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# NOWHERE-ZERO MODULAR EDGE-GRACEFUL GRAPHS

Ryan Jones and Ping Zhang

*Department of Mathematics*
*Western Michigan University*
*Kalamazoo, MI 49008, USA*

**e-mail:** ping.zhang@wmich.edu

**Abstract**

For a connected graph $G$ of order $n \geq 3$, let $f : E(G) \to \mathbb{Z}_n$ be an edge labeling of $G$. The vertex labeling $f' : V(G) \to \mathbb{Z}_n$ induced by $f$ is defined as $f'(u) = \sum_{v \in N(u)} f(uv)$, where the sum is computed in $\mathbb{Z}_n$. If $f'$ is one-to-one, then $f$ is called a modular edge-graceful labeling and $G$ is a modular edge-graceful graph. A modular edge-graceful labeling $f$ of $G$ is nowhere-zero if $f(e) \neq 0$ for all $e \in E(G)$ and in this case, $G$ is a nowhere-zero modular edge-graceful graph. It is shown that a connected graph $G$ of order $n \geq 3$ is nowhere-zero modular edge-graceful if and only if $n \not\equiv 2 \pmod 4$, $G \neq K_3$ and $G$ is not a star of even order. For a connected graph $G$ of order $n \geq 3$, the smallest integer $k \geq n$ for which there exists an edge labeling $f : E(G) \to \mathbb{Z}_k - \{0\}$ such that the induced vertex labeling $f'$ is one-to-one is referred to as the nowhere-zero modular edge-gracefulness of $G$ and this number is determined for every connected graph of order at least 3.

**Keywords:** modular edge-graceful labelings and graphs, nowhere-zero labelings, modular edge-gracefulness.

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## 1. Introduction

Over the past few decades the subject of graph labelings has been growing in popularity. Gallian [7] has compiled a periodically updated survey of many kinds of labelings and numerous results, obtained from well over a thousand referenced research articles. The origin of the study of graph labelings as a major area of graph theory can be traced to a research paper by Rosa [15]. Among the labelings he introduced was a vertex labeling he referred to as a $\beta$-valuation. Let $G$ be a graph of order $n$ and size $m$. A one-to-one function $f : V(G) \to \{0, 1, 2, \dots, m\}$ is called a *$\beta$-valuation* (or a *$\beta$-labeling*) of $G$ if

$$\{|f(u) - f(v)| : uv \in E(G)\} = \{1, 2, \dots, m\}.$$

In 1972 Golomb [9] referred to a $\beta$-labeling as a *graceful labeling* and a graph possessing a graceful labeling as a *graceful graph.* Eventually, it was this terminology that became standard. One of the best known conjectures in this area is Graceful Tree Conjecture, due to Ringel and Kotzig.

**Conjecture.** *Every tree is graceful.*

In 1985 Lo [12] introduced a dual type of labeling — this one an edge labeling. Let $G$ be a connected graph of order $n \geq 2$ and size $m$. For a vertex $v$ of $G$, let $N(v)$ denote the neighborhood of $v$. An *edge-graceful labeling* of $G$ is a bijective function $f : E(G) \to \{1, 2, \dots, m\}$ that gives rise to a bijective function $f' : V(G) \to \{0, 1, 2, \dots, n-1\}$ given by $f'(v) = \sum_{u \in N(v)} f(uv)$, where the sum is computed in $\mathbb{Z}_n$. A graph that admits an edge-graceful labeling is called an *edge-graceful graph.* In the definition of an edge-graceful labeling of a connected graph $G$ of order $n \geq 2$ and size $m$, the edge labeling $f$ is required to be one-to-one. Since, however, the induced vertex labels $f'(v)$ are obtained by addition in $\mathbb{Z}_n$, the function $f$ is actually a function from $E(G)$ to $\mathbb{Z}_n$ and is in general not one-to-one. Dividing $m$ by $n$, we obtain $m = nq + r$, where $q = \lfloor m/n \rfloor$ and $0 \leq r \leq n-1$. Hence in an edge-graceful labeling of $G$, $q+1$ edges are labeled $i$ for each $i$ with $1 \leq i \leq r$ and $q$ edges are labeled $i$ for each $i$ with $r+1 \leq i \leq n$ (in $\mathbb{Z}_n$). Thus this edge labeling $f : E(G) \to \mathbb{Z}_n$ is a one-to-one function only when $m = n-1$ or $m = n$.

In 2008 a vertex coloring of a graph was introduced in [13] in connection with finding a solution to a checkerboard problem posted by Gary Chartrand. For a graph $G$ without isolated vertices, let $c : V(G) \to \mathbb{Z}_k$ ($k \geq 2$) be a vertex coloring of $G$ where adjacent vertices may be colored the same. Then a vertex coloring $c'$ of $G$ is defined such that $c'(v)$ is the sum in $\mathbb{Z}_k$ of the colors of the vertices in the neighborhood of $v$ for each $v \in V(G)$. The coloring $c$ is called a *modular $k$-coloring* of G if $c'(u) \neq c'(v)$ in $\mathbb{Z}_k$ for every pair $u, v$ of adjacent vertices of $G$. The *modular chromatic number* of $G$ is the minimum $k$ for which $G$ has a modular $k$-coloring. This coloring was studied further in [14], which led to a complete solution of the checkerboard problem under investigation. Furthermore, modular colorings are closely related to *sigma colorings* in graphs (see [4]). The modular coloring described above led to an edge version introduced in [10], which was inspired by the research of finding various methods to distinguish every pair of adjacent vertices in a graph by means of edge colorings (see [1, 2, 6, 16] and [5, p. 385], for example). For a graph $G$ without isolated vertices, let $c : E(G) \to \mathbb{Z}_k$ ($k \geq 2$) be an edge coloring of $G$ where adjacent edges may be colored the same.

Then a vertex coloring $c'$ is defined such that $c'(v)$ is the sum in $\mathbb{Z}_k$ of the colors of the edges incident with $v$ for each $v \in V(G)$. An edge coloring $c$ is a *modular $k$-edge coloring* of $G$ if $c'(u) \neq c'(v)$ in $\mathbb{Z}_k$ for all pairs $u, v$ of adjacent vertices of $G$. The *modular chromatic index* of $G$ is the minimum $k$ for which $G$ has a modular $k$-edge coloring. Combining the concepts of graceful labeling and modular edge coloring gives rise to a modular edge-graceful labeling, as we describe next.

Let $G$ be a connected graph of order $n \geq 3$ and let $f : E(G) \to \mathbb{Z}_n$, where $f$ need not be one-to-one. Let $f' : V(G) \to \mathbb{Z}_n$ such that $f'(v) = \sum_{u \in N(v)} f(uv)$, where the sum is computed in $\mathbb{Z}_n$. If $f'$ is one-to-one, then $f$ is called a *modular edge-graceful labeling* and $G$ is a *modular edge-graceful graph.* Consequently, every edge-graceful graph is a modular edge-graceful graph. It turns out that this concept was introduced in 1991 by Jothi [8] under the terminology of *line-graceful graphs* (also see [7]). It was known that if $G$ is a connected graph of order $n \geq 3$ for which $n \equiv 2 \pmod 4$, then $G$ is not modular edge-graceful. Furthermore, it was conjectured that if $T$ is a tree of order $n \geq 3$ for which $n \not\equiv 2 \pmod 4$, then $T$ is modular edge-graceful (see [7]). This conjecture was verified in [11]. In fact, the conjecture is not only true for trees but for all connected graphs.

**Theorem 1.1** [11]. *A connected graph of order $n \geq 3$ is modular edge-graceful if and only if $n \not\equiv 2 \pmod 4$.*

For every connected graph $G$ of order $n$, there is a smallest integer $k \geq n$ for which there exists an edge labeling $f : E(G) \to \mathbb{Z}_k$ such that the induced vertex labeling $f' : V(G) \to \mathbb{Z}_k$ defined by $f'(v) = \sum_{u \in N(v)} f(uv)$, where the sum is computed in $\mathbb{Z}_k$, is one-to-one. The number $k$ is defined in [11] as the *modular edge-gracefulness* $\text{meg}(G)$ of $G$. Thus $\text{meg}(G) \geq n$ and $\text{meg}(G) = n$ if and only if $G$ is a modular edge-graceful graph of order $n$ and if $G$ is not modular edge-graceful, then $\text{meg}(G) \geq n + 1$. In fact, $\text{meg}(G)$ is known for every connected graph $G$, as we state next.

**Theorem 1.2** [11]. *If $G$ is a nontrivial connected graph of order $n \geq 6$ that is not modular edge-graceful, then* $\text{meg}(G) = n + 1$.

