## Note

# ERDÖS-KO-RADO FROM INTERSECTING SHADOWS 

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#### Abstract

A set system is called $t$-intersecting if every two members meet each other in at least $t$ elements. Katona determined the minimum ratio of the shadow and the size of such families and showed that the Erdős-Ko-Rado theorem immediately follows from this result. The aim of this note is to reproduce the proof to obtain a slight improvement in the Kneser graph. We also give a brief overview of corresponding results.


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## 1. Introduction

Throughout the paper we will investigate subsets of an $n$-element underlying set $[n]=\{1,2, \ldots, n\} .\binom{[n]}{k}$ will denote the collection of all $k$-element subsets of $[n]$. A family $\mathcal{F}$ is said to be $k$-uniform if $\mathcal{F} \subseteq\binom{[n]}{k}$.
$\mathcal{F} \subseteq\binom{[n]}{k}$ is called intersecting if it does not contain disjoint sets. In general, $\mathcal{F}$ is $t$-intersecting if $\left|F_{1} \cap F_{2}\right| \geq t$ for all $F_{1}, F_{2} \in \mathcal{F}$.

The Kneser graph, $\operatorname{Kn}(n, k)$, is the graph whose vertices are the $k$-element subsets of $[n]$, i.e. $V(\operatorname{Kn}(n, k))=\binom{[n]}{k}$ and two vertices are connected iff the two corresponding sets are disjoint. A coclique in a graph is a set of vertices, such that no two vertices in the set are adjacent. An intersecting family is a coclique in the corresponding Kneser graph. The maximum size of a coclique in a graph $G$ is denoted by $\alpha(G)$.

The following theorem is one of the famous results in extremal combinatorics:

Theorem 1 (Erdős, Ko, Rado [3]). If $k \leq n / 2$, then

$$
\alpha(\operatorname{Kn}(n, k))=\binom{n-1}{k-1}
$$

Obviously, the family consisting of the $k$-subsets that contain 1 has size $\binom{n-1}{k-1}$, so only the $\leq$ part is interesting.

Let $\mathcal{F} \subseteq\binom{X}{k}$ be a family of $k$-element sets; for $l \leq k$, the $l$-shadow of $\mathcal{F}$ is defined as $\Delta_{l} \mathcal{F}=\{G:|G|=l$, and there exists $F \in \mathcal{F}$ such that $G \subset F\}$. It is clear that $\mathcal{F}=\binom{[2 k-t]}{k}$ is $t$-intersecting and $\Delta_{l} \mathcal{F}=\binom{[2 k-t]}{l}$. The next theorem shows that this is the extremal case in some sense.

Theorem 2 (Katona [5]). Assume that $\mathcal{F}$ is a $k$-uniform, $t$-intersecting family. Then for $l \geq k-t$,

$$
\frac{\left|\Delta_{l} \mathcal{F}\right|}{|\mathcal{F}|} \geq \frac{\binom{2 k-t}{l}}{\binom{2 k-t}{k}}
$$

## 2. A Generalization of the EKR Theorem

In this section we deduce a slight generalization of the EKR theorem from Theorem 2.

For a set $A \subseteq V(\operatorname{Kn}(n, k))$, the neighborhood of $A$ is denoted by $N(A)$. Similarly, for a given $k$-uniform family $\mathcal{F}$, let us introduce the notation $\mathcal{N}(\mathcal{F})=$ $\left\{H \in\binom{[n]}{k}\right.$ : there exists $F \in \mathcal{F}$ such that $\left.H \cap F=\emptyset\right\}$ as the "neighborhood" of $\mathcal{F}$.

Theorem 3. If $k \leq n / 2$ and $C$ is a coclique in the Kneser $\operatorname{graph}, \operatorname{Kn}(n, k)$, then

$$
\frac{|C|}{|C|+|N(C)|} \leq \frac{k}{n}
$$

Since $C$ is a coclique, $C$ and $N(C)$ are disjoint, so $|C|+|N(C)| \leq|V(\operatorname{Kn}(n, k))|=$ $\binom{n}{k}$ and the EKR theorem follows.

