - 4$.*

Lemma 6. *Penultimate vertices of tours W_x separate W into $\tilde{n}$ hamiltonian paths.*

Proof. The penultimate vertex of W_x, say v_x, is omitted by the next tour W_{x+1} because this is clearly true if $x = 0$, with $v_0 = \lfloor \frac{n+1}{2} \rfloor$. Furthermore, the two vertices which immediately follow v_x on W have preimages ∞_0 and 0, respectively; with respective repetition distances $n-1$ and at least $n+1$, which are sufficiently large. ■

Let Z denote the following set of vertices.
$Z = \{z_1 = 0,\ z_2 = \frac{n}{2},\ z_3 = \frac{n+2}{2}, \ldots, z_{\frac{n-2}{2}} = n - 3,\ z_{\frac{n}{2}} = \overline{\infty},\ z_{\frac{n+2}{2}} = \infty\}$ for even n,
$Z = \{z_1 = 0, z_2 = \frac{n+1}{2},\ z_3 = \frac{n+3}{2}, \ldots, z_{\frac{n-1}{2}} = n - 2\}$ for odd n.

Lemma 7. *For even n, starting at any term of W, if the preimage of the term is an $z_i \in Z$, then $\frac{n}{2} + 2 - i$ paths of length $n - 3$ can be cut off from W one by one going forwards along W and $i - 1$ such paths going backwards. If n is odd, however, cutting off such paths from W can be continued in either direction until a path of length 4 remains of W, $4 = (n - 1)^2 \bmod (n - 3)$.*

Proof. Notice that, for every $v \in Z$, either $r(v) \geq n - 2$ or $v \notin W_1$ and $r(v)$ is not defined. Therefore, by Lemma 4, $v = v_0 = z_i \in Z$ is the initial vertex of a certain $v_0 \rightsquigarrow u$ subpath of W of length $n - 3$. Moreover, notation is chosen so that if the initial vertex z_i is not the last vertex in Z then the terminal vertex u of the path has preimage z_{i+1} which immediately follows z_i in Z. In fact, $u = z_{i+1}$ if $v_0 = 0 = z_i$ or if $v_0 = \overline{\infty}$. Otherwise $u = 1 + z_{i+1}$, a vertex of W_1. This is so because, by Definition 1, the tour W_0 has length $n - 1$ and includes the following $z_{i+1} \rightsquigarrow z_i$ paths of length two for each $i < |Z|$. For even n the paths are: $\frac{n}{2} \to \infty \to 0$, $n - k - 1 \to k \to n - k - 2$ $(2 \leq k \leq \frac{n-4}{2})$, $\overline{\infty} \to 1 \to n - 3$ and $\infty \to 0 \to \overline{\infty}$. Similarly, for odd n, the paths: $\frac{n+1}{2} \to \infty \to 0$, $n-k \to k \to n-k-1$

$(2 \le k \le \frac{n-3}{2})$ and $0 \to 1 \to n-2$ are in W_0. Additionally, if n is odd and $v = n-2 = z_{\frac{n-1}{2}}$, which is the last vertex in Z, then $u = 0 = z_1$ because the path $0 \to 1 \to n-2$ is in W_0. Therefore, each term v_1 of W_1 with preimage $v_1 - 1 = z_i \in Z$ such that $i \ne 1$ for even n is also the terminal vertex of a subpath of $W_0 W_1$ of length $n-3$. Due to the rotational structure of W and the fact that $|Z| = \frac{n}{2} + 1$ for even n, the result follows. ■

3. Proofs

***Proof of Theorem* 1.** Assume that $n \ge 5$ due to Theorem C. Let ψ, (and mnemonic letters) τ and θ denote the number of prescribed paths of length $n-1$ (hamiltonian paths), $n-2$, and $n-3$, respectively, in a decomposition of $\mathcal{D}K_n$. Assume that $1 \le \psi < n$ (otherwise we apply Theorems A and B). Notice that the total length of all prescribed nonhamiltonian paths is divisible by $n-1$.

Consider three cases depending on the parity of n and the values of parameters τ and θ.

Case I : n is odd. The case $\psi = n-1$ is fixed by Lemma 6 and formula (1). Assume therefore that $1 \le \psi \le n-2$.