If $G$ is a modular edge-graceful spanning subgraph of a graph $H$, where $G$ and $H$ are connected, then a modular edge-graceful labeling of $G$ can be extended to a modular edge-graceful labeling of $H$ by assigning 0 to each edge of $H$ that does not belong to $G$. Thus modular edge-graceful labelings of a graph that assign 0 to some edges of the graph play an important role in establishing Theorems 1.1 and 1.2. For this reason, we now investigate those modular edge-graceful labelings in which 0 is not permitted. This gives rise to a new concept along with additional challenging problems. More formally, for a connected graph $G$ of order $n \geq 3$ let $f : E(G) \to \mathbb{Z}_n - \{0\}$, where $f$ need not be one-to-one and let $f' : V(G) \to \mathbb{Z}_n$ be defined by $f'(u) = \sum_{v \in N(u)} f(uv)$, where the sum is computed in $\mathbb{Z}_n$. If $f'$

is one-to-one, then $f$ is called a *nowhere-zero modular edge-graceful labeling* and $G$ is a *nowhere-zero modular edge-graceful graph.* In this work, we first show (in Section 2) that if $G$ is a connected graph of order $n \geq 3$ where $n \not\equiv 2 \pmod 4$, then there is a modular edge-graceful labeling $f : E(G) \to \mathbb{Z}_n$ such that $f(e) \neq 0$ for all $e \in E(G)$ with at most one exception. In Section 3 we determine all nowhere-zero modular edge-graceful trees. Finally, we present a characterization of all nowhere-zero modular edge-graceful graphs in Section 4 and determine the nowhere-zero modular edge-gracefulness of every connected graph in Section 5. We refer to the book [3] for graph theory notation and terminology not described in this paper. Henceforth, we assume all graphs under consideration are connected graphs of order at least 3.

## 2. One Zero is Sufficient

In this section, we show that if $G$ is a modular edge-graceful graph of order $n \geq 3$ that is not nowhere-zero, then there is a modular edge-graceful labeling $f : E(G) \to \mathbb{Z}_n$ such that $f(e) \neq 0$ for all $e \in E(G)$ with one exception, that is, one zero is sufficient. Furthermore, for each prescribed edge $e^*$ of $G$, there is a modular edge-graceful labeling $f^* : E(G) \to \mathbb{Z}_n$ such that $f^*(e) \neq 0$ for all $e \in E(G) - \{e^*\}$. First, we present a lemma.

**Lemma 2.1.** *Let $G$ be a connected modular edge-graceful graph of order $n \geq 3$, where $n \not\equiv 2 \pmod 4$ and let $f : E(G) \to \mathbb{Z}_n$ be a given modular edge-graceful labeling of $G$. If $P_k = (v_1, v_2, \ldots, v_k)$ is a path of order $k \geq 3$ in $G$, then there is a modular edge-graceful labeling $g : E(G) \to \mathbb{Z}_n$ of $G$ that satisfies the following four conditions:*

(1) *$g(e) = f(e)$ for all $e \notin E(P_k)$,*

(2) *$g'(v) = f'(v)$ for all $v \notin V(P_k)$,*

(3) *$\{g'(v_i) : 1 \leq i \leq k\} = \{f'(v_i) : 1 \leq i \leq k\}$ and*

(4) *$g(v_i v_{i+1}) \neq 0$ for all $i$ with $1 \leq i \leq k-2$.*

***Proof.*** We proceed by induction on $k$. For $k = 3$, let $P_3 = (v_1, v_2, v_3)$. If $f(v_1v_2) \neq 0$, then let $g = f$. Thus, we may assume that $f(v_1v_2) = 0$. Suppose that $f'(v_1) = a$, $f'(v_2) = b$ and $f'(v_3) = c$. Define a labeling $g : E(G) \to \mathbb{Z}_n$ by

$$g(e) = \begin{cases} f(e) & \text{if } e \notin E(P_3), \\ f(e) + (c-a) & \text{if } e = v_1v_2, \\ f(e) - (c-a) & \text{if } e = v_2v_3. \end{cases}$$

By the definition of $g$, conditions (1) and (2) hold. Since $g(v_1v_2) = f(v_1v_2)+(c-a)$ $= c - a$ and $a \neq c$, it follows that $g(v_1v_2) \neq 0$ and so (3) holds. Furthermore, $g'(v_1) = c$, $g'(v_2) = b$ and $g'(v_1) = a$ and so (4) holds.

Assume for some integer $k \geq 4$ that the result holds for all paths of order $k'$ in $G$ where $3 \leq k' < k$. Let $P_k = (v_1, v_2, \ldots, v_k)$ be a path of order $k \geq 4$ in $G$. First, consider the subpath $P_{k-1} = (v_1, v_2, \ldots, v_{k-1})$ of $P_k$. By the induction hypothesis, there is a modular edge-graceful labeling $h : E(G) \to \mathbb{Z}_n$ of $G$ that satisfies the following four conditions:

(1′) $h(e) = f(e)$ for all $e \notin E(P_{k-1})$,

(2′) $h'(v) = f'(v)$ for all $v \notin V(P_{k-1})$,

(3′) $\{h'(v_i) : 1 \leq i \leq k\} = \{f'(v_i) : 1 \leq i \leq k-1\}$ and

(4′) $h(v_i v_{i+1}) \neq 0$ for all $i$ with $1 \leq i \leq k-3$.

If $h(v_{k-2}v_{k-1}) \neq 0$, then let $g = h$. Thus we may assume that $h(v_{k-2}v_{k-1}) = 0$. Now consider the subpath $P_3 = (v_{k-2}, v_{k-1}, v_k)$ of $P_k$. Applying the induction hypothesis to $P_3$ and the modular edge-graceful labeling $h$ of $G$, we conclude that there is a modular edge-graceful labeling $g : E(G) \to \mathbb{Z}_n$ such that

(1*) $g(e) = h(e)$ for all $e \notin E(P_3)$,

(2*) $g'(v) = h'(v)$ for all $v \notin V(P_3)$,

(3*) $\{g'(v_{k-2}), g'(v_{k-1}), g'(v_k)\} = \{h'(v_{k-2}), h'(v_{k-1}), h'(v_k)\}$ and

(4*) $g(v_{k-2}v_{k-1}) \neq 0$.

Observe that for each integer $j$ with $1 \leq j \leq 4$, conditions $(j')$ and $(j^*)$ give rise to condition $(j)$. Therefore, $g$ and $f$ satisfy conditions (1)–(4). ■

We now present the main result of this section.

**Theorem 2.2.** *Let $G$ be a connected modular edge-graceful graph of order $n \geq 3$, where $n \not\equiv 2 \pmod 4$. Then there is a modular edge-graceful labeling $f : E(G) \to \mathbb{Z}_n$ such that $f(e) \neq 0$ for all $e \in E(G)$ with at most one exception. Furthermore, for a fixed edge $e^*$ of $G$, there is a modular edge-graceful labeling $f : E(G) \to \mathbb{Z}_n$ such that $f(e) \neq 0$ for all $e \in E(G) - \{e^*\}$.*

***Proof.*** It suffices to show that for a fixed edge $e^*$ of $G$, there is a modular edge-graceful labeling $f : E(G) \to \mathbb{Z}_n$ such that $f(e) \neq 0$ for all $e \in E(G) - \{e^*\}$. Among all modular edge-graceful labelings of $G$, let $f : E(G) \to \mathbb{Z}_n$ be one for which the set $S = \{e \in E(G) - \{e^*\} : f(e) = 0\}$ has the smallest possible cardinality. We claim that $S = \emptyset$; for otherwise, let $e' \in S$. Since $G$ is connected, there exists a path $P_k = (v_1, v_2, \ldots, v_k)$ of order $k \geq 3$ such that $e_1 = v_1 v_2$ and $e_2 = v_{k-1}v_k$. By Lemma 2.1, there is a modular edge-graceful labeling $g : E(G) \to \mathbb{Z}_n$ of $G$ that satisfies

(i) $g(e) = f(e)$ for all $e \notin E(P_k)$ and

(ii) $g(v_i v_{i+1}) \neq 0$ for all $i$ with $1 \leq i \leq k-2$.

Therefore, $g(e) \neq 0$ for all $e \in E(G) - \{e^*\}$, which contradicts the defined property of $f$. ■

## 3. TREES

In this section we establish a characterization of nowhere-zero modular edge-graceful trees. More precisely, we show that a tree $T$ of order $n \geq 3$ with $n \not\equiv 2 \pmod 4$ is nowhere-zero modular edge-graceful if and only if $T$ is not a star of even order. We begin with a lemma that determines all nowhere-zero modular edge-graceful paths, stars and double stars.

