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NOWHERE-ZERO MODULAR EDGE-GRACEFUL GRAPHS

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Abstract

For a connected graph G of order $n \geq 3$, let $f : E(G) \to \mathbb{Z}_n$ be an edge labeling of G. The vertex labeling $f' : V(G) \to \mathbb{Z}_n$ induced by f is defined as $f'(u) = \sum_{v \in N(u)} f(uv)$, where the sum is computed in $\mathbb{Z}_n$. If f' is one-to-one, then f is called a modular edge-graceful labeling and G is a modular edge-graceful graph. A modular edge-graceful labeling f of G is nowhere-zero if $f(e) \neq 0$ for all $e \in E(G)$ and in this case, G is a nowhere-zero modular edge-graceful graph. It is shown that a connected graph G of order $n \geq 3$ is nowhere-zero modular edge-graceful if and only if $n \not\equiv 2 \pmod 4$, $G \neq K_3$ and G is not a star of even order. For a connected graph G of order $n \geq 3$, the smallest integer $k \geq n$ for which there exists an edge labeling $f : E(G) \to \mathbb{Z}_k - \{0\}$ such that the induced vertex labeling f' is one-to-one is referred to as the nowhere-zero modular edge-gracefulness of G and this number is determined for every connected graph of order at least 3.

Keywords: modular edge-graceful labelings and graphs, nowhere-zero labelings, modular edge-gracefulness.

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1. Introduction

Over the past few decades the subject of graph labelings has been growing in popularity. Gallian [7] has compiled a periodically updated survey of many kinds of labelings and numerous results, obtained from well over a thousand referenced research articles. The origin of the study of graph labelings as a major area of graph theory can be traced to a research paper by Rosa [15]. Among the labelings he introduced was a vertex labeling he referred to as a β-valuation. Let G be a graph of order n and size m. A one-to-one function $f : V(G) \to \{0, 1, 2, \ldots, m\}$ is called a *β-valuation* (or a *β-labeling*) of G if

$$\{|f(u) - f(v)| : uv \in E(G)\} = \{1, 2, \ldots, m\}.$$

In 1972 Golomb [9] referred to a β-labeling as a *graceful labeling* and a graph possessing a graceful labeling as a *graceful graph*. Eventually, it was this terminology that became standard. One of the best known conjectures in this area is Graceful Tree Conjecture, due to Ringel and Kotzig.

Conjecture. *Every tree is graceful.*

In 1985 Lo [12] introduced a dual type of labeling — this one an edge labeling. Let G be a connected graph of order $n \geq 2$ and size m. For a vertex v of G, let $N(v)$ denote the neighborhood of v. An *edge-graceful labeling* of G is a bijective function $f : E(G) \to \{1, 2, \ldots, m\}$ that gives rise to a bijective function $f' : V(G) \to \{0, 1, 2, \ldots, n-1\}$ given by $f'(v) = \sum_{u \in N(v)} f(uv)$, where the sum is computed in $\mathbb{Z}_n$. A graph that admits an edge-graceful labeling is called an *edge-graceful graph*. In the definition of an edge-graceful labeling of a connected graph G of order $n \geq 2$ and size m, the edge labeling f is required to be one-to-one. Since, however, the induced vertex labels $f'(v)$ are obtained by addition in $\mathbb{Z}_n$, the function f is actually a function from $E(G)$ to $\mathbb{Z}_n$ and is in general not one-to-one. Dividing m by n, we obtain $m = nq + r$, where $q = \lfloor m/n \rfloor$ and $0 \leq r \leq n-1$. Hence in an edge-graceful labeling of G, $q+1$ edges are labeled i for each i with $1 \leq i \leq r$ and q edges are labeled i for each i with $r+1 \leq i \leq n$ (in $\mathbb{Z}_n$). Thus this edge labeling $f : E(G) \to \mathbb{Z}_n$ is a one-to-one function only when $m = n-1$ or $m = n$.

In 2008 a vertex coloring of a graph was introduced in [13] in connection with finding a solution to a checkerboard problem posted by Gary Chartrand. For a graph G without isolated vertices, let $c : V(G) \to \mathbb{Z}_k$ ($k \geq 2$) be a vertex coloring of G where adjacent vertices may be colored the same. Then a vertex coloring c' of G is defined such that $c'(v)$ is the sum in $\mathbb{Z}_k$ of the colors of the vertices in the neighborhood of v for each $v \in V(G)$. The coloring c is called a *modular k-coloring* of G if $c'(u) \neq c'(v)$ in $\mathbb{Z}_k$ for every pair u, v of adjacent vertices of G. The *modular chromatic number* of G is the minimum k for which G has a modular k-coloring. This coloring was studied further in [14], which led to a complete solution of the checkerboard problem under investigation. Furthermore, modular colorings are closely related to *sigma colorings* in graphs (see [4]). The modular coloring described above led to an edge version introduced in [10], which was inspired by the research of finding various methods to distinguish every pair of adjacent vertices in a graph by means of edge colorings (see [1, 2, 6, 16] and [5, p. 385], for example). For a graph G without isolated vertices, let $c : E(G) \to \mathbb{Z}_k$ ($k \geq 2$) be an edge coloring of G where adjacent edges may be colored the same.

Then a vertex coloring c' is defined such that $c'(v)$ is the sum in $\mathbb{Z}_k$ of the colors of the edges incident with v for each $v \in V(G)$. An edge coloring c is a *modular k-edge coloring* of G if $c'(u) \neq c'(v)$ in $\mathbb{Z}_k$ for all pairs u, v of adjacent vertices of G. The *modular chromatic index* of G is the minimum k for which G has a modular k-edge coloring. Combining the concepts of graceful labeling and modular edge coloring gives rise to a modular edge-graceful labeling, as we describe next.

Let G be a connected graph of order $n \geq 3$ and let $f : E(G) \to \mathbb{Z}_n$, where f need not be one-to-one. Let $f' : V(G) \to \mathbb{Z}_n$ such that $f'(v) = \sum_{u \in N(v)} f(uv)$, where the sum is computed in $\mathbb{Z}_n$. If f' is one-to-one, then f is called a *modular edge-graceful labeling* and G is a *modular edge-graceful graph*. Consequently, every edge-graceful graph is a modular edge-graceful graph. It turns out that this concept was introduced in 1991 by Jothi [8] under the terminology of *line-graceful graphs* (also see [7]). It was known that if G is a connected graph of order $n \geq 3$ for which $n \equiv 2 \pmod 4$, then G is not modular edge-graceful. Furthermore, it was conjectured that if T is a tree of order $n \geq 3$ for which $n \not\equiv 2 \pmod 4$, then T is modular edge-graceful (see [7]). This conjecture was verified in [11]. In fact, the conjecture is not only true for trees but for all connected graphs.

Theorem 1.1 [11]**.** *A connected graph of order $n \geq 3$ is modular edge-graceful if and only if $n \not\equiv 2 \pmod 4$.*

For every connected graph G of order n, there is a smallest integer $k \geq n$ for which there exists an edge labeling $f : E(G) \to \mathbb{Z}_k$ such that the induced vertex labeling $f' : V(G) \to \mathbb{Z}_k$ defined by $f'(v) = \sum_{u \in N(v)} f(uv)$, where the sum is computed in $\mathbb{Z}_k$, is one-to-one. The number k is defined in [11] as the *modular edge-gracefulness* $\text{meg}(G)$ of G. Thus $\text{meg}(G) \geq n$ and $\text{meg}(G) = n$ if and only if G is a modular edge-graceful graph of order n and if G is not modular edge-graceful, then $\text{meg}(G) \geq n + 1$. In fact, $\text{meg}(G)$ is known for every connected graph G, as we state next.

Theorem 1.2 [11]**.** *If G is a nontrivial connected graph of order $n \geq 6$ that is not modular edge-graceful, then* $\text{meg}(G) = n + 1$.