Assume that $\tau = 0$. We transform the cycle C into a hamiltonian $\frac{n-1}{2} \rightsquigarrow \frac{n+1}{2} \to \infty$ path, say C^*, and the tour W into an open $\infty \rightsquigarrow \frac{n-1}{2}$ trail, say W^*. To this end, we remove two arcs: the arc $a := (\frac{n+1}{2} \to \frac{n-1}{2})$ from the cycle C and the last arc $\frac{n-1}{2} \to \infty$ from W (it is the last arc of W_{n-2}). Next the last arc $\frac{n+1}{2} \to \infty$ of W_0 is removed and is attached to the path $C - a$ so that the path C^* is constructed, cf. Definition 3. The gap in W_0 is filled in by the arc a followed by $\frac{n-1}{2} \to \infty$. Thus the cycle W_0 (which avoids the vertex $\frac{n-1}{2}$, cf. Definition 1(i)) is transformed into a Hamilton cycle, say W_0^*. Consequently, the tour W becomes just the trail W^*.

Then C^* is one of ψ prescribed hamiltonian paths, the remaining $\psi - 1$ ones are cut off one by one going backwards along W^* from the last vertex $\frac{n-1}{2}$ of W^*, the penultimate vertex of W_{n-2}, cf. Lemma 6. What remains of W^* is a trail which includes W_0^*. The still required paths (of length $n-3$ or less) are cut off going forward along W^*. Note that the repetition distance of the vertex $\frac{n-1}{2}$ along $W_0^* W_1$ is seen to be $n-2$. Hence the girth of W^* is $n-3$, the same as that of W (Lemma 4). Therefore paths shorter than $n-3$ can freely be cut off. However, starting at the first vertex $v = \infty$ of W^* we cut off all θ paths of length $n-3$ one by one first. This can be clearly done if $\theta = 1$ or $n = 5$ and $\theta = 2$. Otherwise, for $n = 5$, after we cut off three paths of length $n-3$ we finish up at the vertex $u = 1$ on W_1. If $n \ge 7$ and $\theta \ge 2$, after we cut off two paths of length $n-3$ we finish up at the vertex $u = \frac{n+7}{2}$ which is the fifth vertex from the last one in W_1. In both cases $u = z+1$ for some $z \in Z$. Thus we can continue cutting off all remaining paths of length $n-3$ and all shorter paths later on.

Assume that $\tau \geq 1$. For increasing values of τ, we construct trails, denoted by $W^{(\tau)}$, which will be cut into required paths of length less than $n-2$ only, i.e., we first show how to get all τ paths of length $n-2$, $\tau \leq 5$, and all ψ hamiltonian paths.

We get the first path of length $n-2$ from the $(n-1)$-cycle C by removal of one arc, the removed arc being $\frac{n+3}{2} \to \frac{n+1}{2}$ if $\tau \leq 2$, otherwise $0 \to n-2$. Due to Lemma 6, starting at the penultimate vertex $v' = \frac{n+3}{2}$ in W_1 we cut off from W all required, ψ, hamiltonian paths one after another, ending up at the penultimate vertex, v'', in $W_{\psi+1}$, $v'' = \frac{n+3}{2} + \psi$. (Going further from v'' we get a path of length $n-2$, which we utilize below in case $\tau = 5$ only.)

Let W' stand for the $v'' \rightsquigarrow v'$ trail which remains of W. If $\tau \leq 2$ then we append the arc $\frac{n+3}{2} \to \frac{n+1}{2}$ to the last vertex v' of W'. Then the resulting trail is just $W^{(1)}$. Since the appended vertex $\frac{n+1}{2}$ is missing in W_1, in the case $\tau = 2$, going backwards along $W^{(1)}$, we cut off the second path of length $n-2$. What is left is clearly $W^{(2)}$.

Consider the case $\tau \geq 3$. Starting at v', the last vertex of W', and going backwards we cut off two paths of length $n-2$ where either path is a section of an $(n-1)$-cycle, W_1 or W_0, since the paths have the vertex ∞_0' in common. Therefore we arrive at the second vertex (in position 1, see Sect. 2.2) in W_0, the vertex being $0 = z_1 \in Z$. Then we append the arc $0 \to n-2$ from C to what is left of W'. Thus we get $W^{(3)}$. Notice that the appended vertex $n-2$ has position 1 in the preceding tour W_{n-2}. Hence, if $\tau \geq 4$, starting at the appended vertex $n-2$ and going backwards we cut off the fourth path of length $n-2$, ending at the vertex $n-3$ in position 3 in W_{n-2}, the preimage of the vertex being $n-2 \in Z$. Therefore, owing to Lemma 7, we continue going backwards and we cut off one after another all prescribed paths of length $n-3$ and next all shorter paths. What remains if $\tau = 5$ is the fifth path of length $n-2$ (as is stated above).