**Lemma 3.1.** *Let $n \geq 3$ be an integer with $n \not\equiv 2 \pmod 4$. Then*

(a) *Every path $P_n$ is nowhere-zero modular edge-graceful.*

(b) *A star of order $n$ is nowhere-zero modular edge-graceful if and only if $n$ is odd.*

(c) *Every double star of order $n \geq 4$ is nowhere-zero modular edge-graceful.*

***Proof.*** Since the proofs of (a) and (b) are relatively straightforward, we only prove (c). Let $T$ be the double star of order $n = a + b + 2 \geq 4$ whose central vertices are $u$ and $v$ where $\deg u = a + 1$ and $\deg v = b + 1$. Let $u_1, u_2, \ldots, u_a$ be end-vertices of $T$ that are adjacent to $u$ and let $v_1, v_2, \ldots, v_b$ be end-vertices of $T$ that are adjacent to $v$.

First, suppose that $n \geq 5$ is odd. We may assume, without loss of generality, that $a$ is odd and $b$ is even. Define a labeling $f : E(T) \to \mathbb{Z}_n - \{0\}$ by

$$f(e) = \begin{cases} \frac{i+1}{2} & \text{if } e = uu_i,\ 1 \leq i \leq a \text{ and } i \text{ is odd,} \\ -\frac{i}{2} & \text{if } e = uu_i,\ 1 \leq i \leq a \text{ and } i \text{ is even,} \\ -\frac{a+1}{2} & \text{if } e = uv, \\ \frac{a+1}{2} + \frac{j+1}{2} & \text{if } e = vv_j,\ 1 \leq j \leq b \text{ and } j \text{ is odd,} \\ -\left(\frac{a+1}{2} + \frac{j}{2}\right) & \text{if } e = vv_j,\ 1 \leq j \leq b \text{ and } j \text{ is even.} \end{cases}$$

Observe in $\mathbb{Z}_n$ that $f'(u) = 0$, $f'(v) = -\frac{a+1}{2}$ and

$$\begin{aligned} \{f'(u_i) : 1 \leq i \leq a\} &= \left\{\pm 1, \pm 2, \ldots, \pm\frac{a-1}{2}, \frac{a+1}{2}\right\}, \\ \{f'(v_j) : 1 \leq j \leq b\} &= \left\{\pm\left(\frac{a+1}{2} + 1\right), \pm\left(\frac{a+1}{2} + 2\right), \ldots, \pm\frac{n-1}{2}\right\}. \end{aligned}$$

Therefore, $f'$ is one-to-one and so $f$ is a nowhere-zero modular edge-graceful labeling.

Next, suppose that $n \geq 4$ is even. Since $n = a + b + 2$, it follows that $a$ and $b$ are of the same parity. We consider two cases.

*Case* 1. *$a$ and $b$ are both odd.* We may assume, without loss of generality, that $a \leq b$. First, suppose that $a = 1$. By (a), we may assume that $b \neq 1$. Since $n = b + 3 \equiv 0 \pmod 4$, it follows that $b \geq 5$. Now define a labeling $f : E(T) \to \mathbb{Z}_n - \{0\}$ such that

(i) $\{f(vv_i):\ 2 \le i \le b\} = \left\{\pm 1, \pm 2, \ldots, \pm \frac{b+1}{2}\right\} - \left\{\pm \frac{n}{4}\right\}$
(where then $f(vv_i) \ne \pm \frac{n}{2}$ for $2 \le i \le b$) and

(ii) $f(uu_1) = \frac{n}{4}$ and $f(uv) = f(vv_1) = \frac{n}{2}$.

By the definition of $f$, $\{f'(v_i) : 2 \le i \le b\} = \left\{\pm 1, \pm 2, \ldots, \pm \frac{b+1}{2}\right\} - \left\{\pm \frac{n}{4}\right\}$ in $\mathbb{Z}_n$ and $f'(v) = 0$, $f'(u_1) = \frac{n}{4}$, $f'(u) = \frac{n}{4} + \frac{n}{2} = \frac{3n}{4} = -\frac{n}{4}$ and $f'(v_1) = \frac{n}{2}$ in $\mathbb{Z}_n$. Thus $f$ is a nowhere-zero modular edge-graceful labeling of $T$.

Next, suppose that $a \ge 3$. Let $a = 2p + 1$ and $b = 2q + 1$ for some positive integers $p$ and $q$ where $p \le q$. Now define a labeling $f : E(T) \to \mathbb{Z}_n - \{0\}$ such that

(i) $\{f(uu_i) : 2 \le i \le a\} = \{\pm 1, \pm 2, \ldots, \pm p\}$,
$\{f(vv_j) : 2 \le j \le b\} = \{\pm(p+1), \pm(p+3), \ldots, \pm(p+q+1)\} - \left\{\pm \frac{n}{4}\right\}$.
(where then $f(uu_i) \ne \pm\frac{n}{2}$ and $f(vv_j) \ne \pm\frac{n}{2}$ for $2 \le i \le a$ and $2 \le j \le b$) and

(ii) $f(uu_1) = \frac{n}{4}$ and $f(uv) = f(vv_1) = \frac{n}{2}$.

By the definition of $f$,

$$\{f'(u_i) : 2 \le i \le a\} \cup \{f'(v_j) : 2 \le j \le b\} = \mathbb{Z}_n - \left\{0, \tfrac{n}{4}, \tfrac{n}{2}, \tfrac{3n}{4} = -\tfrac{n}{4}\right\} \quad \text{in } \mathbb{Z}_n.$$

Furthermore, $f'(u_1) = \frac{n}{4}$, $f'(v_1) = \frac{n}{2}$, $f'(u) = -\frac{n}{4}$ and $f'(v) = 0$ in $\mathbb{Z}_n$. Thus $f$ is a nowhere-zero modular edge-graceful labeling of $T$.

*Case* 2. *$a$ and $b$ are both even.* Because $n \equiv 0 \pmod 4$ and $n = a + b + 2$, we may assume, without loss of generality, that $a \equiv 0 \pmod 4$ and $b \equiv 2 \pmod 4$. Since $a > 0$ and $b > 0$, it follows that $a \ge 4$ and $b \ge 2$. Define the sets $U$ and $W$ of edges of $T$ by

$$U = \{e = uu_i : 3 \le i \le a\} \text{ and } W = \{e = vv_i : 1 \le i \le b\}.$$

Then $|U| = a - 2$ and $|W| = b$ are both even and so $|U \cup W| = a + b - 2 = n - 4$. Furthermore, let $S = \mathbb{Z}_n - \left\{0, \frac{n}{4}, \frac{n}{2}, \frac{3n}{4}\right\}$ and so $|S| = n - 4 = |U \cup W|$. Let $g : U \cup W \to S$ be any bijective function with the property that $g(uu_i) = r \in S$ where $3 \le i \le a$ if and only if $g(uu_j) = -r \in S$ for some $j$ with $i \ne j$ and $3 \le j \le a$. This implies that $g(vv_i) = r' \in S$ where $1 \le i \le b$ if and only if $g(vv_j) = -r' \in S$ for some $j$ with $i \ne j$ and $1 \le j \le b$. Now define a labeling $f : E(T) \to \mathbb{Z}_n$ in terms of $g$ by $f(uu_1) = \frac{n}{2}$, $f(uu_2) = \frac{n}{4}$, $f(uv) = 0$ and $f(e) = g(e)$ for $e \in U \cup W$. By the definitions of $f$ and $g$, it follows that

$$\{f'(u_i) : 3 \le i \le a\} \cup \{f'(v_j) : 1 \le j \le b\} = S = \mathbb{Z}_n - \left\{0, \tfrac{n}{4}, \tfrac{n}{2}, \tfrac{3n}{4}\right\}.$$

Furthermore, $f'(u_1) = \frac{n}{2}$, $f'(u_2) = \frac{n}{4}$, $f'(u) = -\frac{n}{4}$ and $f'(v) = 0$ in $\mathbb{Z}_n$. Thus $f$ is a modular edge-graceful labeling of $T$ but $f$ is not nowhere-zero.