If G is a modular edge-graceful spanning subgraph of a graph H, where G and H are connected, then a modular edge-graceful labeling of G can be extended to a modular edge-graceful labeling of H by assigning 0 to each edge of H that does not belong to G. Thus modular edge-graceful labelings of a graph that assign 0 to some edges of the graph play an important role in establishing Theorems 1.1 and 1.2. For this reason, we now investigate those modular edge-graceful labelings in which 0 is not permitted. This gives rise to a new concept along with additional challenging problems. More formally, for a connected graph G of order $n \geq 3$ let $f : E(G) \to \mathbb{Z}_n - \{0\}$, where f need not be one-to-one and let $f' : V(G) \to \mathbb{Z}_n$ be defined by $f'(u) = \sum_{v \in N(u)} f(uv)$, where the sum is computed in $\mathbb{Z}_n$. If f'

is one-to-one, then f is called a *nowhere-zero modular edge-graceful labeling* and G is a *nowhere-zero modular edge-graceful graph.* In this work, we first show (in Section 2) that if G is a connected graph of order $n \geq 3$ where $n \not\equiv 2 \pmod 4$, then there is a modular edge-graceful labeling $f : E(G) \to \mathbb{Z}_n$ such that $f(e) \neq 0$ for all $e \in E(G)$ with at most one exception. In Section 3 we determine all nowhere-zero modular edge-graceful trees. Finally, we present a characterization of all nowhere-zero modular edge-graceful graphs in Section 4 and determine the nowhere-zero modular edge-gracefulness of every connected graph in Section 5. We refer to the book [3] for graph theory notation and terminology not described in this paper. Henceforth, we assume all graphs under consideration are connected graphs of order at least 3.

2. One Zero is Sufficient

In this section, we show that if G is a modular edge-graceful graph of order $n \geq 3$ that is not nowhere-zero, then there is a modular edge-graceful labeling $f : E(G) \to \mathbb{Z}_n$ such that $f(e) \neq 0$ for all $e \in E(G)$ with one exception, that is, one zero is sufficient. Furthermore, for each prescribed edge e^* of G, there is a modular edge-graceful labeling $f^* : E(G) \to \mathbb{Z}_n$ such that $f^*(e) \neq 0$ for all $e \in E(G) - \{e^*\}$. First, we present a lemma.

Lemma 2.1. *Let G be a connected modular edge-graceful graph of order $n \geq 3$, where $n \not\equiv 2 \pmod 4$ and let $f : E(G) \to \mathbb{Z}_n$ be a given modular edge-graceful labeling of G. If $P_k = (v_1, v_2, \ldots, v_k)$ is a path of order $k \geq 3$ in G, then there is a modular edge-graceful labeling $g : E(G) \to \mathbb{Z}_n$ of G that satisfies the following four conditions:*

(1) *$g(e) = f(e)$ for all $e \notin E(P_k)$,*

(2) *$g'(v) = f'(v)$ for all $v \notin V(P_k)$,*

(3) *$\{g'(v_i) : 1 \leq i \leq k\} = \{f'(v_i) : 1 \leq i \leq k\}$ and*

(4) *$g(v_i v_{i+1}) \neq 0$ for all i with $1 \leq i \leq k-2$.*

Proof. We proceed by induction on k. For $k = 3$, let $P_3 = (v_1, v_2, v_3)$. If $f(v_1v_2) \neq 0$, then let $g = f$. Thus, we may assume that $f(v_1v_2) = 0$. Suppose that $f'(v_1) = a$, $f'(v_2) = b$ and $f'(v_3) = c$. Define a labeling $g : E(G) \to \mathbb{Z}_n$ by

$$g(e) = \begin{cases} f(e) & \text{if } e \notin E(P_3), \\ f(e) + (c-a) & \text{if } e = v_1v_2, \\ f(e) - (c-a) & \text{if } e = v_2v_3. \end{cases}$$

By the definition of g, conditions (1) and (2) hold. Since $g(v_1v_2) = f(v_1v_2)+(c-a)$ $= c - a$ and $a \neq c$, it follows that $g(v_1v_2) \neq 0$ and so (3) holds. Furthermore, $g'(v_1) = c$, $g'(v_2) = b$ and $g'(v_1) = a$ and so (4) holds.

Assume for some integer $k \geq 4$ that the result holds for all paths of order k' in G where $3 \leq k' < k$. Let $P_k = (v_1, v_2, \ldots, v_k)$ be a path of order $k \geq 4$ in G. First, consider the subpath $P_{k-1} = (v_1, v_2, \ldots, v_{k-1})$ of P_k. By the induction hypothesis, there is a modular edge-graceful labeling $h : E(G) \to \mathbb{Z}_n$ of G that satisfies the following four conditions:

(1′) $h(e) = f(e)$ for all $e \notin E(P_{k-1})$,

(2′) $h'(v) = f'(v)$ for all $v \notin V(P_{k-1})$,

(3′) $\{h'(v_i) : 1 \leq i \leq k\} = \{f'(v_i) : 1 \leq i \leq k-1\}$ and

(4′) $h(v_i v_{i+1}) \neq 0$ for all i with $1 \leq i \leq k-3$.

If $h(v_{k-2}v_{k-1}) \neq 0$, then let $g = h$. Thus we may assume that $h(v_{k-2}v_{k-1}) = 0$. Now consider the subpath $P_3 = (v_{k-2}, v_{k-1}, v_k)$ of P_k. Applying the induction hypothesis to P_3 and the modular edge-graceful labeling h of G, we conclude that there is a modular edge-graceful labeling $g : E(G) \to \mathbb{Z}_n$ such that

(1*) $g(e) = h(e)$ for all $e \notin E(P_3)$,

(2*) $g'(v) = h'(v)$ for all $v \notin V(P_3)$,

(3*) $\{g'(v_{k-2}), g'(v_{k-1}), g'(v_k)\} = \{h'(v_{k-2}), h'(v_{k-1}), h'(v_k)\}$ and

(4*) $g(v_{k-2}v_{k-1}) \neq 0$.

Observe that for each integer j with $1 \leq j \leq 4$, conditions (j') and (j^*) give rise to condition (j). Therefore, g and f satisfy conditions (1)–(4). ∎

We now present the main result of this section.

Theorem 2.2. *Let G be a connected modular edge-graceful graph of order $n \geq 3$, where $n \not\equiv 2 \pmod 4$. Then there is a modular edge-graceful labeling $f : E(G) \to \mathbb{Z}_n$ such that $f(e) \neq 0$ for all $e \in E(G)$ with at most one exception. Furthermore, for a fixed edge e^* of G, there is a modular edge-graceful labeling $f : E(G) \to \mathbb{Z}_n$ such that $f(e) \neq 0$ for all $e \in E(G) - \{e^*\}$.*

Proof. It suffices to show that for a fixed edge e^* of G, there is a modular edge-graceful labeling $f : E(G) \to \mathbb{Z}_n$ such that $f(e) \neq 0$ for all $e \in E(G) - \{e^*\}$. Among all modular edge-graceful labelings of G, let $f : E(G) \to \mathbb{Z}_n$ be one for which the set $S = \{e \in E(G) - \{e^*\} : f(e) = 0\}$ has the smallest possible cardinality. We claim that $S = \emptyset$; for otherwise, let $e' \in S$. Since G is connected, there exists a path $P_k = (v_1, v_2, \ldots, v_k)$ of order $k \geq 3$ such that $e_1 = v_1v_2$ and $e_2 = v_{k-1}v_k$. By Lemma 2.1, there is a modular edge-graceful labeling $g : E(G) \to \mathbb{Z}_n$ of G that satisfies

(i) $g(e) = f(e)$ for all $e \notin E(P_k)$ and

(ii) $g(v_i v_{i+1}) \neq 0$ for all i with $1 \leq i \leq k-2$.

Therefore, $g(e) \neq 0$ for all $e \in E(G) - \{e^*\}$, which contradicts the defined property of f. ∎

3. Trees

In this section we establish a characterization of nowhere-zero modular edge-graceful trees. More precisely, we show that a tree T of order $n \geq 3$ with $n \not\equiv 2 \pmod 4$ is nowhere-zero modular edge-graceful if and only if T is not a star of even order. We begin with a lemma that determines all nowhere-zero modular edge-graceful paths, stars and double stars.

