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# THE TOTAL $\{k\}$-DOMATIC NUMBER OF DIGRAPHS

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**Abstract**

For a positive integer $k$, a *total* $\{k\}$*-dominating function* of a digraph $D$ is a function $f$ from the vertex set $V(D)$ to the set $\{0, 1, 2, \ldots, k\}$ such that for any vertex $v \in V(D)$, the condition $\sum_{u \in N^-(v)} f(u) \geq k$ is fulfilled, where $N^-(v)$ consists of all vertices of $D$ from which arcs go into $v$. A set $\{f_1, f_2, \ldots, f_d\}$ of total $\{k\}$-dominating functions of $D$ with the property that $\sum_{i=1}^{d} f_i(v) \leq k$ for each $v \in V(D)$, is called a *total* $\{k\}$*-dominating family* (of functions) on $D$. The maximum number of functions in a total $\{k\}$-dominating family on $D$ is the *total* $\{k\}$*-domatic number* of $D$, denoted by $d_t^{\{k\}}(D)$. Note that $d_t^{\{1\}}(D)$ is the classic total domatic number $d_t(D)$. In this paper we initiate the study of the total $\{k\}$-domatic number in digraphs, and we present some bounds for $d_t^{\{k\}}(D)$. Some of our results are extensions of well-know properties of the total domatic number of digraphs and the total $\{k\}$-domatic number of graphs.

**Keywords:** digraph, total $\{k\}$-dominating function, total $\{k\}$-domination number, total $\{k\}$-domatic number.

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## 1. Introduction

In this paper, $D$ is a finite and simple digraph with vertex set $V = V(D)$ and arc set $A = A(D)$. The order $|V|$ of $D$ is denoted by $n = n(D)$. We write $d_D^+(v) = d^+(v)$ for the *outdegree* of a vertex $v$ and $d_D^-(v) = d^-(v)$ for its *indegree*. The *minimum* and *maximum indegree* are $\delta^-(D)$ and $\Delta^-(D)$. The sets $N^+(v) = \{x|(v,x) \in A(D)\}$ and $N^-(v) = \{x|(x,v) \in A(D)\}$ are called the *outset* and *inset* of the vertex $v$. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by $X$. For an arc $(x,y) \in A(D)$, the vertex $y$ is an *outer neighbor* of $x$ and $x$ is an *inner neighbor* of $y$. We write $K_n^*$ for the *complete digraph* of order $n$. Consult [5] for the notation and terminology which are not defined here.

For a positive integer $k$, a *total $\{k\}$-dominating function* (T$\{k\}$DF) of a digraph $D$ with $\delta^-(D) \geq 1$ is a function $f$ from the vertex set $V(D)$ to the set $\{0,1,2,\ldots,k\}$ such that for any vertex $v \in V(D)$, the condition $\sum_{u\in N^-(v)} f(u) \geq k$ is fulfilled. The *weight* of a T$\{k\}$DF $f$ is the value $\omega(f) = \sum_{v\in V(D)} f(v)$. The *total $\{k\}$-domination number* of a digraph $D$, denoted by $\gamma_t^{\{k\}}(D)$, is the minimum weight of a T$\{k\}$DF of $D$. A $\gamma_t^{\{k\}}(D)$*-function* is a total $\{k\}$-dominating function of $D$ with weight $\gamma_t^{\{k\}}(D)$. Note that $\gamma_t^{\{1\}}(D)$ is the classical total domination number $\gamma_t(D)$. If $F$ is a minimum total dominating set of a digraph $D$ with $\delta^-(D) \geq 1$, then the function $f$ from $V(D)$ to $\{0,1,2,\ldots,k\}$ with $f(v) = k$ for $v \in F$ and $f(x) = 0$ for $x \in V(D) - F$ is a total $\{k\}$-dominating function of $D$ and therefore

$$\gamma_t^{\{k\}}(D) \leq k|F| = k\gamma_t(D).$$

In this paper we always assume that $D$ is a digraph with $\delta^-(D) \geq 1$.

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct total $\{k\}$-dominating functions of $D$ with the property that $\sum_{i=1}^d f_i(v) \leq k$ for each $v \in V(D)$, is called a *total $\{k\}$-dominating family* (of functions) on $D$. The maximum number of functions in a total $\{k\}$-dominating family (T$\{k\}$D family) on $D$ is the *total $\{k\}$-domatic number* of $D$, denoted by $d_t^{\{k\}}(D)$. The total $\{k\}$-domatic number is well-defined and

$$d_t^{\{k\}}(D) \geq 1, \text{ for all digraphs } D \text{ with } \delta^-(D) \geq 1, \tag{1}$$

since the set consisting of the function $f : V(D) \to \{0,1,2,\ldots,k\}$ defined by $f(v) = k$ for each $v \in V(D)$, forms a T$\{k\}$D family on $D$. The total domatic number of a digraph was introduced by Jacob and Arumugam in [6].