It remains to complete the cases $\tau \leq 3$. Then the vertex v'' with preimage $\frac{n+1}{2} \in Z$ is the initial vertex of each trail $W^{(\tau)}$. Therefore, due to Lemma 7, going forwards from v'' we cut off prescribed paths of length $n-3$ and next remaining ones, which ends the proof for odd n.

Case II : n is even and $\theta + \lfloor \tau/2 \rfloor \geq \frac{n+4}{2}$. Hence, since $\tau \leq 5$, $\theta \geq n/2$. On the other hand, since $\psi \geq 1$, $\tau + \theta \leq n+1$ for $n \geq 8$ and $\tau + \theta \leq n+2$ for $n = 6$. We first construct three long paths. To this end we use all arcs of C', one arc of C'', and most arcs of W_0 and C. We construct a path, say P^*, of length $n-3$ which comprises the section $\frac{n-4}{2} \to \frac{n-6}{2} \rightsquigarrow \frac{n}{2}$ of C followed by the arc $\frac{n}{2} \to \infty$ (cut off from W_0; notice that P^* avoids the vertices $\frac{n-2}{2}$ and $\overline{\infty}$). If $\tau \geq 2$ then P^* is transformed into a path of length $n-2$ by attaching the arc $\frac{n-2}{2} \to \frac{n-4}{2}$ (cut off from C). The hamiltonian path, say $P^\wedge$, is formed by the arcs $\overline{\infty} \to \infty$ from

C'', $\infty \to 0$ taken from W_0, $0 \to 1$ from C', the path $1 \rightsquigarrow \frac{n-4}{2}$ (of length $n-6$) cut off from W_0 (notice that this path omits the vertices: ∞, $\overline{\infty}$, 0, $\frac{n-2}{2}$ and $\frac{n}{2}$) and then the length-2 path $\frac{n-4}{2} \to \frac{n-2}{2} \to \frac{n}{2}$ cut off from C'. What remains of C' glued together by the subpath $0 \to \overline{\infty} \to 1$ of W_0 gives the path, say $P^{\vee}$, $\frac{n}{2} \rightsquigarrow \frac{n-4}{2}$ of length $n-3$, which avoids the vertices $\frac{n-2}{2}$ and ∞. If $4 \le \tau \le 5$ then we transform $P^{\wedge}$ and $P^{\vee}$ into two paths of length $n-2$ by moving the arc $\frac{n-2}{2} \to \frac{n}{2}$ from $P^{\wedge}$ to $P^{\vee}$. Notice that $2 - \lfloor \tau/2 \rfloor$ paths of length $n-3$ have been constructed.

Let P be the length-3 path with arcs $\frac{n-4}{2} \to \frac{n}{2}$ (from W_0), $\frac{n}{2} \to \frac{n-2}{2}$ (from C) and $\frac{n-2}{2} \to \infty$ (cut off from W_{n-3}). Let W^* be the trail $\frac{n-4}{2} \rightsquigarrow \frac{n-2}{2}$ which is P followed by what remained of W. Only the arcs of W^*, the arc $\infty \to \overline{\infty}$ (from C''), and—only if $\tau < 2$—the arc $\frac{n-2}{2} \to \frac{n-4}{2}$ (from C) are still left. The repetition distance, say r, of the vertex $u = \frac{n-2}{2}$ (which was avoided by W_0 and is the third vertex of P) in PW_1 is seen to be $r = n-2$. Therefore the first vertex, say v, of P ($v = \frac{n-4}{2}$) has repetition distance $n-2$ in PW_1 which is one greater than $r(v)$ in W_0W_1, see proof of Lemma 4 for $r(v) = n-3$. On the other hand, the second vertex of P, $\frac{n}{2}$, is avoided by W_1. Therefore the girth of the trail W^* remains $n-3$. Moreover, if necessary, all (if $n \ge 8$) or all but one (if $n = 6$, $\tau = 0$ and $\theta = 8$) of $\theta + \lfloor \tau/2 \rfloor - \frac{n+4}{2}$ (up to $\frac{n-2}{2}$) paths of length $n-3$ can be cut off from the initial section of W^*. To this end, notice that if a path of length $n-3$ is removed from the initial section of W^* then its terminal vertex w on W_1 is $w = \infty$ if $n = 6$, $w = \overline{\infty}$ if $n = 8$, and $w = \frac{n+6}{2}$ if $n \ge 10$. Therefore the preimage of w is z_4 whence, by Lemma 7, altogether up to $1 + \frac{n-4}{2}$ paths of length $n-3$ can really be cut off one after another.