Lemma 3.1. *Let $n \geq 3$ be an integer with $n \not\equiv 2 \pmod 4$. Then*

(a) *Every path P_n is nowhere-zero modular edge-graceful.*

(b) *A star of order n is nowhere-zero modular edge-graceful if and only if n is odd.*

(c) *Every double star of order $n \geq 4$ is nowhere-zero modular edge-graceful.*

Proof. Since the proofs of (a) and (b) are relatively straightforward, we only prove (c). Let T be the double star of order $n = a + b + 2 \geq 4$ whose central vertices are u and v where $\deg u = a + 1$ and $\deg v = b + 1$. Let $u_1, u_2, \ldots, u_a$ be end-vertices of T that are adjacent to u and let $v_1, v_2, \ldots, v_b$ be end-vertices of T that are adjacent to v.

First, suppose that $n \geq 5$ is odd. We may assume, without loss of generality, that a is odd and b is even. Define a labeling $f : E(T) \to \mathbb{Z}_n - \{0\}$ by

$$f(e) = \begin{cases} \frac{i+1}{2} & \text{if } e = uu_i,\ 1 \leq i \leq a \text{ and } i \text{ is odd,} \\ -\frac{i}{2} & \text{if } e = uu_i,\ 1 \leq i \leq a \text{ and } i \text{ is even,} \\ -\frac{a+1}{2} & \text{if } e = uv, \\ \frac{a+1}{2} + \frac{j+1}{2} & \text{if } e = vv_j,\ 1 \leq j \leq b \text{ and } j \text{ is odd,} \\ -\left(\frac{a+1}{2} + \frac{j}{2}\right) & \text{if } e = vv_j,\ 1 \leq j \leq b \text{ and } j \text{ is even.} \end{cases}$$

Observe in $\mathbb{Z}_n$ that $f'(u) = 0$, $f'(v) = -\frac{a+1}{2}$ and

$$\begin{aligned} \{f'(u_i) : 1 \leq i \leq a\} &= \left\{\pm 1, \pm 2, \ldots, \pm\frac{a-1}{2}, \frac{a+1}{2}\right\}, \\ \{f'(v_j) : 1 \leq j \leq b\} &= \left\{\pm\left(\frac{a+1}{2} + 1\right), \pm\left(\frac{a+1}{2} + 2\right), \ldots, \pm\frac{n-1}{2}\right\}. \end{aligned}$$

Therefore, f' is one-to-one and so f is a nowhere-zero modular edge-graceful labeling.

Next, suppose that $n \geq 4$ is even. Since $n = a + b + 2$, it follows that a and b are of the same parity. We consider two cases.

Case 1. *a and b are both odd.* We may assume, without loss of generality, that $a \leq b$. First, suppose that $a = 1$. By (a), we may assume that $b \neq 1$. Since $n = b + 3 \equiv 0 \pmod 4$, it follows that $b \geq 5$. Now define a labeling $f : E(T) \to \mathbb{Z}_n - \{0\}$ such that

(i) $\{f(vv_i):\ 2 \le i \le b\} = \left\{\pm 1, \pm 2, \ldots, \pm\frac{b+1}{2}\right\} - \left\{\pm\frac{n}{4}\right\}$
(where then $f(vv_i) \neq \pm\frac{n}{2}$ for $2 \le i \le b$) and

(ii) $f(uu_1) = \frac{n}{4}$ and $f(uv) = f(vv_1) = \frac{n}{2}$.

By the definition of f, $\{f'(v_i) : 2 \le i \le b\} = \left\{\pm 1, \pm 2, \ldots, \pm\frac{b+1}{2}\right\} - \left\{\pm\frac{n}{4}\right\}$ in $\mathbb{Z}_n$ and $f'(v) = 0$, $f'(u_1) = \frac{n}{4}$, $f'(u) = \frac{n}{4} + \frac{n}{2} = \frac{3n}{4} = -\frac{n}{4}$ and $f'(v_1) = \frac{n}{2}$ in $\mathbb{Z}_n$. Thus f is a nowhere-zero modular edge-graceful labeling of T.

Next, suppose that $a \ge 3$. Let $a = 2p+1$ and $b = 2q+1$ for some positive integers p and q where $p \le q$. Now define a labeling $f : E(T) \to \mathbb{Z}_n - \{0\}$ such that

(i) $\{f(uu_i) : 2 \le i \le a\} = \{\pm 1, \pm 2, \ldots, \pm p\}$,
$\{f(vv_j) : 2 \le j \le b\} = \{\pm(p+1), \pm(p+3), \ldots, \pm(p+q+1)\} - \left\{\pm\frac{n}{4}\right\}$.
(where then $f(uu_i) \neq \pm\frac{n}{2}$ and $f(vv_j) \neq \pm\frac{n}{2}$ for $2 \le i \le a$ and $2 \le j \le b$) and

(ii) $f(uu_1) = \frac{n}{4}$ and $f(uv) = f(vv_1) = \frac{n}{2}$.

By the definition of f,

$$\{f'(u_i) : 2 \le i \le a\} \cup \{f'(v_j) : 2 \le j \le b\} = \mathbb{Z}_n - \left\{0, \tfrac{n}{4}, \tfrac{n}{2}, \tfrac{3n}{4} = -\tfrac{n}{4}\right\} \quad \text{in } \mathbb{Z}_n.$$

Furthermore, $f'(u_1) = \frac{n}{4}$, $f'(v_1) = \frac{n}{2}$, $f'(u) = -\frac{n}{4}$ and $f'(v) = 0$ in $\mathbb{Z}_n$. Thus f is a nowhere-zero modular edge-graceful labeling of T.

Case 2. *a and b are both even.* Because $n \equiv 0 \pmod 4$ and $n = a+b+2$, we may assume, without loss of generality, that $a \equiv 0 \pmod 4$ and $b \equiv 2 \pmod 4$. Since $a > 0$ and $b > 0$, it follows that $a \ge 4$ and $b \ge 2$. Define the sets U and W of edges of T by

$$U = \{e = uu_i : 3 \le i \le a\} \text{ and } W = \{e = vv_i : 1 \le i \le b\}.$$

Then $|U| = a-2$ and $|W| = b$ are both even and so $|U \cup W| = a+b-2 = n-4$. Furthermore, let $S = \mathbb{Z}_n - \left\{0, \frac{n}{4}, \frac{n}{2}, \frac{3n}{4}\right\}$ and so $|S| = n-4 = |U \cup W|$. Let $g : U \cup W \to S$ be any bijective function with the property that $g(uu_i) = r \in S$ where $3 \le i \le a$ if and only if $g(uu_j) = -r \in S$ for some j with $i \neq j$ and $3 \le j \le a$. This implies that $g(vv_i) = r' \in S$ where $1 \le i \le b$ if and only if $g(vv_j) = -r' \in S$ for some j with $i \neq j$ and $1 \le j \le b$. Now define a labeling $f : E(T) \to \mathbb{Z}_n$ in terms of g by $f(uu_1) = \frac{n}{2}$, $f(uu_2) = \frac{n}{4}$, $f(uv) = 0$ and $f(e) = g(e)$ for $e \in U \cup W$. By the definitions of f and g, it follows that

$$\{f'(u_i) : 3 \le i \le a\} \cup \{f'(v_j) : 1 \le j \le b\} = S = \mathbb{Z}_n - \left\{0, \tfrac{n}{4}, \tfrac{n}{2}, \tfrac{3n}{4}\right\}.$$

Furthermore, $f'(u_1) = \frac{n}{2}$, $f'(u_2) = \frac{n}{4}$, $f'(u) = -\frac{n}{4}$ and $f'(v) = 0$ in $\mathbb{Z}_n$. Thus f is a modular edge-graceful labeling of T but f is not nowhere-zero.

We now construct a nowhere-zero modular edge-graceful labeling h of T from f as follows. Suppose that $f(vv_1) = s$ for some $s \in S$. It follows by the definition of the set S that $s \neq \frac{3n}{4}$ in $\mathbb{Z}_n$. Define $h : E(T) \to \mathbb{Z}_n - \{0\}$ by

$$h(e) = \begin{cases} f(e) & \text{if } e \neq uv \text{ and } e \neq vv_1, \\ s - \frac{3n}{4} & \text{if } e = uv, \\ \frac{3n}{4} & \text{if } e = vv_1. \end{cases}$$

Observe that $h'(u) = s$, $h'(v) = 0$, $h'(v_1) = \frac{3n}{4}$ and $h'(w) = f'(w)$ if $w \neq u, v_1$. Therefore, h is a nowhere-zero modular edge-graceful labeling of T. ■

We now consider trees in general whose diameter is at least 4.