We now construct a nowhere-zero modular edge-graceful labeling $h$ of $T$ from $f$ as follows. Suppose that $f(vv_1) = s$ for some $s \in S$. It follows by the definition of the set $S$ that $s \ne \frac{3n}{4}$ in $\mathbb{Z}_n$. Define $h : E(T) \to \mathbb{Z}_n - \{0\}$ by

$$h(e) = \begin{cases} f(e) & \text{if } e \neq uv \text{ and } e \neq vv_1, \\ s - \frac{3n}{4} & \text{if } e = uv, \\ \frac{3n}{4} & \text{if } e = vv_1. \end{cases}$$

Observe that $h'(u) = s$, $h'(v) = 0$, $h'(v_1) = \frac{3n}{4}$ and $h'(w) = f'(w)$ if $w \neq u, v_1$. Therefore, $h$ is a nowhere-zero modular edge-graceful labeling of $T$. ■

We now consider trees in general whose diameter is at least 4.

**Theorem 3.2.** *If $T$ is a tree of order $n \geq 5$ with $n \not\equiv 2 \pmod 4$ and diameter at least 4, then $T$ is nowhere-zero modular edge-graceful.*

***Proof.*** Assume, to the contrary, that there is a tree $T$ of order $n \geq 5$ with $n \not\equiv 2 \pmod 4$ and diameter at least 4 but $T$ is not nowhere-zero modular edge-graceful. Let $v_0$ be an end-vertex of $T$ for which the eccentricity $e(v_0)$ of $v_0$ is at least 4 and let $P = (v_0, v_1, v_2, v_3, v_4)$ be a $v_0 - v_4$ path in $T$. By Theorems 1.1 and 2.2, there is a modular edge-graceful labeling $f : E(T) \to \mathbb{Z}_n$ such that $f(e) \neq 0$ for all $e \in E(T) - \{v_0v_1\}$. Since $T$ is not nowhere-zero modular edge-graceful, $f(v_0v_1) = 0$ and so $f'(v_0) = 0$. Suppose that $f'(v_i) = x_i$ for $1 \leq i \leq 4$. Then $x_i \neq 0$ for $1 \leq i \leq 4$. We now construct a sequence of four edge labelings $g, h, i, j$ of $T$ recursively as follows.

First, define $g : E(T) \to \mathbb{Z}_n$ from the labeling $f$ by

$$g(e) = \begin{cases} x_2 & \text{if } e = v_0v_1, \\ f(e) - x_2 & \text{if } e = v_1v_2, \\ f(e) & \text{otherwise.} \end{cases}$$

Because $g'(v_0) = x_2 = f'(v_2)$, $g'(v_2) = 0 = f'(v_0)$ and $g'(v) = f'(v)$ for all $v \in V(G) - \{v_0, v_2\}$, it follows that $g$ is a modular edge-graceful labeling of $T$. Since $f(e) = g(e)$ for all $e \in E(T) - \{v_0v_1, v_1v_2\}$ and $g(v_0v_1) = x_2 \neq 0$, it follows that $g(e) \neq 0$ for all $e \in E(T) - \{v_1v_2\}$. Again, since $T$ is not nowhere-zero modular edge-graceful, $g(v_1v_2) = f(v_1v_2) - x_2 = 0$, implying that $f(v_1v_2) = x_2$.

Secondly, define $h : E(T) \to \mathbb{Z}_n$ from the labeling $g$ by

$$h(e) = \begin{cases} x_3 - x_1 & \text{if } e = v_1v_2, \\ g(e) - (x_3 - x_1) & \text{if } e = v_2v_3, \\ g(e) & \text{otherwise.} \end{cases}$$

Then $h'(v) = g'(v) = f'(v)$ for all $v \in V(G) - \{v_1, v_2, v_3\}$ and

$$\begin{aligned} h'(v_1) &= g'(v_1) + (x_3 - x_1) = x_1 + (x_3 - x_1) = x_3 = g'(v_3), \\ h'(v_2) &= g'(v_2) + (x_3 - x_1) - (x_3 - x_1) = g'(v_2) = 0, \\ h'(v_3) &= g'(v_3) - (x_3 - x_1) = x_3 - (x_3 - x_1) = x_1 = g'(v_1). \end{aligned}$$

Hence $h$ is a modular edge-graceful labeling of $T$. Since $h(e) \neq 0$ for all $e \in E(T) - \{v_2v_3\}$ and $T$ is not nowhere-zero modular edge-graceful, $h(v_2v_3) = 0$ and so $f(v_2v_3) = x_3 - x_1$.
Next, define $i : E(T) \to \mathbb{Z}_n$ from the labeling $h$ by

$$i(e) = \begin{cases} x_4 & \text{if } e = v_2v_3, \\ h(e) - x_4 & \text{if } e = v_3v_4, \\ h(e) & \text{otherwise.} \end{cases}$$

Then $i'(v) = h'(v)$ for all $v \in V(G) - \{v_2, v_4\}$ and

$$\begin{aligned} i'(v_2) &= h'(v_2) + x_4 = 0 + x_4 = x_4 = h'(v_4), \\ i'(v_4) &= h'(v_4) - x_4 = x_4 - x_4 = 0 = h'(v_2). \end{aligned}$$

Hence $i$ is a modular edge-graceful labeling of $T$. Since $i(e) \neq 0$ for all $e \in E(T) - \{v_3v_4\}$ and $T$ is not nowhere-zero modular edge-graceful, $i(v_3v_4) = 0$ and so $f(v_3v_4) = x_4$. Therefore, $f(v_1v_2) = x_2$, $f(v_2v_3) = x_3 - x_1$ and $f(v_3v_4) = x_4$. Since $f$ is a modular edge-graceful labeling of $T$, $x_1 \neq x_4$ and hence $x_4 - x_1 \neq 0$. Thus $x_3 + x_4 - x_1 \neq x_3$, which implies that $\deg v_3 \neq 2$. Let $v_5 \in V(T) - V(P)$ that is adjacent to $v_3$ and let $f'(v_5) = x_5$. Applying the same argument to the path $(v_0, v_1, v_2, v_3, v_5)$ and the modular edge-graceful labeling $f$ in which $f(e) \neq 0$ for all $e \in E(T) - \{v_0v_1\}$, we obtain that $f(v_3v_5) = x_5$. Now observe that at most one of $x_3 - x_1 - x_2 + x_4$ and $x_3 - x_1 - x_2 + x_5$ is 0. We may assume, without loss of generality, that $x_3 - x_1 - x_2 + x_4 \neq 0$.

We now define a labeling $j : E(T) \to \mathbb{Z}_n$ from the labeling $f$ by

$$j(e) = \begin{cases} x_4 & \text{if } e = v_0v_1, \\ x_2 - x_4 & \text{if } e = v_1v_2, \\ x_4 + x_3 - x_2 - x_1 & \text{if } e = v_2v_3, \\ x_2 & \text{if } e = v_3v_4, \\ f(e) & \text{otherwise.} \end{cases}$$

Since $x_3 - x_1 - x_2 + x_4 \neq 0$ by assumption and $f(e) \neq 0$ for all $e \in E(T) - E(P)$, it follows that $j(e) \neq 0$ for all $e \in E(T)$. Furthermore, $j'(v) = f'(v)$ for all $v \in V(T) - V(P)$ and

$$\begin{aligned} j'(v_0) &= x_4, \\ j'(v_1) &= f'(v_1) - x_4 + x_4 = f'(v_1) = x_1, \\ j'(v_2) &= f'(v_2) - x_4 + (x_4 - x_2) = x_2 - x_4 + (x_4 - x_2) = 0, \\ j'(v_3) &= f'(v_3) + (x_2 - x_4) + (x_4 - x_2) = f'(v_3) = x_3, \\ j'(v_4) &= f'(v_4) + (x_2 - x_4) = x_4 + (x_2 - x_4) = x_2. \end{aligned}$$

Thus $\{j'(v) : v \in V(P)\} = \{f'(v) : v \in V(P)\} = \{0, x_1, x_2, x_3, x_4\}$ and so $\{j'(v) : v \in V(T)\} = \{f'(v) : v \in V(T)\}$. Therefore, $j$ is a nowhere-zero modular edge-graceful labeling of $T$, which is a contradiction. ∎

Combining Lemma 3.1 and Theorem 3.2 establishes the main result of this section.

**Theorem 3.3.** *A tree $T$ of order $n \geq 3$ with $n \not\equiv 2 \pmod 4$ is nowhere-zero modular edge-graceful if and only if $T$ is not a star of even order.*

## 4. Connected Graphs with Cycles

We now consider connected graphs of order $n \geq 3$ that are not trees, that is, connected graphs with cycles. We first determine those connected graphs with even cycles that are modular edge-graceful. We begin with a lemma.