Proof of Theorem 3. To apply Theorem 2 , let $\mathcal{F}$ be the intersecting $k$-uniform family that corresponds to $C$. Let $\mathcal{F}^{c}$ be the family of complements, i.e. $\mathcal{F}^{c}=$ $\{[n] \backslash F: F \in \mathcal{F}\} \subseteq\binom{[n]}{n-k}$. For each pair $F_{1}, F_{2} \in \mathcal{F}$, we have $\left|F_{1} \cup F_{2}\right| \leq 2 k-1$, thus $\mathcal{F}^{c}$ is $t$-intersecting for $t=n-2 k+1$. By Theorem 2,

$$
\frac{\left|\Delta_{k} \mathcal{F}^{c}\right|}{\left|\mathcal{F}^{c}\right|} \geq \frac{\binom{2(n-k)-(n-2 k+1)}{k}}{\binom{2(n-k)-(n-2 k+1)}{n-k}}=\frac{n-k}{k} .
$$

$\left|\mathcal{F}^{c}\right|=|\mathcal{F}|$ and $\Delta_{k} \mathcal{F}^{c} \subseteq \mathcal{N}(\mathcal{F})$, because for every $H \in \Delta_{k} \mathcal{F}^{c}, H \subseteq[n] \backslash F$ for some $F \in \mathcal{F}$ and $H \cap F=\emptyset$. Thus,

$$
\frac{|N(C)|}{|C|}=\frac{|\mathcal{N}(\mathcal{F})|}{|\mathcal{F}|} \geq \frac{n-k}{k}
$$

and we are done.

## 3. Similar Results

Let $A \subseteq V(\operatorname{Kn}(n, k))$. For another slight generalization, we denote by $I(A)$ the family of isolated points in $A$, that is,

$$
I(A)=\{a \in A:(a, b) \notin E(\operatorname{Kn}(n, k)) \text { for all } b \in A\}
$$

In his paper, Borg [1] extended Daykin's proof [2] of the EKR theorem to obtain the following improvement:

Theorem 4 (Borg). If $A \subseteq V(\operatorname{Kn}(n, k))$ and $k \leq n / 2$, then

$$
|I(A)|+\frac{k}{n}|A \backslash I(A)| \leq\binom{ n-1}{k-1}
$$

It is easy to see that Theorems 3 and 4 are equivalent.
First, let $A$ be an arbitrary subgraph of $\operatorname{Kn}(n, k) . C:=I(A)$ is a coclique, so by Theorem $3, \frac{|I(A)|}{|I(A)|+|N(I(A))|} \leq \frac{k}{n}$.
By definition, $I(A), A \backslash I(A)$ and $N(I(A))$ are disjoint, hence $|I(A)|+|A \backslash I(A)|+$ $|N(I(A))| \leq\binom{ n}{k}$.
These two inequalities now imply Theorem 4.
On the other hand, if $C$ is a coclique, let $A:=V(\operatorname{Kn}(n, k)) \backslash N(C)$. By definition, $C$ and $N(C)$ are disjoint, and $C \subseteq I(A)$. Thus, by Theorem 4,

$$
|C|+\frac{k}{n}|V(\operatorname{Kn}(n, k)) \backslash N(C) \backslash C| \leq\binom{ n-1}{k-1}
$$

and Theorem 3 follows.
Remember that though the two theorems are equivalent, their proofs are quite different: while Theorem 3 is proved as a consequence of the theorem on shadows of intersecting families, Borg uses the Kruskal-Katona theorem $[6,7]$ to verify Theorem 4.

Remark 5. In [1], Borg also showed that Theorem 4 (and so Theorem 3) yields Hilton's theorem [4] for cross-intersecting sub-families of $\binom{[n]}{k}$.

Recently, J. Wang and H. Zhang [8, 9] investigated similar problems in general circumstances. A graph $G=(V, E)$ is called vertex-transitive if its automorphism group, $\operatorname{Aut}(G)$, acts transitively on $V$, i.e. for every $u, v \in V$ there exists a $\gamma \in \operatorname{Aut}(G)$ such that $\gamma(u)=v$.

The following theorem is the analogue of Theorem 3 for arbitrary vertextransitive graph.

Theorem 6 (Zhang). Let $G=(V, E)$ be a vertex-transitive simple graph. If $C \subseteq V$ is a coclique, then

$$
\frac{|C|}{|C|+|N(C)|} \leq \frac{\alpha(G)}{|V|}
$$

Note that the EKR theorem and Theorem 6 together imply Theorem 3.

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