Theorem 3.2. *If T is a tree of order $n \geq 5$ with $n \not\equiv 2 \pmod 4$ and diameter at least 4, then T is nowhere-zero modular edge-graceful.*

Proof. Assume, to the contrary, that there is a tree T of order $n \geq 5$ with $n \not\equiv 2 \pmod 4$ and diameter at least 4 but T is not nowhere-zero modular edge-graceful. Let v_0 be an end-vertex of T for which the eccentricity $e(v_0)$ of v_0 is at least 4 and let $P = (v_0, v_1, v_2, v_3, v_4)$ be a $v_0 - v_4$ path in T. By Theorems 1.1 and 2.2, there is a modular edge-graceful labeling $f : E(T) \to \mathbb{Z}_n$ such that $f(e) \neq 0$ for all $e \in E(T) - \{v_0v_1\}$. Since T is not nowhere-zero modular edge-graceful, $f(v_0v_1) = 0$ and so $f'(v_0) = 0$. Suppose that $f'(v_i) = x_i$ for $1 \leq i \leq 4$. Then $x_i \neq 0$ for $1 \leq i \leq 4$. We now construct a sequence of four edge labelings g, h, i, j of T recursively as follows.

First, define $g : E(T) \to \mathbb{Z}_n$ from the labeling f by

$$g(e) = \begin{cases} x_2 & \text{if } e = v_0v_1, \\ f(e) - x_2 & \text{if } e = v_1v_2, \\ f(e) & \text{otherwise.} \end{cases}$$

Because $g'(v_0) = x_2 = f'(v_2)$, $g'(v_2) = 0 = f'(v_0)$ and $g'(v) = f'(v)$ for all $v \in V(G) - \{v_0, v_2\}$, it follows that g is a modular edge-graceful labeling of T. Since $f(e) = g(e)$ for all $e \in E(T) - \{v_0v_1, v_1v_2\}$ and $g(v_0v_1) = x_2 \neq 0$, it follows that $g(e) \neq 0$ for all $e \in E(T) - \{v_1v_2\}$. Again, since T is not nowhere-zero modular edge-graceful, $g(v_1v_2) = f(v_1v_2) - x_2 = 0$, implying that $f(v_1v_2) = x_2$.

Secondly, define $h : E(T) \to \mathbb{Z}_n$ from the labeling g by

$$h(e) = \begin{cases} x_3 - x_1 & \text{if } e = v_1v_2, \\ g(e) - (x_3 - x_1) & \text{if } e = v_2v_3, \\ g(e) & \text{otherwise.} \end{cases}$$

Then $h'(v) = g'(v) = f'(v)$ for all $v \in V(G) - \{v_1, v_2, v_3\}$ and

$$\begin{aligned} h'(v_1) &= g'(v_1) + (x_3 - x_1) = x_1 + (x_3 - x_1) = x_3 = g'(v_3), \\ h'(v_2) &= g'(v_2) + (x_3 - x_1) - (x_3 - x_1) = g'(v_2) = 0, \\ h'(v_3) &= g'(v_3) - (x_3 - x_1) = x_3 - (x_3 - x_1) = x_1 = g'(v_1). \end{aligned}$$

Hence h is a modular edge-graceful labeling of T. Since $h(e) \neq 0$ for all $e \in E(T) - \{v_2v_3\}$ and T is not nowhere-zero modular edge-graceful, $h(v_2v_3) = 0$ and so $f(v_2v_3) = x_3 - x_1$.
Next, define $i : E(T) \to \mathbb{Z}_n$ from the labeling h by

$$i(e) = \begin{cases} x_4 & \text{if } e = v_2v_3, \\ h(e) - x_4 & \text{if } e = v_3v_4, \\ h(e) & \text{otherwise.} \end{cases}$$

Then $i'(v) = h'(v)$ for all $v \in V(G) - \{v_2, v_4\}$ and

$$\begin{aligned} i'(v_2) &= h'(v_2) + x_4 = 0 + x_4 = x_4 = h'(v_4), \\ i'(v_4) &= h'(v_4) - x_4 = x_4 - x_4 = 0 = h'(v_2). \end{aligned}$$

Hence i is a modular edge-graceful labeling of T. Since $i(e) \neq 0$ for all $e \in E(T) - \{v_3v_4\}$ and T is not nowhere-zero modular edge-graceful, $i(v_3v_4) = 0$ and so $f(v_3v_4) = x_4$. Therefore, $f(v_1v_2) = x_2$, $f(v_2v_3) = x_3 - x_1$ and $f(v_3v_4) = x_4$. Since f is a modular edge-graceful labeling of T, $x_1 \neq x_4$ and hence $x_4 - x_1 \neq 0$. Thus $x_3 + x_4 - x_1 \neq x_3$, which implies that $\deg v_3 \neq 2$. Let $v_5 \in V(T) - V(P)$ that is adjacent to v_3 and let $f'(v_5) = x_5$. Applying the same argument to the path $(v_0, v_1, v_2, v_3, v_5)$ and the modular edge-graceful labeling f in which $f(e) \neq 0$ for all $e \in E(T) - \{v_0v_1\}$, we obtain that $f(v_3v_5) = x_5$. Now observe that at most one of $x_3 - x_1 - x_2 + x_4$ and $x_3 - x_1 - x_2 + x_5$ is 0. We may assume, without loss of generality, that $x_3 - x_1 - x_2 + x_4 \neq 0$.

We now define a labeling $j : E(T) \to \mathbb{Z}_n$ from the labeling f by

$$j(e) = \begin{cases} x_4 & \text{if } e = v_0v_1, \\ x_2 - x_4 & \text{if } e = v_1v_2, \\ x_4 + x_3 - x_2 - x_1 & \text{if } e = v_2v_3, \\ x_2 & \text{if } e = v_3v_4, \\ f(e) & \text{otherwise.} \end{cases}$$

Since $x_3 - x_1 - x_2 + x_4 \neq 0$ by assumption and $f(e) \neq 0$ for all $e \in E(T) - E(P)$, it follows that $j(e) \neq 0$ for all $e \in E(T)$. Furthermore, $j'(v) = f'(v)$ for all $v \in V(T) - V(P)$ and

$$\begin{aligned} j'(v_0) &= x_4, \\ j'(v_1) &= f'(v_1) - x_4 + x_4 = f'(v_1) = x_1, \\ j'(v_2) &= f'(v_2) - x_4 + (x_4 - x_2) = x_2 - x_4 + (x_4 - x_2) = 0, \\ j'(v_3) &= f'(v_3) + (x_2 - x_4) + (x_4 - x_2) = f'(v_3) = x_3, \\ j'(v_4) &= f'(v_4) + (x_2 - x_4) = x_4 + (x_2 - x_4) = x_2. \end{aligned}$$

Thus $\{j'(v) : v \in V(P)\} = \{f'(v) : v \in V(P)\} = \{0, x_1, x_2, x_3, x_4\}$ and so $\{j'(v) : v \in V(T)\} = \{f'(v) : v \in V(T)\}$. Therefore, j is a nowhere-zero modular edge-graceful labeling of T, which is a contradiction. ■

Combining Lemma 3.1 and Theorem 3.2 establishes the main result of this section.

Theorem 3.3. *A tree T of order $n \geq 3$ with $n \not\equiv 2 \pmod 4$ is nowhere-zero modular edge-graceful if and only if T is not a star of even order.*

4. Connected Graphs with Cycles

We now consider connected graphs of order $n \geq 3$ that are not trees, that is, connected graphs with cycles. We first determine those connected graphs with even cycles that are modular edge-graceful. We begin with a lemma.