**Lemma 4.1.** *Let $G$ be a connected modular edge-graceful graph of order $n \geq 4$ containing an even cycle $C$ and let $f$ be a modular edge-graceful labeling. Suppose that there is $a \in \mathbb{Z}_n$ that satisfies one of the following two conditions:*

(1) *$f(e) \neq \pm a$ for each $e \in E(C)$,*

(2) *if $a \neq -a$, then $f(e) = a$ for exactly one $e \in E(C)$ and $f(e) \neq -a$ for each $e \in E(C)$.*

*Then $G$ has a modular edge-graceful labeling $g$ for which*

(i) *$g(e) = f(e)$ for each $e \notin E(C)$,*

(ii) *$g(e) \neq 0$ for all $e \in E(C)$ and*

(iii) *$g'(v) = f'(v)$ for all $v \in V(G)$.*

***Proof.*** If $a = 0$, then $g = f$ satisfies conditions (i)–(iii). Thus, we may assume that $a \neq 0$. Let $C = (v_1, v_2, \ldots, v_k, v_{k+1} = v_1)$ where $k \geq 4$ is even.

First, suppose that condition (1) holds, that is, $f(e) \neq \pm a$ for each $e \in E(C)$. Define a labeling $g : E(G) \to \mathbb{Z}_n$ by

$$g(e) = \begin{cases} f(e) & \text{if } e \notin E(C), \\ f(e) + a & \text{if } e = v_i v_{i+1} \text{ and } i \text{ is odd}, \\ f(e) - a & \text{if } e = v_i v_{i+1} \text{ and } i \text{ is even}. \end{cases} \tag{1}$$

Thus $g(e) = f(e) \pm a \neq 0$ for each $e \in E(C)$ and $g'(v) = f'(v)$ for all $v \in V(G)$. Thus $g$ satisfies conditions (i)–(iii).

Next suppose that (2) holds, that is, $f(e) = a$ for exactly one $e \in E(C)$. We may assume, without loss of generality, that $e = v_1 v_2$. Then the labeling $g$ defined in (1) satisfies conditions (i)-(iii). ■

**Theorem 4.2.** *Let $G$ be a connected modular edge-graceful graph of order at least 4 that contains an even cycle $C$. For each modular edge-graceful labeling $g$ of $G$, there is a modular edge-graceful labeling $f : E(G) \to \mathbb{Z}_n$ of $G$ such that $f(e) = g(e)$ for each $e \in E(G) - E(C)$ and $f(e) \neq 0$ for each $e \in E(C)$.*

***Proof.*** Suppose that the order of $G$ is $n \geq 4$. Let $C = (v_1, v_2, \ldots, v_k, v_{k+1} = v_1)$ be an even cycle in $G$, where then $k \geq 4$ is even and let $g : E(G) \to \mathbb{Z}_n$ be a modular edge-graceful labeling of $G$. Since $G$ is modular edge-graceful, $n \not\equiv 2 \pmod 4$.

First, assume that $n$ is odd. Since $k$ is even, $k < n$. For each integer $i$ with $0 \leq i \leq \frac{n-1}{2}$, let $S_i$ be the set of edges $e$ in $C$ for which $g(e) = i$ or $g(e) = -i$. If $S_0 = \emptyset$, then let $f = g$. Thus we may assume that $S_0 \neq \emptyset$. We claim that there is an integer $i$ with $1 \leq i \leq \frac{n-1}{2}$ such that $|S_i| \leq 1$. Assume, to the contrary, that $|S_i| \geq 2$ for all $i$ with $1 \leq i \leq \frac{n-1}{2}$. Since $|S_0| \geq 1$, it follows that $k = \sum_{i=0}^{\frac{n-1}{2}} |S_i| \geq 2 \cdot \left(\frac{n-1}{2}\right) + 1 = n$. Since $k < n$ in this case, a contradiction is produced. Thus, as claimed, there is an integer $i$ with $1 \leq i \leq \frac{n-1}{2}$ such that $|S_i| \leq 1$. It then follows by Lemma 4.1 that there is a modular edge-graceful labeling $f$ of $G$ such that $f(e) \neq 0$ for each $e \in E(C)$ and $f(e) = g(e)$ for each $e \notin E(C)$.

Next, assume that $n$ is even. For each integer $i$ with $0 \leq i \leq \frac{n}{2} - 1$, let $S_i$ be the set of edges $e$ in $C$ for which $g(e) = i$ or $g(e) = -i$ and let $S_{\frac{n}{2}}$ be the set of edges $e$ in $C$ for which $g(e) = \frac{n}{2}$. If $S_0 = \emptyset$, then let $f = g$. Thus we may assume that $S_0 \neq \emptyset$ and so $|S_0| \geq 1$. We consider two cases, according to whether $k < n$ or $k = n$.

*Case* 1. $k < n$. We claim that there is an integer $i$ with $1 \leq i < \frac{n}{2}$ such that $|S_i| \leq 1$ or $|S_{\frac{n}{2}}| = 0$. If this is not the case, then $|S_i| \geq 2$ for all $i$ with $1 \leq i < \frac{n}{2}$, $|S_{\frac{n}{2}}| \geq 1$, and $|S_0| \geq 1$. However then, $k = \sum_{i=0}^{\frac{n}{2}} |S_i| \geq 1 + \sum_{i=1}^{\frac{n}{2}-1} |S_i| + 1 \geq 2 + 2\left(\frac{n}{2} - 1\right) = n$ which is impossible. Thus, as claimed, if $k < n$, then $|S_i| \leq 1$ for some integer $i$ with $1 \leq i < \frac{n}{2}$ or $|S_{\frac{n}{2}}| = 0$. Hence by Lemma 4.1, there is a modular edge-graceful labeling $f$ of $G$ such that $f(e) \neq 0$ for each $e \in E(C)$ and $f(e) = g(e)$ for each $e \notin E(C)$.

*Case* 2. $k = n$. Then $C = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ is a Hamiltonian cycle of $G$. By the discussion above, if $|S_0| \neq 0$, $|S_{\frac{n}{2}}| \neq 0$ and $|S_{\frac{n}{2}}| \geq 2$ for all $i$ with $1 \leq i \leq \frac{n}{2} - 1$, then $|S_0| = |S_{\frac{n}{2}}| = 1$ and $|S_i| = 2$ for all $i$ with $1 \leq i \leq \frac{n}{2} - 1$. Assume, without loss of generality, that $g(v_1 v_2) = 0$. Consider the set $A = \{g(v_i v_{i+1}) :\ i \text{ is odd and } 1 \leq i \leq n-1\}$.

Notice that $|A| \leq \frac{n}{2}$. Since $0 \in A$, there exists $a \in \mathbb{Z}_n - A$ such that $1 \leq a \leq \frac{n}{2}$. Define a labeling $f : E(G) \to \mathbb{Z}_n$ by

$$f(e) = \begin{cases} g(e) & \text{if } e \notin E(C) \text{ or } e = v_i v_{i+1},\ i \text{ is even and } 1 \leq i \leq n, \\ g(e) - a & \text{if } e = v_i v_{i+1},\ i \text{ is odd and } 1 \leq i \leq n-1. \end{cases}$$

Clearly, $f(e) = g(e)$ for each $e \notin E(C)$. Since $C$ is a Hamiltonian cycle of $G$, $f'(v) = g'(v) - a$ for each $v \in V(G)$ and so $f$ is a modular edge-graceful labeling of $G$. Furthermore, since $g(v_i v_{i+1}) \neq a$ for all odd integers $i$ with $1 \leq i \leq n-1$, it follows that $f(v_i v_{i+1}) = g(v_i v_{i+1}) - a \neq 0$ for all odd integers $i$ with $1 \leq i \leq n-1$.

Finally, since $|S_0| = 1$, we have $g(v_i v_{i+1}) = 0$ if and only if $i = 1$. Hence $f(e) \neq 0$ for each $e \in E(C)$. ■

The following corollary is an immediate consequence of Theorem 4.2.

**Corollary 4.3.** *If $G$ is a connected modular edge-graceful graph of order at least 3 and $g : E(G) \to \mathbb{Z}_n$ is a modular edge-graceful labeling of $G$, then there is a modular edge-graceful labeling $f$ of $G$ such that $f(e) \neq 0$ for each edge $e$ that lies on an even cycle of $G$ and $f(e) = g(e)$ for each edge $e$ that does not lie on an even cycle of $G$.*

With the aid of Theorem 2.2 and Corollary 4.3, we can now establish the following result on connected graphs with even cycles.