In what follows in case II we cut off paths from the trail W^* going backwards along W^*. It can be seen that the number of still required paths of length $n-3$ equals $n/2$ unless $n = 6$, $\tau = 0$, and $\theta = 8$, in which case the number is 4.

Let $\tau \le 1$. Then starting at the last term of W^*, which is $\frac{n-2}{2}$, the penultimate vertex on W_{n-3}, we cut off the path, say $\tilde{P}$, of length $n-4$ with initial vertex being $\overline{\infty}$ on W_{n-3}. Since the vertex $\frac{n-4}{2}$ is omitted by W_{n-3}, we get a path of length $n-3$ by appending the available arc $\frac{n-2}{2} \to \frac{n-4}{2}$ (from C) to $\tilde{P}$. Since the preimage of the vertex $\overline{\infty}$ is $\overline{\infty} = z_{\frac{n}{2}}$, therefore going further backwards from $\overline{\infty}$ (which is on W_{n-3}), due to Lemma 7, we cut off $\frac{n-4}{2}$ paths of length $n-3$ one after another. Thus we end up at the penultimate vertex of $W_{\frac{n-2}{2}}$, the vertex being $v = 1$. We continue going backwards and, due to Lemma 6, we cut off all of required $\psi - 1$ hamiltonian paths, ending at the vertex $u = 2 - \psi$ on $W_{\frac{n}{2}-\psi}$.

Assume that $\tau \ge 2$. Then starting at the last term $\frac{n-2}{2}$ of W^*, which is the penultimate vertex on W_{n-3}, we cut off the path of length $n-2$ whose initial vertex is ∞ on W_{n-3}. Since the preimage of ∞ is $\infty = z_{\frac{n+2}{2}}$, therefore going backwards from ∞ on W_{n-3}, due to Lemma 7, we cut off $\frac{n-2}{2}$ paths of length $n-3$ one after another. Thus we end up at the penultimate vertex of $W_{\frac{n-4}{2}}$, the vertex being $v = 0$. We continue going backwards and, due to Lemma 6, we cut off all of required hamiltonian paths, ending at the vertex $u = 1 - \psi$ on $W_{\frac{n-2}{2}-\psi}$ (if $\tau < 4$) or $u = n - 2 - \psi$ on $W_{\frac{n-4}{2}-\psi}$ (if $\tau \geq 4$).

In order to complete the construction of long paths we consider two subcases.

Let τ be even ($\tau = 0, 2, 4$). Then all required paths of length $n-2$ have been constructed. Starting at the vertex u we cut off a path, say $\overline{P}$, of length $n-4$, with initial vertex $\overline{\infty}$. Since ∞ is not a vertex of $\overline{P}$, appending $\overline{P}$ to the available arc $\infty \to \overline{\infty}$ results in a next path of length $n-3$. One possibly still lacking path of length $n-3$ (only if $n = 6$, $\tau = 0$ and $\theta = 8$) can be cut off starting at the vertex $\overline{\infty}$ (on $W_{\frac{n}{2}-\psi}$).

Let τ be odd. Then we need one each path of lengths $n-2$ and $n-3$. Therefore starting at the vertex u we first cut off the path of length $n-2$, whose initial vertex is ∞. Next we remove a path, say $\tilde{P}$, of length $n-4$, whose initial vertex is in position 3. Hence $\overline{\infty}$ is not a vertex of $\tilde{P}$. That is why appending the available arc $\infty \to \overline{\infty}$ to $\tilde{P}$ we get a path of length $n-3$.

What remains of W^* is cut into required short paths (of length at most $n-4$).

Case III : n is even and $\theta + \lfloor \tau/2 \rfloor \leq \frac{n+2}{2}$. Recall that $1 \leq \psi \leq n-1$ and $\tau \leq 5$. As in the Case II, the last arc $\frac{n}{2} \to \infty$ of W_0 is replaced by length-2 path comprising the arc $a := (\frac{n}{2} \to \frac{n-2}{2})$ (removed from the cycle C) and the arc $\frac{n-2}{2} \to \infty$, the last arc of W_{n-3} (as well as of W). In place of W_0 we thus get a hamiltonian cycle, say W_0^*, because the added vertex $\frac{n-2}{2}$ is avoided by W_0, cf. Definitions 1 and 2.