Lemma 4.1. *Let G be a connected modular edge-graceful graph of order $n \geq 4$ containing an even cycle C and let f be a modular edge-graceful labeling. Suppose that there is $a \in \mathbb{Z}_n$ that satisfies one of the following two conditions:*

(1) *$f(e) \neq \pm a$ for each $e \in E(C)$,*

(2) *if $a \neq -a$, then $f(e) = a$ for exactly one $e \in E(C)$ and $f(e) \neq -a$ for each $e \in E(C)$.*

Then G has a modular edge-graceful labeling g for which

(i) *$g(e) = f(e)$ for each $e \notin E(C)$,*

(ii) *$g(e) \neq 0$ for all $e \in E(C)$ and*

(iii) *$g'(v) = f'(v)$ for all $v \in V(G)$.*

Proof. If $a = 0$, then $g = f$ satisfies conditions (i)–(iii). Thus, we may assume that $a \neq 0$. Let $C = (v_1, v_2, \ldots, v_k, v_{k+1} = v_1)$ where $k \geq 4$ is even.

First, suppose that condition (1) holds, that is, $f(e) \neq \pm a$ for each $e \in E(C)$. Define a labeling $g : E(G) \to \mathbb{Z}_n$ by

$$g(e) = \begin{cases} f(e) & \text{if } e \notin E(C), \\ f(e) + a & \text{if } e = v_i v_{i+1} \text{ and } i \text{ is odd}, \\ f(e) - a & \text{if } e = v_i v_{i+1} \text{ and } i \text{ is even}. \end{cases} \tag{1}$$

Thus $g(e) = f(e) \pm a \neq 0$ for each $e \in E(C)$ and $g'(v) = f'(v)$ for all $v \in V(G)$. Thus g satisfies conditions (i)–(iii).

Next suppose that (2) holds, that is, $f(e) = a$ for exactly one $e \in E(C)$. We may assume, without loss of generality, that $e = v_1 v_2$. Then the labeling g defined in (1) satisfies conditions (i)-(iii). ∎

Theorem 4.2. *Let G be a connected modular edge-graceful graph of order at least 4 that contains an even cycle C. For each modular edge-graceful labeling g of G, there is a modular edge-graceful labeling $f : E(G) \to \mathbb{Z}_n$ of G such that $f(e) = g(e)$ for each $e \in E(G) - E(C)$ and $f(e) \neq 0$ for each $e \in E(C)$.*

Proof. Suppose that the order of G is $n \geq 4$. Let $C = (v_1, v_2, \ldots, v_k, v_{k+1} = v_1)$ be an even cycle in G, where then $k \geq 4$ is even and let $g : E(G) \to \mathbb{Z}_n$ be a modular edge-graceful labeling of G. Since G is modular edge-graceful, $n \not\equiv 2 \pmod 4$.

First, assume that n is odd. Since k is even, $k < n$. For each integer i with $0 \leq i \leq \frac{n-1}{2}$, let S_i be the set of edges e in C for which $g(e) = i$ or $g(e) = -i$. If $S_0 = \emptyset$, then let $f = g$. Thus we may assume that $S_0 \neq \emptyset$. We claim that there is an integer i with $1 \leq i \leq \frac{n-1}{2}$ such that $|S_i| \leq 1$. Assume, to the contrary, that $|S_i| \geq 2$ for all i with $1 \leq i \leq \frac{n-1}{2}$. Since $|S_0| \geq 1$, it follows that $k = \sum_{i=0}^{\frac{n-1}{2}} |S_i| \geq 2 \cdot \left(\frac{n-1}{2}\right) + 1 = n$. Since $k < n$ in this case, a contradiction is produced. Thus, as claimed, there is an integer i with $1 \leq i \leq \frac{n-1}{2}$ such that $|S_i| \leq 1$. It then follows by Lemma 4.1 that there is a modular edge-graceful labeling f of G such that $f(e) \neq 0$ for each $e \in E(C)$ and $f(e) = g(e)$ for each $e \notin E(C)$.

Next, assume that n is even. For each integer i with $0 \leq i \leq \frac{n}{2} - 1$, let S_i be the set of edges e in C for which $g(e) = i$ or $g(e) = -i$ and let $S_{\frac{n}{2}}$ be the set of edges e in C for which $g(e) = \frac{n}{2}$. If $S_0 = \emptyset$, then let $f = g$. Thus we may assume that $S_0 \neq \emptyset$ and so $|S_0| \geq 1$. We consider two cases, according to whether $k < n$ or $k = n$.

Case 1. $k < n$. We claim that there is an integer i with $1 \leq i < \frac{n}{2}$ such that $|S_i| \leq 1$ or $|S_{\frac{n}{2}}| = 0$. If this is not the case, then $|S_i| \geq 2$ for all i with $1 \leq i < \frac{n}{2}$, $|S_{\frac{n}{2}}| \geq 1$, and $|S_0| \geq 1$. However then, $k = \sum_{i=0}^{\frac{n}{2}} |S_i| \geq 1 + \sum_{i=1}^{\frac{n}{2}-1} |S_i| + 1 \geq 2 + 2\left(\frac{n}{2} - 1\right) = n$ which is impossible. Thus, as claimed, if $k < n$, then $|S_i| \leq 1$ for some integer i with $1 \leq i < \frac{n}{2}$ or $|S_{\frac{n}{2}}| = 0$. Hence by Lemma 4.1, there is a modular edge-graceful labeling f of G such that $f(e) \neq 0$ for each $e \in E(C)$ and $f(e) = g(e)$ for each $e \notin E(C)$.

Case 2. $k = n$. Then $C = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ is a Hamiltonian cycle of G. By the discussion above, if $|S_0| \neq 0$, $|S_{\frac{n}{2}}| \neq 0$ and $|S_{\frac{n}{2}}| \geq 2$ for all i with $1 \leq i \leq \frac{n}{2} - 1$, then $|S_0| = |S_{\frac{n}{2}}| = 1$ and $|S_i| = 2$ for all i with $1 \leq i \leq \frac{n}{2} - 1$. Assume, without loss of generality, that $g(v_1v_2) = 0$. Consider the set $A = \{g(v_iv_{i+1}) : \ i \text{ is odd and } 1 \leq i \leq n-1\}$.

Notice that $|A| \leq \frac{n}{2}$. Since $0 \in A$, there exists $a \in \mathbb{Z}_n - A$ such that $1 \leq a \leq \frac{n}{2}$. Define a labeling $f : E(G) \to \mathbb{Z}_n$ by

$$f(e) = \begin{cases} g(e) & \text{if } e \notin E(C) \text{ or } e = v_iv_{i+1},\ i \text{ is even and } 1 \leq i \leq n, \\ g(e) - a & \text{if } e = v_iv_{i+1},\ i \text{ is odd and } 1 \leq i \leq n-1. \end{cases}$$

Clearly, $f(e) = g(e)$ for each $e \notin E(C)$. Since C is a Hamiltonian cycle of G, $f'(v) = g'(v) - a$ for each $v \in V(G)$ and so f is a modular edge-graceful labeling of G. Furthermore, since $g(v_iv_{i+1}) \neq a$ for all odd integers i with $1 \leq i \leq n-1$, it follows that $f(v_iv_{i+1}) = g(v_iv_{i+1}) - a \neq 0$ for all odd integers i with $1 \leq i \leq n-1$.

Finally, since $|S_0| = 1$, we have $g(v_i v_{i+1}) = 0$ if and only if $i = 1$. Hence $f(e) \neq 0$ for each $e \in E(C)$. ■

The following corollary is an immediate consequence of Theorem 4.2.

Corollary 4.3. *If G is a connected modular edge-graceful graph of order at least 3 and $g : E(G) \to \mathbb{Z}_n$ is a modular edge-graceful labeling of G, then there is a modular edge-graceful labeling f of G such that $f(e) \neq 0$ for each edge e that lies on an even cycle of G and $f(e) = g(e)$ for each edge e that does not lie on an even cycle of G.*

With the aid of Theorem 2.2 and Corollary 4.3, we can now establish the following result on connected graphs with even cycles.