**Theorem 4.4.** *If $G$ is a connected modular edge-graceful graph of order $n \geq 4$ that contains an even cycle, then $G$ is nowhere-zero modular edge-graceful.*

***Proof.*** Let $G$ be a connected modular edge-graceful graph of order $n \geq 4$, where then $n \not\equiv 2 \pmod 4$, such that $G$ contains an even cycle $C$ and let $e^* \in E(C)$. By Theorem 2.2, there is a modular edge-graceful labeling $g : E(G) \to \mathbb{Z}_n$ such that $g(e) \neq 0$ for all $e \in E(G) - \{e^*\}$. By Corollary 4.3, there is a modular edge-graceful labeling $f$ of $G$ such that $f(e) \neq 0$ for each edge $e$ that lies on $C$ and $f(e) = g(e)$ for each edge $e$ that does not lie on $C$. Since $g(e) \neq 0$ for all $e \in E(G) - \{e^*\}$, it follows that $f(e) \neq 0$ for all $e \in E(G)$. Hence $G$ is nowhere-zero modular edge-graceful. ■

We now consider connected graphs with cycles in general. First, we present two lemmas. The proof of the first lemma is relatively standard and is therefore omitted.

**Lemma 4.5.** *For each integer $n \geq 3$, the cycle $C_n$ is nowhere-zero modular edge-graceful if and only if $n \geq 4$ and $n \not\equiv 2 \pmod 4$.*

**Lemma 4.6.** *Let $G$ be a modular edge-graceful graph of order $n \geq 3$ that is not nowhere-zero modular edge-graceful. Let $v_1 v_2$ be an edge of $G$ and let $f : E(G) \to \mathbb{Z}_n$ be a modular edge-graceful labeling of $G$ such that $f(v_1 v_2) = 0$ and $f(e) \neq 0$ for all $e \in E(G) - \{v_1 v_2\}$. If $P_k = (v_1, v_2, \ldots, v_k)$ is a path of order $k \geq 3$ in $G$ such that $f'(v_i) = x_i$ for $1 \leq i \leq k$ and $f(v_i v_{i+1}) = y_i$ for $1 \leq i \leq k-1$ where $y_1 = 0$, then*

$$y_i = \begin{cases} x_{i+1} - x_1 & \text{if } 2 \leq i \leq k-1 \text{ and } i \text{ is even,} \\ x_{i+1} - x_2 & \text{if } 3 \leq i \leq k-1 \text{ and } i \text{ is odd.} \end{cases} \tag{2}$$

*Furthermore, there is a modular edge-graceful labeling $g : E(G) \to \mathbb{Z}_n$ of $G$ that satisfies the following conditions:*

- *$g(e) \neq 0$ for all $e \in E(G) - \{v_{k-1}v_k\}$ and $g(v_{k-1}v_k) = 0$,*
- *$g(e) = f(e)$ for all $e \in E(G) - E(P_k)$, and*
- *if $k$ is odd, then $g'(v_k) = x_1$ and $g'(v_{k-1}) = x_2$; while if $k$ is even, then $g'(v_k) = x_2$ and $g'(v_{k-1}) = x_1$.*

***Proof.*** We proceed by induction on $k$. We first consider the two base cases when $k = 3$ and $k = 4$.

First, suppose that $k = 3$. Let $f : E(G) \to \mathbb{Z}_n$ be a modular edge-graceful labeling of $G$ such that $f(v_1v_2) = 0$ and $f(e) \neq 0$ for all $e \in E(G) - \{v_1v_2\}$ and let $P_3 = (v_1, v_2, v_3)$. Suppose that $f'(v_i) = x_i$ for $1 \leq i \leq 3$. We shall show that $f(v_2v_3) = y_2 = x_3 - x_1$. Define a labeling $g : E(G) \to \mathbb{Z}_n$ of $G$ by

$$g(e) = \begin{cases} f(e) & \text{if} \quad e \neq v_1v_2, v_2v_3, \\ x_3 - x_1 & \text{if} \quad e = v_1v_2, \\ y_2 - (x_3 - x_1) & \text{if} \quad e = v_2v_3. \end{cases}$$

Observe that $g'(v) = f'(v)$ if $v \in V(G) - \{v_1, v_2, v_3\}$ and

$$\begin{aligned} g'(v_1) &= f'(v_1) + (x_3 - x_1) = x_1 + (x_3 - x_1) = x_3, \\ g'(v_2) &= f'(v_2) + (x_3 - x_1) - (x_3 - x_1) = f'(v_2) = x_2, \\ g'(v_3) &= f'(v_3) - (x_3 - x_1) = x_3 - (x_3 - x_1) = x_1. \end{aligned}$$

Thus $\{g'(v_i) : 1 \leq i \leq 3\} = \{f'(v_i) : 1 \leq i \leq 3\} = \{x_1, x_2, x_3\}$ and so $g$ is a modular edge-graceful labeling of $G$. Since $g(e) = f(e)$ for all $e \in E(G) - E(P_3)$, it follows that $g(e) \neq 0$ if $e \neq v_1v_2, v_2v_3$. Also, $x_3 \neq x_1$ and so $g(v_1v_2) = x_3 - x_1 \neq 0$. Since $G$ is not nowhere-zero modular edge-graceful, $g(v_2v_3) = y_2 - (x_3 - x_1) = 0$ and so $y_2 = x_3 - x_1$. Thus (2) holds. Furthermore, the modular edge-graceful labeling $g : E(G) \to \mathbb{Z}_n$ of $G$ satisfies the following conditions:

- $g(e) \neq 0$ for all $e \in E(G) - \{v_2v_3\}$ and $g(v_2v_3) = 0$,
- $g(e) = f(e)$ for all $e \in E(G) - E(P_3)$, and
- $g'(v_3) = x_1$ and $g'(v_2) = x_2$.

Hence the result holds for $k = 3$.

Next suppose that $k = 4$. Let $f : E(G) \to \mathbb{Z}_n$ be a modular edge-graceful labeling of $G$ such that $f(v_1v_2) = 0$ and $f(e) \neq 0$ for all $e \in E(G) - \{v_1v_2\}$ and let $P_4 = (v_1, v_2, v_3, v_4)$. Again, suppose that $f'(v_i) = x_i$ for $1 \leq i \leq 4$ and $f(v_iv_{i+1}) = y_i$ for $1 \leq i \leq 3$, where then $y_1 = 0$. By the case when $k = 3$, there is a modular edge-graceful labeling $g : E(G) \to \mathbb{Z}_n$ of $G$ such that $g(e) \neq 0$ for all $e \in E(G) - \{v_2v_3\}$, $g(v_3v_4) = y_3 = f(v_3v_4)$, $g'(v_1) = x_3$, $g'(v_3) = x_1$, and $g'(v) = f'(v)$ for all $v \in V(G) - \{v_1, v_3\}$. Define a labeling $h : E(G) \to \mathbb{Z}_n$ of $G$ from the modular edge-graceful labeling $g$ of $G$ by

$$h(e) = \begin{cases} g(e) & \text{if} \quad e \neq v_2v_3, v_3v_4, \\ x_4 - x_2 & \text{if} \quad e = v_2v_3, \\ y_3 - (x_4 - x_2) & \text{if} \quad e = v_3v_4. \end{cases}$$

Observe that

$$\begin{aligned} h'(v) &= g'(v) \text{ if } v \in V(G) - \{v_2, v_3, v_4\} \text{ and}\\ h'(v_2) &= g'(v_2) + (x_4 - x_2) = x_2 + (x_4 - x_2) = x_4,\\ h'(v_3) &= g'(v_3) + (x_4 - x_2) - (x_4 - x_2) = g'(v_3) = x_1,\\ h'(v_4) &= g'(v_4) - (x_4 - x_2) = x_4 - (x_4 - x_2) = x_2. \end{aligned}$$

Thus $\{h'(v_i) : 2 \le i \le 4\} = \{g'(v_i) : 2 \le i \le 4\} = \{x_1, x_2, x_4\}$ and so $h$ is a modular edge-graceful labeling of $G$. Since $h(e) = g(e)$ for all $e \in E(G) - \{v_2v_3, v_3v_4\}$, it follows that $h(e) \ne 0$ if $e \notin \{v_2v_3, v_3v_4\}$. Also, $x_4 \ne x_2$ and so $h(v_2v_3) = x_4 - x_2 \ne 0$. Since $G$ is not nowhere-zero modular edge-graceful, $h(v_3v_4) = y_3 - (x_4 - x_2) = 0$ and so $y_3 = x_4 - x_2$. Thus (2) holds. Furthermore, the modular edge-graceful labeling $h : E(G) \to \mathbb{Z}_n$ of $G$ satisfies the following conditions:

- $h(e) \ne 0$ for all $e \in E(G) - \{v_3v_4\}$ and $h(v_3v_4) = 0$,
- $h(e) = g(e) = f(e)$ for all $e \in E(G) - E(P_4)$, and
- $h'(v_4) = x_2$ and $h'(v_3) = x_1$.