Let W^* stand for the resulting image of W, W^* being an open trail $\infty \to 0 \rightsquigarrow \frac{n-2}{2}$, with W_0^* being the initial section of W^*. One can easily see (cp. the preceding case II) that the girth of the tour $W_0^* W_1$ is $n-2$ whence the girth of the trail W^* is $n-3$. Let C^* be the first decomposition part, a hamiltonian path in fact, obtained from the path $C - a$ by appending the arc $\frac{n}{2} \to \infty$ (from W_0) and next the arc $\infty \to \overline{\infty}$ (taken from C'').

Let $\psi' = n-4$ if $\psi \geq n-2$, otherwise $\psi' = \psi - 1$. Now ψ' hamiltonian paths are cut off one by one going backwards along W^* from the last vertex $\frac{n-2}{2}$ of W^*, the penultimate vertex of W_{n-3}, cf. Lemma 6. Then we stop at the vertex which (in u, v encoding, Sect. 2.2) is $v_{n-3-\psi'} = \frac{n-2}{2} - \psi'$ on $W_{n-3-\psi'}$, and this is $v_1 = \frac{n+2}{2}$ on W_1 if $\psi \geq n-3$. Thus $\psi' + 1$ hamiltonian paths are already cut off, that is, all required ones if $\psi \leq n-3$. Let W^{**} denote what remains of W^*.

Assume that $\psi \leq n-3$ and $n \geq 8$. The idea is to construct two or four long enough paths comprising all or most of arcs remaining from C'' and C' as well as all arcs of a certain initial section of W^{**} so that only a single tour with girth $n-3$ and containing the rest of W^{**} as a (terminal) section could remain to be dealt with.

Let $\tau' = \tau - 1$ if τ is odd and $\tau' = \tau$ otherwise. If τ is odd then a path of length $n-2$ is cut off starting at the last vertex ($v_{n-3-\psi'}$ on $W_{n-3-\psi'}$) of W^{**} and going backwards. Then the initial vertex of the path is $u_{n-3-\psi'} = \infty$, the first vertex on $W_{n-3-\psi'}$. Thus the number of paths of length $n-2$ still to be cut off is τ'. Moreover, if $\theta + \lfloor \frac{\tau}{2} \rfloor = \frac{n+2}{2}$ then a path of length $n-3$ is cut off going backwards again and starting either at $v_{n-3-\psi'}$ (if τ is even) or at $u_{n-3-\psi'}$ (if τ is odd). Let $\hat{W}$ denote what remains of W^{**}.

Let $\tau' = 4$, 4 being the largest possible value of τ'. Then $\psi \le n-4$ can be seen. A path of length $n-2$ (which we now construct) comprises the arcs: $\overline{\infty} \to \infty$ (from C''), $\infty \to 0$ (from W_0^*), $0 \to 1$ (from C'), the path $1 \rightsquigarrow \frac{n-4}{2}$ of length $n-6$ from W_0^* and the arc $\frac{n-4}{2} \to \frac{n-2}{2}$ (from C'). The second path is built of the arcs: $\frac{n-2}{2} \to \frac{n+2}{2}$ (from W_1), the path $\frac{n+2}{2} \rightsquigarrow 0$ of length $\frac{n-6}{2}$ from C', $0 \to \overline{\infty} \to 1$ (from W_0^*), the path $1 \rightsquigarrow \frac{n-4}{2}$ of length $\frac{n-6}{2}$ from C' and the arc $\frac{n-4}{2} \to \frac{n}{2}$ (from W_0^*). The next path of length $n-2$ consists of the arcs $\frac{n}{2} \to \frac{n-2}{2} \to \infty$ (from W_0^*) and the path $\infty \rightsquigarrow \frac{n+4}{2}$ of length $n-4$ from W_1. The last of those paths is built of the arcs: $\frac{n+4}{2} \to \frac{n-2}{2}$ (from W_1), $\frac{n-2}{2} \to \frac{n}{2} \to \frac{n+2}{2}$ (from C'), $\frac{n+2}{2} \to \infty$ (from W_1) and the path $\infty \rightsquigarrow \frac{n+8}{2}$ of length $n-6$ from W_2. Notice that the preimage of $\frac{n+8}{2}$ on W_2 is $\frac{n+4}{2} = z_4$ whence, by Lemma 7, up to $\frac{n-4}{2}$ paths of length $n-3$ can easily be cut off one after another going forwards along $\hat{W}$.