Theorem 4.4. *If G is a connected modular edge-graceful graph of order $n \geq 4$ that contains an even cycle, then G is nowhere-zero modular edge-graceful.*

Proof. Let G be a connected modular edge-graceful graph of order $n \geq 4$, where then $n \not\equiv 2 \pmod 4$, such that G contains an even cycle C and let $e^* \in E(C)$. By Theorem 2.2, there is a modular edge-graceful labeling $g : E(G) \to \mathbb{Z}_n$ such that $g(e) \neq 0$ for all $e \in E(G) - \{e^*\}$. By Corollary 4.3, there is a modular edge-graceful labeling f of G such that $f(e) \neq 0$ for each edge e that lies on C and $f(e) = g(e)$ for each edge e that does not lie on C. Since $g(e) \neq 0$ for all $e \in E(G) - \{e^*\}$, it follows that $f(e) \neq 0$ for all $e \in E(G)$. Hence G is nowhere-zero modular edge-graceful. ■

We now consider connected graphs with cycles in general. First, we present two lemmas. The proof of the first lemma is relatively standard and is therefore omitted.

Lemma 4.5. *For each integer $n \geq 3$, the cycle C_n is nowhere-zero modular edge-graceful if and only if $n \geq 4$ and $n \not\equiv 2 \pmod 4$.*

Lemma 4.6. *Let G be a modular edge-graceful graph of order $n \geq 3$ that is not nowhere-zero modular edge-graceful. Let v_1v_2 be an edge of G and let $f : E(G) \to \mathbb{Z}_n$ be a modular edge-graceful labeling of G such that $f(v_1v_2) = 0$ and $f(e) \neq 0$ for all $e \in E(G) - \{v_1v_2\}$. If $P_k = (v_1, v_2, \ldots, v_k)$ is a path of order $k \geq 3$ in G such that $f'(v_i) = x_i$ for $1 \leq i \leq k$ and $f(v_i v_{i+1}) = y_i$ for $1 \leq i \leq k-1$ where $y_1 = 0$, then*

$$y_i = \begin{cases} x_{i+1} - x_1 & \text{if } 2 \leq i \leq k-1 \text{ and } i \text{ is even,} \\ x_{i+1} - x_2 & \text{if } 3 \leq i \leq k-1 \text{ and } i \text{ is odd.} \end{cases} \tag{2}$$

Furthermore, there is a modular edge-graceful labeling $g : E(G) \to \mathbb{Z}_n$ of G that satisfies the following conditions:

- *$g(e) \neq 0$ for all $e \in E(G) - \{v_{k-1}v_k\}$ and $g(v_{k-1}v_k) = 0$,*
- *$g(e) = f(e)$ for all $e \in E(G) - E(P_k)$, and*
- *if k is odd, then $g'(v_k) = x_1$ and $g'(v_{k-1}) = x_2$; while if k is even, then $g'(v_k) = x_2$ and $g'(v_{k-1}) = x_1$.*

Proof. We proceed by induction on k. We first consider the two base cases when $k = 3$ and $k = 4$.

First, suppose that $k = 3$. Let $f : E(G) \to \mathbb{Z}_n$ be a modular edge-graceful labeling of G such that $f(v_1v_2) = 0$ and $f(e) \neq 0$ for all $e \in E(G) - \{v_1v_2\}$ and let $P_3 = (v_1, v_2, v_3)$. Suppose that $f'(v_i) = x_i$ for $1 \leq i \leq 3$. We shall show that $f(v_2v_3) = y_2 = x_3 - x_1$. Define a labeling $g : E(G) \to \mathbb{Z}_n$ of G by

$$g(e) = \begin{cases} f(e) & \text{if } e \neq v_1v_2, v_2v_3, \\ x_3 - x_1 & \text{if } e = v_1v_2, \\ y_2 - (x_3 - x_1) & \text{if } e = v_2v_3. \end{cases}$$

Observe that $g'(v) = f'(v)$ if $v \in V(G) - \{v_1, v_2, v_3\}$ and

$$\begin{aligned} g'(v_1) &= f'(v_1) + (x_3 - x_1) = x_1 + (x_3 - x_1) = x_3, \\ g'(v_2) &= f'(v_2) + (x_3 - x_1) - (x_3 - x_1) = f'(v_2) = x_2, \\ g'(v_3) &= f'(v_3) - (x_3 - x_1) = x_3 - (x_3 - x_1) = x_1. \end{aligned}$$

Thus $\{g'(v_i) : 1 \leq i \leq 3\} = \{f'(v_i) : 1 \leq i \leq 3\} = \{x_1, x_2, x_3\}$ and so g is a modular edge-graceful labeling of G. Since $g(e) = f(e)$ for all $e \in E(G) - E(P_3)$, it follows that $g(e) \neq 0$ if $e \neq v_1v_2, v_2v_3$. Also, $x_3 \neq x_1$ and so $g(v_1v_2) = x_3 - x_1 \neq 0$. Since G is not nowhere-zero modular edge-graceful, $g(v_2v_3) = y_2 - (x_3 - x_1) = 0$ and so $y_2 = x_3 - x_1$. Thus (2) holds. Furthermore, the modular edge-graceful labeling $g : E(G) \to \mathbb{Z}_n$ of G satisfies the following conditions:

- $g(e) \neq 0$ for all $e \in E(G) - \{v_2v_3\}$ and $g(v_2v_3) = 0$,
- $g(e) = f(e)$ for all $e \in E(G) - E(P_3)$, and
- $g'(v_3) = x_1$ and $g'(v_2) = x_2$.

Hence the result holds for $k = 3$.

Next suppose that $k = 4$. Let $f : E(G) \to \mathbb{Z}_n$ be a modular edge-graceful labeling of G such that $f(v_1v_2) = 0$ and $f(e) \neq 0$ for all $e \in E(G) - \{v_1v_2\}$ and let $P_4 = (v_1, v_2, v_3, v_4)$. Again, suppose that $f'(v_i) = x_i$ for $1 \leq i \leq 4$ and $f(v_iv_{i+1}) = y_i$ for $1 \leq i \leq 3$, where then $y_1 = 0$. By the case when $k = 3$, there is a modular edge-graceful labeling $g : E(G) \to \mathbb{Z}_n$ of G such that $g(e) \neq 0$ for all $e \in E(G) - \{v_2v_3\}$, $g(v_3v_4) = y_3 = f(v_3v_4)$, $g'(v_1) = x_3$, $g'(v_3) = x_1$, and $g'(v) = f'(v)$ for all $v \in V(G) - \{v_1, v_3\}$. Define a labeling $h : E(G) \to \mathbb{Z}_n$ of G from the modular edge-graceful labeling g of G by

$$h(e) = \begin{cases} g(e) & \text{if } e \neq v_2v_3, v_3v_4, \\ x_4 - x_2 & \text{if } e = v_2v_3, \\ y_3 - (x_4 - x_2) & \text{if } e = v_3v_4. \end{cases}$$

Observe that

$$\begin{aligned}
&h'(v) = g'(v) \text{ if } v \in V(G) - \{v_2, v_3, v_4\} \text{ and}\\
&h'(v_2) = g'(v_2) + (x_4 - x_2) = x_2 + (x_4 - x_2) = x_4,\\
&h'(v_3) = g'(v_3) + (x_4 - x_2) - (x_4 - x_2) = g'(v_3) = x_1,\\
&h'(v_4) = g'(v_4) - (x_4 - x_2) = x_4 - (x_4 - x_2) = x_2.
\end{aligned}$$

Thus $\{h'(v_i) : 2 \le i \le 4\} = \{g'(v_i) : 2 \le i \le 4\} = \{x_1, x_2, x_4\}$ and so h is a modular edge-graceful labeling of G. Since $h(e) = g(e)$ for all $e \in E(G) - \{v_2v_3, v_3v_4\}$, it follows that $h(e) \ne 0$ if $e \notin \{v_2v_3, v_3v_4\}$. Also, $x_4 \ne x_2$ and so $h(v_2v_3) = x_4 - x_2 \ne 0$. Since G is not nowhere-zero modular edge-graceful, $h(v_3v_4) = y_3 - (x_4 - x_2) = 0$ and so $y_3 = x_4 - x_2$. Thus (2) holds. Furthermore, the modular edge-graceful labeling $h : E(G) \to \mathbb{Z}_n$ of G satisfies the following conditions:

- $h(e) \ne 0$ for all $e \in E(G) - \{v_3v_4\}$ and $h(v_3v_4) = 0$,
- $h(e) = g(e) = f(e)$ for all $e \in E(G) - E(P_4)$, and
- $h'(v_4) = x_2$ and $h'(v_3) = x_1$.