Hence the result holds for $k = 4$.

Now suppose that the result holds for some integer $k \ge 4$. Let $f : E(G) \to \mathbb{Z}_n$ be a modular edge-graceful labeling of $G$ such that $f(v_1v_2) = 0$ and $f(e) \ne 0$ for all $e \in E(G) - \{v_1v_2\}$ and $P_{k+1} = (v_1, v_2, \ldots, v_{k+1})$ be a path of order $k + 1 \ge 5$ in $G$. Suppose that $f'(v_i) = x_i$ for $1 \le i \le k+1$ and $f(v_iv_{i+1}) = y_i$ for $1 \le i \le k$ where $y_1 = 0$. Let $P_k = (v_1, v_2, \ldots, v_k)$ be the subpath of order $k$ in $P_{k+1}$. We consider two cases, according to whether $k$ is even or $k$ is odd.

*Case* 1. *$k$ is even or $k + 1$ is odd.* By the induction hypothesis of $f$ on the path $P_k$, we have

$$y_i = \begin{cases} x_{i+1} - x_1 & \text{if } 2 \le i \le k-2 \text{ and } i \text{ is even},\\ x_{i+1} - x_2 & \text{if } 3 \le i \le k-1 \text{ and } i \text{ is odd}. \end{cases} \tag{3}$$

Furthermore, there is a modular edge-graceful labeling $g : E(G) \to \mathbb{Z}_n$ of $G$ such that

- $g(e) \ne 0$ for all $e \in E(G) - \{v_{k-1}v_k\}$ and $g(v_{k-1}v_k) = 0$,
- $g(e) = f(e)$ for all $e \in E(G) - E(P_k)$, and
- $g'(v_k) = x_2$ and $g'(v_{k-1}) = x_1$.

Hence $g(v_kv_{k+1}) = y_k = f(v_kv_{k+1})$. Define a labeling $h : E(G) \to \mathbb{Z}_n$ of $G$ from the modular edge-graceful labeling $g$ of $G$ by

$$h(e) = \begin{cases} g(e) & \text{if } \ e \ne v_{k-1}v_k, v_kv_{k+1},\\ x_{k+1} - x_1 & \text{if } \ e = v_{k-1}v_k,\\ y_k - (x_{k+1} - x_1) & \text{if } \ e = v_kv_{k+1}. \end{cases}$$

Observe that

(i) $h'(v) = g'(v)$ if $v \in V(G) - \{v_{k-1}, v_k, v_{k+1}\}$ and

(ii) by (3),

$$\begin{aligned} h'(v_{k-1}) &= g'(v_{k-1}) + (x_{k+1} - x_1) = x_1 + (x_{k+1} - x_1) = x_{k+1}, \\ h'(v_k) &= g'(v_k) + (x_{k+1} - x_1) - (x_{k+1} - x_1) = g'(v_k) = x_2, \\ h'(v_{k+1}) &= g'(v_{k+1}) - (x_{k+1} - x_1) = x_{k+1} - (x_{k+1} - x_1) = x_1. \end{aligned}$$

Thus $\{h'(v_i) : k-1 \le i \le k+1\} = \{g'(v_i) : k-1 \le i \le k+1\}$ and so $h$ is a modular edge-graceful labeling of $G$. Since $h(e) = g(e)$ for all $e \in E(G) - \{v_{k-1}v_k, v_kv_{k+1}\}$, it follows that $h(e) \neq 0$ if $e \neq v_{k-1}v_k, v_kv_{k+1}$. Also, $x_{k+1} \neq x_1$ and so $h(v_{k-1}v_k) = x_{k+1} - x_1 \neq 0$. Since $G$ is not nowhere-zero modular edge-graceful, $h(v_kv_{k+1}) = y_k - (x_{k+1} - x_1) = 0$ and so $y_k = x_{k+1} - x_1$. Hence (2) holds. Furthermore, the modular edge-graceful labeling $h : E(G) \to \mathbb{Z}_n$ of $G$ satisfies the following conditions:

- $h(e) \neq 0$ for all $e \in E(G) - \{v_kv_{k+1}\}$ and $h(v_kv_{k+1}) = 0$,
- $h(e) = g(e) = f(e)$ for all $e \notin E(P_{k+1})$, and
- $h'(v_{k+1}) = x_1$ and $h'(v_k) = x_2$.

Hence the result holds for $k$ is even (or $k+1$ is odd).

*Case* 2. *$k$ is odd or $k+1$ is even.* By the induction hypothesis of $f$ on the path $P_k$, we have

$$y_i = \begin{cases} x_{i+1} - x_1 & \text{if } 2 \le i \le k-1 \text{ and } i \text{ is even,} \\ x_{i+1} - x_2 & \text{if } 3 \le i \le k-2 \text{ and } i \text{ is odd.} \end{cases} \tag{4}$$

Furthermore, there is a modular edge-graceful labeling $g : E(G) \to \mathbb{Z}_n$ of $G$ such that

- $g(e) \neq 0$ for all $e \in E(G) - \{v_{k-1}v_k\}$ and $g(v_{k-1}v_k) = 0$,
- $g(e) = f(e)$ for all $e \in E(G) - E(P_k)$, and
- $g'(v_k) = x_1$ and $g'(v_{k-1}) = x_2$.

Hence $g(v_kv_{k+1}) = y_k = f(v_kv_{k+1})$. Define a labeling $h : E(G) \to \mathbb{Z}_n$ of $G$ from the modular edge-graceful labeling $g$ of $G$ by

$$h(e) = \begin{cases} g(e) & \text{if } e \neq v_{k-1}v_k, v_kv_{k+1}, \\ x_{k+1} - x_2 & \text{if } e = v_{k-1}v_k, \\ y_k - (x_{k+1} - x_2) & \text{if } e = v_kv_{k+1}. \end{cases}$$

Observe that

(i) $h'(v) = g'(v)$ if $v \in V(G) - \{v_{k-1}, v_k, v_{k+1}\}$ and

(ii) by (4),

$$\begin{aligned} h'(v_{k-1}) &= g'(v_{k-1}) + (x_{k+1} - x_2) = x_2 + (x_{k+1} - x_2) = x_{k+1}, \\ h'(v_k) &= g'(v_k) + (x_{k+1} - x_2) - (x_{k+1} - x_2) = g'(v_k) = x_1, \\ h'(v_{k+1}) &= g'(v_{k+1}) - (x_{k+1} - x_2) = x_{k+1} - (x_{k+1} - x_2) = x_2. \end{aligned}$$

Thus $\{h'(v_i) : k-1 \le i \le k+1\} = \{g'(v_i) : k-1 \le i \le k+1\}$ and so $h$ is a modular edge-graceful labeling of $G$. Since $h(e) = g(e)$ for all $e \in E(G) - \{v_{k-1}v_k, v_kv_{k+1}\}$, it follows that $h(e) \neq 0$ if $e \neq v_{k-1}v_k, v_kv_{k+1}$. Also, $x_{k+1} \neq x_2$ and so $h(v_{k-1}v_k) = x_{k+1} - x_2 \neq 0$. Since $G$ is not nowhere-zero modular edge-graceful, $h(v_kv_{k+1}) = y_k - (x_{k+1} - x_2) = 0$ and so $y_k = x_{k+1} - x_2$. Hence (2) holds. Furthermore, the modular edge-graceful labeling $h : E(G) \to \mathbb{Z}_n$ of $G$ satisfies the following conditions:

- $h(e) \neq 0$ for all $e \in E(G) - \{v_kv_{k+1}\}$ and $h(v_{k1}v_{k+1}) = 0$,
- $h(e) = g(e) = f(e)$ for all $e \notin E(P_{k+1})$, and
- $h'(v_{k+1}) = x_2$ and $h'(v_k) = x_1$.

Hence the result holds for $k$ is odd (or $k+1$ is even). ■

**Theorem 4.7.** *If $G$ is a connected modular edge-graceful graph of order $n \ge 4$ that is not a star, then $G$ is nowhere-zero modular edge-graceful.*

***Proof.*** Assume, to the contrary, that there is a connected modular edge-graceful graph $G$ of order $n \ge 4$ that is not a star such that $G$ is not nowhere-zero modular edge-graceful. By Theorem 3.3, $G$ is not a tree. By Theorem 4.4, $G$ does not contain an even cycle. By Lemma 4.5, $G$ is not an odd cycle. Since $G$ is connected, it follows that $G$ contains a unicyclic subgraph $H$ that is obtained from an odd cycle by adding a pendant edge. We may assume that $V(H) = \{v_1, v_2, v_3, \ldots, v_{2k+2}\}$, where $C_{2k+1} = (v_2, v_3, \ldots, v_{2k+2}, v_2)$ is the odd cycle of $H$ and $v_1v_2$ is the pendant edge of $H$.