Assume that $\tau' < 4$. Hence there are either two or none paths of length $n-2$ which remain to be cut off. Let $M = (m_i)_{i=1}^{i=t}$ be the nonincreasing sequence of all, say t, designed lengths, m_i, of remaining paths (all nonhamiltonian) in the decomposition of $\mathcal{D}K_n$. Hence $m_1, m_2 \le n-2$ and $m_i \le n-3$ for all $i = 3, 4, \ldots, t$. If m_2, the second largest of those lengths, is small, $m_2 < n/2$, then we find a positive integer r such that $r < t$ and a sum $S_r = \sum_{i=r+1}^{t} m_i$ satisfies $\frac{n}{2} \le m_2 + S_r \le n-2$. Otherwise (if $m_2 \ge n/2$) we put $r = t$ and $S_r = 0$. We proceed analogously if $m_1 < n/2$ to find an integer p and a sum $S_p = \sum_{i=p+1}^{r} m_i$ such that $p < r$ and $\frac{n}{2} \le m_1 + S_p \le n-2$; if $m_1 \ge n/2$ we take $p = r$ and $S_p = 0$. Let $\overline{M} = (\overline{m})_{i=1}^{i=p}$ be the nonincreasing sequence obtained from the initial p-subsequence of M by replacing m_2 and m_1 with the the sums $m_2 + S_r$ and $m_1 + S_p$, respectively. It is clear that $\overline{m}_3 \le n-3$ and moreover $\overline{m}_3 < n/2$ if $\overline{M} \ne M$; thus $\overline{M} = M$ if $m_2 \ge n/2$. The idea behind this modification is clear. It is enough to find a decomposition prescribed by $\overline{M}$ because two too long paths can be split freely later on. Hence in what follows we assume that

$$k = \overline{m}_1 + \overline{m}_2$$

is the sum of lengths of two longest nonhamiltonian paths whence $n \le k \le 2n-4$.

Let $k = n$. Notice that, in particular, this is the case when $\theta \ge 2$ and $n = 6$. We easily build up two paths of length $\frac{n}{2}$. The first of them includes the arcs $\overline{\infty} \to \infty$ from C'', $\infty \to 0$ from W_0^*, and the path $0 \rightsquigarrow \frac{n-4}{2}$ from C'. The second path of length $\frac{n}{2}$ is the remaining section $\frac{n-4}{2} \rightsquigarrow 0$ of C'. Then we freely cut what remains of $\hat{W}$ into remaining (nonhamiltonian) paths, each of length at most $\frac{n}{2}$ $(\geq \overline{m}_3)$.

Assume that $k > n$. Let $s = 1$ if k is odd and $s = 0$ otherwise. Let $l = k - n + 1 + s$. Then l is odd and $3 \leq l \leq n-3$. Moreover, by the definition of k, $\overline{m}_1 - l = n - \overline{m}_2 - 1 - s$ and $\overline{m}_1 - l > 0$ because $\overline{m}_2 \leq n-2$.

We first build a path of length $\overline{m}_1$ by gluing together three arcs, namely, $\overline{\infty} \to \infty$ from C'', $\infty \to 0$ from W_0^*, and $0 \to 1$ from C', next the subpath $\pi_l := (1 \rightsquigarrow \frac{l-1}{2})$ of W_0^*, and the subpath $\gamma_l := (\frac{l-1}{2} \rightsquigarrow \overline{m}_1 - \frac{l+1}{2})$ of C'. This construction is correct because numbers $\overline{m}_1 - l$ and $l-3$ are lengths of γ_l and π_l, respectively, and, for $l > 3$, vertices of π_l are $1, 2, \ldots, \frac{l-1}{2}$ and from $n-3$ down to $n - \frac{l+1}{2}$ $(> \overline{m}_1 - \frac{l+1}{2})$ whence all vertices of γ_l are just in the in-between gap. Now the path $0 \to \overline{\infty} \to 1$ which has been left (separated from $\hat{W}$) together with two sections of the rest of C', namely the sections $(\overline{m}_p - \frac{l+1}{2}) \rightsquigarrow 0$ and $1 \rightsquigarrow \frac{l-1}{2} - s$, form a required path of length just $\overline{m}_2$. The length is so because all of $n-2$ arcs of C' have been used unless $s = 1$, and then the only arc which is still left is $v - 1 \to v$ where $v = \frac{l-1}{2}$. Let $\tilde{W}$ be the subtrail of $\hat{W}$ which still remains. Note that v is the initial vertex of $\tilde{W}$.