Hence the result holds for $k = 4$.

Now suppose that the result holds for some integer $k \ge 4$. Let $f : E(G) \to \mathbb{Z}_n$ be a modular edge-graceful labeling of G such that $f(v_1v_2) = 0$ and $f(e) \ne 0$ for all $e \in E(G) - \{v_1v_2\}$ and $P_{k+1} = (v_1, v_2, \ldots, v_{k+1})$ be a path of order $k+1 \ge 5$ in G. Suppose that $f'(v_i) = x_i$ for $1 \le i \le k+1$ and $f(v_iv_{i+1}) = y_i$ for $1 \le i \le k$ where $y_1 = 0$. Let $P_k = (v_1, v_2, \ldots, v_k)$ be the subpath of order k in P_{k+1}. We consider two cases, according to whether k is even or k is odd.

Case 1. *k is even or $k+1$ is odd.* By the induction hypothesis of f on the path P_k, we have

$$y_i = \begin{cases} x_{i+1} - x_1 & \text{if } 2 \le i \le k-2 \text{ and } i \text{ is even,}\\ x_{i+1} - x_2 & \text{if } 3 \le i \le k-1 \text{ and } i \text{ is odd.}\end{cases} \tag{3}$$

Furthermore, there is a modular edge-graceful labeling $g : E(G) \to \mathbb{Z}_n$ of G such that

- $g(e) \ne 0$ for all $e \in E(G) - \{v_{k-1}v_k\}$ and $g(v_{k-1}v_k) = 0$,
- $g(e) = f(e)$ for all $e \in E(G) - E(P_k)$, and
- $g'(v_k) = x_2$ and $g'(v_{k-1}) = x_1$.

Hence $g(v_kv_{k+1}) = y_k = f(v_kv_{k+1})$. Define a labeling $h : E(G) \to \mathbb{Z}_n$ of G from the modular edge-graceful labeling g of G by

$$h(e) = \begin{cases} g(e) & \text{if } \ e \ne v_{k-1}v_k, v_kv_{k+1},\\ x_{k+1} - x_1 & \text{if } \ e = v_{k-1}v_k,\\ y_k - (x_{k+1} - x_1) & \text{if } \ e = v_kv_{k+1}.\end{cases}$$

Observe that

(i) $h'(v) = g'(v)$ if $v \in V(G) - \{v_{k-1}, v_k, v_{k+1}\}$ and

(ii) by (3),

$$\begin{aligned} h'(v_{k-1}) &= g'(v_{k-1}) + (x_{k+1} - x_1) = x_1 + (x_{k+1} - x_1) = x_{k+1}, \\ h'(v_k) &= g'(v_k) + (x_{k+1} - x_1) - (x_{k+1} - x_1) = g'(v_k) = x_2, \\ h'(v_{k+1}) &= g'(v_{k+1}) - (x_{k+1} - x_1) = x_{k+1} - (x_{k+1} - x_1) = x_1. \end{aligned}$$

Thus $\{h'(v_i) : k-1 \leq i \leq k+1\} = \{g'(v_i) : k-1 \leq i \leq k+1\}$ and so h is a modular edge-graceful labeling of G. Since $h(e) = g(e)$ for all $e \in E(G) - \{v_{k-1}v_k, v_kv_{k+1}\}$, it follows that $h(e) \neq 0$ if $e \neq v_{k-1}v_k, v_kv_{k+1}$. Also, $x_{k+1} \neq x_1$ and so $h(v_{k-1}v_k) = x_{k+1} - x_1 \neq 0$. Since G is not nowhere-zero modular edge-graceful, $h(v_kv_{k+1}) = y_k - (x_{k+1} - x_1) = 0$ and so $y_k = x_{k+1} - x_1$. Hence (2) holds. Furthermore, the modular edge-graceful labeling $h : E(G) \to \mathbb{Z}_n$ of G satisfies the following conditions:

- $h(e) \neq 0$ for all $e \in E(G) - \{v_kv_{k+1}\}$ and $h(v_kv_{k+1}) = 0$,
- $h(e) = g(e) = f(e)$ for all $e \notin E(P_{k+1})$, and
- $h'(v_{k+1}) = x_1$ and $h'(v_k) = x_2$.

Hence the result holds for k is even (or $k+1$ is odd).

Case 2. *k is odd or $k+1$ is even.* By the induction hypothesis of f on the path P_k, we have

$$y_i = \begin{cases} x_{i+1} - x_1 & \text{if } 2 \leq i \leq k-1 \text{ and } i \text{ is even}, \\ x_{i+1} - x_2 & \text{if } 3 \leq i \leq k-2 \text{ and } i \text{ is odd}. \end{cases} \tag{4}$$

Furthermore, there is a modular edge-graceful labeling $g : E(G) \to \mathbb{Z}_n$ of G such that

- $g(e) \neq 0$ for all $e \in E(G) - \{v_{k-1}v_k\}$ and $g(v_{k-1}v_k) = 0$,
- $g(e) = f(e)$ for all $e \in E(G) - E(P_k)$, and
- $g'(v_k) = x_1$ and $g'(v_{k-1}) = x_2$.

Hence $g(v_kv_{k+1}) = y_k = f(v_kv_{k+1})$. Define a labeling $h : E(G) \to \mathbb{Z}_n$ of G from the modular edge-graceful labeling g of G by

$$h(e) = \begin{cases} g(e) & \text{if } e \neq v_{k-1}v_k, v_kv_{k+1}, \\ x_{k+1} - x_2 & \text{if } e = v_{k-1}v_k, \\ y_k - (x_{k+1} - x_2) & \text{if } e = v_kv_{k+1}. \end{cases}$$

Observe that

(i) $h'(v) = g'(v)$ if $v \in V(G) - \{v_{k-1}, v_k, v_{k+1}\}$ and

(ii) by (4),

$$\begin{aligned} h'(v_{k-1}) &= g'(v_{k-1}) + (x_{k+1} - x_2) = x_2 + (x_{k+1} - x_2) = x_{k+1}, \\ h'(v_k) &= g'(v_k) + (x_{k+1} - x_2) - (x_{k+1} - x_2) = g'(v_k) = x_1, \\ h'(v_{k+1}) &= g'(v_{k+1}) - (x_{k+1} - x_2) = x_{k+1} - (x_{k+1} - x_2) = x_2. \end{aligned}$$

Thus $\{h'(v_i) : k-1 \le i \le k+1\} = \{g'(v_i) : k-1 \le i \le k+1\}$ and so h is a modular edge-graceful labeling of G. Since $h(e) = g(e)$ for all $e \in E(G) - \{v_{k-1}v_k, v_kv_{k+1}\}$, it follows that $h(e) \neq 0$ if $e \neq v_{k-1}v_k, v_kv_{k+1}$. Also, $x_{k+1} \neq x_2$ and so $h(v_{k-1}v_k) = x_{k+1} - x_2 \neq 0$. Since G is not nowhere-zero modular edge-graceful, $h(v_kv_{k+1}) = y_k - (x_{k+1} - x_2) = 0$ and so $y_k = x_{k+1} - x_2$. Hence (2) holds. Furthermore, the modular edge-graceful labeling $h : E(G) \to \mathbb{Z}_n$ of G satisfies the following conditions:

- $h(e) \neq 0$ for all $e \in E(G) - \{v_kv_{k+1}\}$ and $h(v_{k1}v_{k+1}) = 0$,
- $h(e) = g(e) = f(e)$ for all $e \notin E(P_{k+1})$, and
- $h'(v_{k+1}) = x_2$ and $h'(v_k) = x_1$.

Hence the result holds for k is odd (or $k+1$ is even). ■

Theorem 4.7. *If G is a connected modular edge-graceful graph of order $n \ge 4$ that is not a star, then G is nowhere-zero modular edge-graceful.*

Proof. Assume, to the contrary, that there is a connected modular edge-graceful graph G of order $n \ge 4$ that is not a star such that G is not nowhere-zero modular edge-graceful. By Theorem 3.3, G is not a tree. By Theorem 4.4, G does not contain an even cycle. By Lemma 4.5, G is not an odd cycle. Since G is connected, it follows that G contains a unicyclic subgraph H that is obtained from an odd cycle by adding a pendant edge. We may assume that $V(H) = \{v_1, v_2, v_3, \ldots, v_{2k+2}\}$, where $C_{2k+1} = (v_2, v_3, \ldots, v_{2k+2}, v_2)$ is the odd cycle of H and v_1v_2 is the pendant edge of H.