Since $G$ is not nowhere-zero modular edge-graceful, by Lemma 4.6, there is a modular edge-graceful labeling $f : E(G) \to \mathbb{Z}_n$ of $G$ such that $f(v_1v_2) = 0$ and $f(e) \neq 0$ for all $e \in E(G) - \{v_1v_2\}$. Furthermore, if $P_{2k+2} = (v_1, v_2, v_3, \ldots, v_{2k+2})$ such that $f'(v_i) = x_i$ for $1 \le i \le 2k+2$ and $f(v_iv_{i+1}) = y_i$ for $1 \le i \le 2k+1$ where $y_1 = 0$, then

$$y_i = \begin{cases} x_{i+1} - x_1 & \text{if } 2 \le i \le 2k \text{ and } i \text{ is even,} \\ x_{i+1} - x_2 & \text{if } 3 \le i \le 2k+1 \text{ and } i \text{ is odd.} \end{cases} \tag{5}$$

Now consider the two paths $Q_1$ and $Q_2$ with initial edge $v_1v_2$ in $H$, namely,

$$\begin{aligned} Q_1 &= (v_1, v_2, v_3, v_4, \ldots, v_{k+2}, v_{k+3}), \\ Q_2 &= (v_1, v_2, v_{2k+2}, v_{2k+1}, v_{2k}, \ldots, v_{k+3}, v_{k+2}). \end{aligned}$$

Thus, $|V(Q_1)| = |V(Q_2)| = k+3$ and $E(Q_1) \cap E(Q_2) = \{v_1v_2, v_{k+2}v_{k+3}\}$.

First, assume that $k+2$ is odd. By traversing along the path $Q_1$ and applying (5), we obtain $y_{k+2} = x_{k+3} - x_1$. On the other hand, by traversing along the path $Q_2$ and applying Lemma 4.6, we obtain $y_{k+2} = x_{k+2} - x_1$. This implies that $x_{k+3} = x_{k+2}$, which is a contradiction.

Next, assume that $k+2$ is even. Then a similar argument shows that $x_{k+3} = x_{k+2}$, which is a contradiction. ■

Theorems 3.3, 4.4 and 4.7 then provide a characterization of connected nowhere-zero modular edge-graceful graphs.

**Theorem 4.8.** *A connected graph $G$ of order $n \geq 3$ is nowhere-zero modular edge-graceful if and only if*

(i) $n \not\equiv 2 \pmod 4$,

(ii) $G \neq K_3$ *and*

(iii) *$G$ is not a star of even order.*

## 5. Nowhere-zero Modular Edge-gracefulness

For every connected graph $G$ of order $n$, there is a smallest integer $k \geq n$ for which there exists an edge labeling $f: E(G) \to \mathbb{Z}_k - \{0\}$ such that the induced vertex labeling $f': V(G) \to \mathbb{Z}_k$ defined by $f'(v) = \sum_{u \in N(v)} f(uv)$, where the sum is computed in $\mathbb{Z}_k$, is one-to-one. This number $k$ is referred to as the *nowhere-zero modular edge-gracefulness* of $G$ and is denoted by $\mathrm{nzg}(G)$. Thus $\mathrm{nzg}(G) = n$ if and only if $G$ is nowhere-zero modular edge-graceful. Next, we determine $\mathrm{nzg}(G)$ for those graphs $G$ that are not nowhere-zero modular edge-graceful. We first consider connected graphs of order $n \geq 3$ where $n \not\equiv 2 \pmod 4$. By Theorem 4.8, it suffices to determine the nowhere-zero gracefulness of $K_3$ or a star of even order. Since $\mathrm{nzg}(K_3) = 4$, it remains to consider a star of even order.

**Corollary 5.1.** *For each odd integer $s \geq 3$,* $\mathrm{nzg}(K_{1,s}) = s + 3$.

***Proof.*** By Theorem 4.8, $\mathrm{nzg}(K_{1,s}) \geq s + 2$. First, we show that $\mathrm{nzg}(K_{1,s}) \neq s + 2$ for all odd integers $s \geq 3$. Assume, to the contrary, that there is an edge labeling $f: E(K_{1,s}) \to \mathbb{Z}_{s+2} - \{0\}$ such that the induced vertex labeling $f': V(K_{1,s}) \to \mathbb{Z}_{s+2}$ is an injective function. Since every edge of $K_{1,s}$ is incident with an end-vertex, it follows that $f$ is injective as well. Because $|\mathbb{Z}_{s+2} - \{0\}| = s + 1 = |E(K_{1,s})| + 1$, there is a unique $a \in \mathbb{Z}_{s+2} - \{0\}$ such that $f(e) \neq a$ for all $e \in E(K_{1,s})$. Since $s + 2$ is odd, $a \neq -a$ for all $a \in \mathbb{Z}_{s+2}$. This implies that there is $e = uv \in E(K_{1,s})$ such that $f(e) = -a$. Suppose that $u$ is an end-vertex of $K_{1,s}$ and $v$ is the central vertex of $K_{1,s}$. Then $f'(u) = f'(v) = -a$, which is a contradiction. Therefore, $\mathrm{nzg}(K_{1,s}) \neq s + 2$ and so $\mathrm{nzg}(K_{1,s}) \geq s + 3$.

To show that $\mathrm{nzg}(K_{1,s}) \leq s + 3$, we define an edge labeling $g: E(K_{1,s}) \to \mathbb{Z}_{s+3} - \{0\}$ such that the induced vertex labeling $g': V(K_{1,s}) \to \mathbb{Z}_{s+3}$ is an injective function. Let $V(K_{1,s}) = \{v, v_1, v_2, \ldots, v_s\}$ where $v$ is the central vertex of $K_{1,s}$. For $s = 3, 5$, let $g(vv_i) = i$ for $1 \leq i \leq s$; while for $s \geq 7$, let

$$g(vv_i) = \begin{cases} i + 1 & \text{if } 1 \leq i \leq \frac{s-1}{2} - 2, \\ i + 2 & \text{if } \frac{s-1}{2} - 1 \leq i \leq s. \end{cases}$$

In each case, $g'$ is injective and so $\mathrm{nzg}(K_{1,s}) = s + 3$. ■

By Theorem 4.8, $\operatorname{nzg}(K_3) = 4$ and Proposition 5.1, we have the following.

**Theorem 5.2.** *If $G$ is a connected graph of order $n \geq 3$ with $n \not\equiv 2 \pmod 4$ that is not nowhere-zero modular edge-graceful, then $\operatorname{nzg}(G) \in \{n+1, n+2\}$. Furthermore, $\operatorname{nzg}(G) = n+1$ if and only if $G = K_3$ and $\operatorname{nzg}(G) = n+2$ if and only if $G$ is a star of even order.*

By Theorem 1.1, if $G$ is a connected graph of order $n \geq 6$ where $n \equiv 2 \pmod 4$, then $G$ is not modular edge-graceful. Consequently, $G$ is not nowhere-zero modular edge-graceful and so $\operatorname{nzg}(G) \geq n+1$. Proceeding as above under the hypothesis that $n \equiv 2 \pmod 4$ rather than $n \not\equiv 2 \pmod 4$, then with a similar argument, both in length and method, the following can be established.

**Theorem 5.3.** *If $G$ is a connected graph of order $n \geq 6$ where $n \equiv 2 \pmod 4$, then $\operatorname{nzg}(G) \in \{n+1, n+2\}$. Furthermore, $\operatorname{nzg}(G) = n+2$ if and only if $G$ is a star.*

In summary then, we have the following.

**Theorem 5.4.** *If $G$ is a connected graph of order $n \geq 3$, then $\operatorname{nzg}(G) \in \{n, n+1, n+2\}$. Furthermore,*

- *$\operatorname{nzg}(G) = n$ if and only if $G$ is nowhere-zero modular edge-graceful,*
- *$\operatorname{nzg}(G) = n+1$ if and only if $G = K_3$ or $n \equiv 2 \pmod 4$ and $G$ is not a star of even order.*
- *$\operatorname{nzg}(G) = n+2$ if and only if $G$ is a star of even order.*

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