Let $s = 1$. Assume that the available arc $v-1 \to v$ is attached to the beginning of $\tilde{W}$ and that $\tilde{W}'$ denotes the resulting trail. Then the repetition distance of the vertex $v-1$ in $\tilde{W}'$ is equal to $r(v-1)$ (where $r(v-1) = n-3$, the equality being determined in the proof of Lemma 4) because the new distance, 1, between $v-1$ and v is exactly one smaller than that on W_0. Suppose that $\overline{m}_3 = n-3$. Then $\overline{M} = M$ and either $m_1 = m_2 = n-2$ or $m_1 = m_2 = n-3$. Hence $k = 2n-4$ or $k = 2n-6$ is even and $s = 0$, a contradiction. Thus $\tilde{W}'$ can be freely cut into paths of length at most $n-4$.

Let $s = 0$. Suppose that $\overline{M} = M$ and $m_3 = n-3$. Since either $l = n-3$ (if $m_1 = m_2 = n-2$) or $l = n-5$ (if $m_1 = m_2 = n-3$), the initial vertex of $\tilde{W}$ is $v = \frac{l-1}{2} = \frac{n-4}{2}$ or $v = \frac{n-6}{2}$, respectively. As the girth of $W_0^* W_1$ is $n-2$, we cut off from $\tilde{W}$ the path, $v \rightsquigarrow u$, of length $n-3$ starting at v. Then its terminal vertex u is in W_1. For $l = n-3$: $u = \infty$ if $n = 6$, $u = \overline{\infty}$ if $n = 8$, $u = \frac{n+6}{2}$ if $n \geq 10$. Moreover, for $l = n-5$: $u = \infty$ if $n = 8$, $u = \overline{\infty}$ if $n = 10$, $u = \frac{n+8}{2}$ if $n \geq 12$. Notice that the preimage of u is z_4 if $l = n-3$ and z_5 if $l = n-5$. Hence, by Lemma 7, starting at u and going forwards along $\tilde{W}$, we can cut off up to $\frac{n-4}{2}$ if $l = n-3$ or $\frac{n-6}{2}$ if $l = n-5$ lacking paths of length $n-3$. Finally, shorter paths may be cut off.

Assume that $n-2 \leq \psi \leq n-1$. Then $\tau \leq 2$. The next hamiltonian path is obtained from what remains of W_1 (which avoids the vertex $\frac{n}{2}$) by replacing its last arc $\frac{n-2}{2} \to \frac{n+2}{2}$ by the path $\frac{n-2}{2} \to \frac{n}{2} \to \frac{n+2}{2}$ of length two and removed from C'. Only one more hamiltonian path is required if $\psi = n-1$.

Let $\psi = n-2$. We proceed similarly to the above. Let $k = \overline{m}_1 + \overline{m}_2$, where

$n \le k \le 2n-4$. Moreover, let $s' = 1$ if k is even and $s' = 0$ otherwise. Let $l' = k - n + 2 + s'$. Thus l' is odd and $l' \ge 3$. To cut off two paths of length $\overline{m}_1$ and $\overline{m}_2$ we proceed analogously as above putting l' and s' in place of l and s, respectively. Another difference is only that the available arc from W_1 is used to replace the path removed from C'. Namely, both of still remaining sections of W_0^* and C' are cut into four pieces so that together with available single arcs from C'' and W_1 they give a single trail, say $\hat{W}$, consisting of the arcs $\overline{\infty} \to \infty$, $\infty \to 0$, $0 \to 1$, together with the subpath $1 \rightsquigarrow \frac{l'-1}{2}$ of W_0^*, the subpath $\frac{l'-1}{2} \rightsquigarrow \frac{n-2}{2}$ of C', the arc $\frac{n-2}{2} \to \frac{n+2}{2}$ removed from W_1, the path $\frac{n+2}{2} \rightsquigarrow 0$ cut off from C', the path $0 \to \overline{\infty} \to 1$ from W_0^*, the path $1 \rightsquigarrow \frac{l'-1}{2}$ cut off from C' and the path $\frac{l'-1}{2} \rightsquigarrow \infty$ from W_0^*. One can check (just as above) that starting at the first vertex of $\tilde{W}$ we are able to cut off a path of length $\overline{m}_1$, next of length $\overline{m}_2$, and then all required short paths.