Our purpose in this paper is to initiate the study of the total $\{k\}$-domatic number in digraphs. We first study basic properties and bounds for the total $\{k\}$-domatic number of a digraph. In addition, we determine the total $\{k\}$-domatic number of some classes of digraphs. Some of our results are extensions of well-know properties of the total domatic number of digraphs and the total $\{k\}$-domatic number of graphs (see, for example, [2, 3, 4, 6, 8]).

We start with the following observation.

**Observation 1.** *Let $k$ be an integer, and let $D$ be a digraph with $\delta^-(D) \geq 1$. Then $\gamma_t^{\{k\}}(D) \geq k+1$, with equality if and only if there exists a subset $S \subseteq V(D)$ of size $k+1$ such that $D[S]$ is a complete digraph, and each vertex $x \in V(D) - S$ has at least $k$ inner neighbors in $S$.*

***Proof.*** Let $f$ be a $\gamma_t^{\{k\}}(D)$-function, and let $v \in V(D)$ be an arbitrary vertex. The definition implies that $\sum_{x \in N^-(v)} f(x) \geq k$. If $\sum_{x \in N^-(v)} f(x) \geq k+1$, then $\gamma_t^{\{k\}}(D) \geq k+1$. If $\sum_{x \in N^-(v)} f(x) = k$, then let $u \in N^-(v)$ be a vertex such that $f(u) \geq 1$. Since $\sum_{x \in N^-(u)} f(x) \geq k$ and $u \notin N^-(u)$, we deduce that $\omega(f) = \sum_{x \in V(D)} f(v) \geq \sum_{x \in (N^-(u) \cup \{u\})} f(x) \geq k+1$ and therefore $\gamma_t^{\{k\}}(D) \geq k+1$.

Assume that $\gamma_t^{\{k\}}(D) = k+1$. Let $f$ be a $\gamma_t^{\{k\}}(D)$-function. If there exists a vertex $v$ such that $f(v) \geq 2$, then we obtain the contradiction $\sum_{x \in N^-(v)} f(x) \leq k+1-2 = k-1$. Hence $f(x) = 1$ or $f(x) = 0$ for each vertex $x \in V(D)$. Let $S \subseteq V(D)$ such that $f(x) = 1$ for each $x \in S$. Then $|S| = k+1$, $D[S]$ is a complete digraph, and each vertex $x \in V(D) - S$ has at least $k$ inner neighbors in $S$.

Conversely, assume that there exists a subset $S \subseteq V(D)$ of size $k+1$ such that $D[S]$ is a complete digraph, and each vertex $x \in V(D) - S$ has at least $k$ inner neighbors in $S$. Define the function $f$ by $f(x) = 1$ for $x \in S$ and $f(x) = 0$ for $x \in V(D) - S$. Then $f$ is a total $\{k\}$-dominating function of $D$ such that $\omega(f) = k+1$. Since $\gamma_t^{\{k\}}(D) \geq k+1$, we deduce that $\gamma_t^{\{k\}}(D) = k+1$. ∎

## 2. Properties of the $\{k\}$-domatic Number

In this section we mainly present basic properties of $d_t^{\{k\}}(D)$ and bounds on the total $\{k\}$-domatic number of a digraph.

**Theorem 2.** *If $D$ is a digraph of order $n$, then $\gamma_t^{\{k\}}(D) \cdot d_t^{\{k\}}(D) \leq kn$. Moreover, if $\gamma_t^{\{k\}}(D) \cdot d_t^{\{k\}}(D) = kn$, then for each T$\{k\}$D family $\{f_1, f_2, \ldots, f_d\}$ on $D$ with $d = d_t^{\{k\}}(D)$, each function $f_i$ is a $\gamma_t^{\{k\}}(D)$-function and $\sum_{i=1}^d f_i(v) = k$ for all $v \in V(D)$.*

***Proof.*** Let $\{f_1, f_2, \ldots, f_d\}$ be a T$\{k\}$D family on $D$ such that $d = d_t^{\{k\}}(D)$. Then

$$\begin{aligned} d \cdot \gamma_t^{\{k\}}(D) &= \sum_{i=1}^d \gamma_t^{\{k\}}(D) \leq \sum_{i=1}^d \sum_{v \in V(D)} f_i(v) = \sum_{v \in V(D)} \sum_{i=1}^d f_i(v) \\ &\leq \sum_{v \in V(D)} k = kn. \end{aligned}$$