Since G is not nowhere-zero modular edge-graceful, by Lemma 4.6, there is a modular edge-graceful labeling $f : E(G) \to \mathbb{Z}_n$ of G such that $f(v_1v_2) = 0$ and $f(e) \neq 0$ for all $e \in E(G) - \{v_1v_2\}$. Furthermore, if $P_{2k+2} = (v_1, v_2, v_3, \ldots, v_{2k+2})$ such that $f'(v_i) = x_i$ for $1 \le i \le 2k+2$ and $f(v_iv_{i+1}) = y_i$ for $1 \le i \le 2k+1$ where $y_1 = 0$, then

$$y_i = \begin{cases} x_{i+1} - x_1 & \text{if } 2 \le i \le 2k \text{ and } i \text{ is even,} \\ x_{i+1} - x_2 & \text{if } 3 \le i \le 2k+1 \text{ and } i \text{ is odd.} \end{cases} \tag{5}$$

Now consider the two paths Q_1 and Q_2 with initial edge v_1v_2 in H, namely,

$$\begin{aligned} Q_1 &= (v_1, v_2, v_3, v_4, \ldots, v_{k+2}, v_{k+3}), \\ Q_2 &= (v_1, v_2, v_{2k+2}, v_{2k+1}, v_{2k}, \ldots, v_{k+3}, v_{k+2}). \end{aligned}$$

Thus, $|V(Q_1)| = |V(Q_2)| = k+3$ and $E(Q_1) \cap E(Q_2) = \{v_1v_2, v_{k+2}v_{k+3}\}$.

First, assume that $k+2$ is odd. By traversing along the path Q_1 and applying (5), we obtain $y_{k+2} = x_{k+3} - x_1$. On the other hand, by traversing along the path Q_2 and applying Lemma 4.6, we obtain $y_{k+2} = x_{k+2} - x_1$. This implies that $x_{k+3} = x_{k+2}$, which is a contradiction.

Next, assume that $k+2$ is even. Then a similar argument shows that $x_{k+3} = x_{k+2}$, which is a contradiction. ■

Theorems 3.3, 4.4 and 4.7 then provide a characterization of connected nowhere-zero modular edge-graceful graphs.

Theorem 4.8. *A connected graph G of order $n \geq 3$ is nowhere-zero modular edge-graceful if and only if*

(i) $n \not\equiv 2 \pmod 4$,

(ii) $G \neq K_3$ *and*

(iii) G *is not a star of even order.*

5. Nowhere-zero Modular Edge-gracefulness

For every connected graph G of order n, there is a smallest integer $k \geq n$ for which there exists an edge labeling $f: E(G) \to \mathbb{Z}_k - \{0\}$ such that the induced vertex labeling $f': V(G) \to \mathbb{Z}_k$ defined by $f'(v) = \sum_{u \in N(v)} f(uv)$, where the sum is computed in $\mathbb{Z}_k$, is one-to-one. This number k is referred to as the *nowhere-zero modular edge-gracefulness* of G and is denoted by $\mathrm{nzg}(G)$. Thus $\mathrm{nzg}(G) = n$ if and only if G is nowhere-zero modular edge-graceful. Next, we determine $\mathrm{nzg}(G)$ for those graphs G that are not nowhere-zero modular edge-graceful. We first consider connected graphs of order $n \geq 3$ where $n \not\equiv 2 \pmod 4$. By Theorem 4.8, it suffices to determine the nowhere-zero gracefulness of K_3 or a star of even order. Since $\mathrm{nzg}(K_3) = 4$, it remains to consider a star of even order.

Corollary 5.1. *For each odd integer $s \geq 3$,* $\mathrm{nzg}(K_{1,s}) = s + 3$.

Proof. By Theorem 4.8, $\mathrm{nzg}(K_{1,s}) \geq s+2$. First, we show that $\mathrm{nzg}(K_{1,s}) \neq s+2$ for all odd integers $s \geq 3$. Assume, to the contrary, that there is an edge labeling $f: E(K_{1,s}) \to \mathbb{Z}_{s+2} - \{0\}$ such that the induced vertex labeling $f': V(K_{1,s}) \to \mathbb{Z}_{s+2}$ is an injective function. Since every edge of $K_{1,s}$ is incident with an end-vertex, it follows that f is injective as well. Because $|\mathbb{Z}_{s+2} - \{0\}| = s+1 = |E(K_{1,s})| + 1$, there is a unique $a \in \mathbb{Z}_{s+2} - \{0\}$ such that $f(e) \neq a$ for all $e \in E(K_{1,s})$. Since $s+2$ is odd, $a \neq -a$ for all $a \in \mathbb{Z}_{s+2}$. This implies that there is $e = uv \in E(K_{1,s})$ such that $f(e) = -a$. Suppose that u is an end-vertex of $K_{1,s}$ and v is the central vertex of $K_{1,s}$. Then $f'(u) = f'(v) = -a$, which is a contradiction. Therefore, $\mathrm{nzg}(K_{1,s}) \neq s+2$ and so $\mathrm{nzg}(K_{1,s}) \geq s+3$.

To show that $\mathrm{nzg}(K_{1,s}) \leq s+3$, we define an edge labeling $g: E(K_{1,s}) \to \mathbb{Z}_{s+3} - \{0\}$ such that the induced vertex labeling $g': V(K_{1,s}) \to \mathbb{Z}_{s+3}$ is an injective function. Let $V(K_{1,s}) = \{v, v_1, v_2, \ldots, v_s\}$ where v is the central vertex of $K_{1,s}$. For $s = 3, 5$, let $g(vv_i) = i$ for $1 \leq i \leq s$; while for $s \geq 7$, let

$$g(vv_i) = \begin{cases} i+1 & \text{if } 1 \leq i \leq \frac{s-1}{2} - 2, \\ i+2 & \text{if } \frac{s-1}{2} - 1 \leq i \leq s. \end{cases}$$

In each case, g' is injective and so $\mathrm{nzg}(K_{1,s}) = s+3$. ∎

By Theorem 4.8, $\mathrm{nzg}(K_3) = 4$ and Proposition 5.1, we have the following.

Theorem 5.2. *If G is a connected graph of order $n \geq 3$ with $n \not\equiv 2 \pmod 4$ that is not nowhere-zero modular edge-graceful, then $\mathrm{nzg}(G) \in \{n+1, n+2\}$. Furthermore, $\mathrm{nzg}(G) = n+1$ if and only if $G = K_3$ and $\mathrm{nzg}(G) = n+2$ if and only if G is a star of even order.*

By Theorem 1.1, if G is a connected graph of order $n \geq 6$ where $n \equiv 2 \pmod 4$, then G is not modular edge-graceful. Consequently, G is not nowhere-zero modular edge-graceful and so $\mathrm{nzg}(G) \geq n+1$. Proceeding as above under the hypothesis that $n \equiv 2 \pmod 4$ rather than $n \not\equiv 2 \pmod 4$, then with a similar argument, both in length and method, the following can be established.

Theorem 5.3. *If G is a connected graph of order $n \geq 6$ where $n \equiv 2 \pmod 4$, then $\mathrm{nzg}(G) \in \{n+1, n+2\}$. Furthermore, $\mathrm{nzg}(G) = n+2$ if and only if G is a star.*

In summary then, we have the following.

Theorem 5.4. *If G is a connected graph of order $n \geq 3$, then $\mathrm{nzg}(G) \in \{n, n+1, n+2\}$. Furthermore,*

- *$\mathrm{nzg}(G) = n$ if and only if G is nowhere-zero modular edge-graceful,*
- *$\mathrm{nzg}(G) = n+1$ if and only if $G = K_3$ or $n \equiv 2 \pmod 4$ and G is not a star of even order.*
- *$\mathrm{nzg}(G) = n+2$ if and only if G is a star of even order.*

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