If $\psi = n-1$ then we take $l' = n-1$ and we construct the trail $\hat{W}$ as above. The last of required hamiltonian paths is the initial section of $\hat{W}$. What remains of $\hat{W}$ is the trail $\frac{n-2}{2} \rightsquigarrow \frac{n-2}{2} \to \infty$ which, in fact, is a cycle of length $n-2$ with one pendant arc. Therefore all still required (only nonhamiltonian) paths can be easily cut off. ■

***Proof of Theorem* 3.** Let $M^* = (m_i^*)_{i=1}^{i=t}$ be a non-increasing sequence of prescribed lengths $m_i^* \le n-1$, $m_i^* \ne n-2$, in a would-be path decomposition of a given ${}^{\lambda}\mathcal{D}K_n$ where t is a number of paths. Construct a new sequence M, $M = (m_j)$, recursively from M^* by possibly splitting each of some, less than λ, original terms into two new ones so that (m_j) could be the concatenation of λ sections $(m_{k_{i-1}+1}, m_{k_{i-1}+2}, \ldots, m_{k_i})$, where m_{k_i} is the last length in section i, $i = 1, \ldots, \lambda$, $k_0 := 0 < k_1 < \cdots < k_\lambda$ with $k_\lambda \ge t$, and such that

(i) the terms m_j in each section sum up to the size $n(n-1)$ of a complete n-vertex subdigraph,

(ii) the first and last terms of the two sequences mutually coincide ($m_1 = m_1^*$ and $m_{k_\lambda} = m_t^*$),

(iii) removing two terms, the first and the last, from any of the λ sections gives a section of the original sequence M^*, and

(iv) any two neighboring extreme terms m_{k_i}, m_{k_i+1} of neighboring sections either are neighboring terms in M^* or their sum $m_{k_i} + m_{k_i+1}$ is a term there.

Consider an ordered decomposition of the complete multidigraph into λ copies of $\mathcal{D}K_n$. Match consecutive sections of the sequence M with consecutive copies of $\mathcal{D}K_n$. Decompose one by one the consecutive copies into paths as prescribed by the path lengths in the corresponding sections. Such decompositions exist due to Corollary 2. For consecutive pairs of neighboring sections i and $i+1$, if the two neighboring extreme terms of the sections are obtained by splitting a term of M^* then we only permute vertices of the $(i+1)st$ complete subdigraph so that the first path in its decomposition glued together to the last path in the decomposition of the preceding complete subdigraph could make up a path of originally prescribed length. ■

Though the proofs in this paper seem to be similar to those in [6, 7] there are substantial differences. The essential difference is that in the former papers the initial tour W_0 has n arcs (and is even a hamiltonian cycle of $\mathcal{D}K_n$ if n is odd) so that the final tour, W, is an Euler tour and of girth $n-2$. Consequently, constructions in the present paper are more involved but perhaps more instructive about how to deal with what still remains to be done.

4. Concluding remarks

The constructions used in [6] for odd n are extended in [7] to prove our Conjecture if the numbers of long paths are large enough. Proofs in [7] make use of the above Corollary 2.

As above the symbols ψ and τ denote the number of prescribed hamiltonian paths and those of length $n-2$, respectively, in a decomposition of $\mathcal{D}K_n$.

Proposition D (Meszka and Skupień [7, Corollaries 2 and 4])**.** *For odd n, the conjecture is true if either $n \leq 15$ or else $\psi \leq 2$, $\psi \geq n-5$, $\tau \leq 5$, or $\tau \geq (n-3)/2$.*

Theorem E (Meszka and Skupień [7, Theorem 3])**.** *For odd n, the complete n-vertex digraph $\mathcal{D}K_n$ is decomposable into paths of arbitrarily prescribed lengths provided that $\tau \geq \frac{n+3-\psi}{2}$ and the lengths sum up to the size $n\,(n-1)$ of $\mathcal{D}K_n$.*

Conjecture stated in Introduction still remains open. Nevertheless, for odd n, Conjecture remains unsettled only when $n \geq 17$, $3 \leq \psi \leq n-6$ and $6 \leq \tau < \min\{(n-3)/2,\ (n+3-\psi)/2\}$, cf. Proposition D and Theorem E. Moreover, for any $n \leq 19$, Conjecture has been verified by a computer. Therefore, a counterexample, if exists, must have more than 19 vertices.

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