If $\gamma_t^{\{k\}}(D) \cdot d_t^{\{k\}}(D) = kn$, then the two inequalities occurring in the proof become equalities. Hence for the T$\{k\}$D family $\{f_1, f_2, \ldots, f_d\}$ on $D$ and for each $i$,

$\sum_{v\in V(D)} f_i(v) = \gamma_t^{\{k\}}(D)$. Thus each function $f_i$ is a $\gamma_t^{\{k\}}(D)$-function, and $\sum_{i=1}^{d} f_i(v) = k$ for all $v \in V(D)$. ■

The special case $k = 1$ in Theorem 2 can be found in [6].

**Corollary 3.** *Let $k, n$ be two positive integers. If $k+1$ is a divisor of $n$ and $\frac{n}{k+1} \geq 2$, then $d_t^{\{k\}}(K_n^*) = \frac{kn}{k+1}$.*

***Proof.*** Applying Observation 1 and Theorem 2, we see that $d_t^{\{k\}}(K_n^*) \leq \frac{kn}{k+1}$.

Now we consider a partition of $V(K_n^*)$ into $s = \frac{n}{k+1}$ sets $V_1, V_2, \ldots, V_s$ such that $|V_i| = k+1$ for each $i$. Let $V_i = \{v_1^i, v_2^i, \ldots, v_{k+1}^i\}$ for $1 \leq i \leq s$. Define, for $1 \leq i \leq s$ and $1 \leq j \leq k$,

$$f_i^j(v_1^i) = \cdots = f_i^j(v_j^i) = 1,\ f_i^j(v_{j+1}^{i+1}) = \cdots = f_i^j(v_{k+1}^{i+1}) = 1 \text{ and}$$
$$f_i^j(x) = 0 \text{ otherwise, where the indices } i+1 \text{ are taken modulo } s.$$

It is easy see that $\{f_i^j \mid 1 \leq i \leq \frac{n}{k+1}, 1 \leq j \leq k,\}$ is a T$\{k\}$D family on $K_n^*$, and therefore $d_t^{\{k\}}(K_n^*) \geq \frac{kn}{k+1}$. Since $k+1$ is a divisor of $n$, the proof is complete. ■

A further consequence of Theorem 2 and Observation 1 now follows.

**Corollary 4.** *If $k \geq 2$ is an integer, and $D$ is a digraph of order $k+1$, then $d_t^{\{k\}}(D) \leq k-1$.*

***Proof.*** Since $\gamma_t^{\{k\}}(D) \geq k+1$, it follows from Theorem 2 that $d_t^{\{k\}}(D) \leq k$. If $\gamma_t^{\{k\}}(D) \geq k+2$, then Theorem 2 implies $d_t^{\{k\}}(D) \leq k-1$ immediately. If $\gamma_t^{\{k\}}(D) = k+1$ and $d_t^{\{k\}}(D) = k$, then for the T$\{k\}$D family $\{f_1, f_2, \ldots, f_k\}$ on $D$, each function $f_i$ is a $\gamma_t^{\{k\}}(D)$-function, and Observation 1 leads to the contradiction that $f_1 \equiv f_2 \equiv \cdots \equiv f_k$. This completes the proof. ■

**Corollary 5.** *If $k$ is a positive integer, and $D$ is a digraph of order $n$, then $d_t^{\{k\}}(D) \leq \frac{kn}{k+1}$, with equality only if $k+1$ is a divisor of $n$ and $\frac{n}{k+1} \geq 2$ when $k \geq 2$.*

***Proof.*** Since $\gamma_t^{\{k\}}(D) \geq k+1$, it follows from Theorem 2 that $d_t^{\{k\}}(D) \leq \frac{kn}{\gamma_t^{\{k\}}(D)} \leq \frac{kn}{k+1}$, and this is the desired inequality.

Assume that $d_t^{\{k\}}(D) = \frac{kn}{k+1}$. Since $(k, k+1) = 1$, $k+1$ must be a divisor of $n$. If $k \geq 2$, then it follows from Corollary 4 that $\frac{n}{k+1} \geq 2$. ■

Corollary 3 demonstrates that Corollary 5 is sharp.

**Theorem 6.** *If $D$ is a digraph of order $n$ and $k$ a positive integer, then*

$$\gamma_t^{\{k\}}(D) + d_t^{\{k\}}(D) \leq nk + 1.$$

***Proof.*** Applying Theorem 2, we obtain $\gamma_t^{\{k\}}(D) + d_t^{\{k\}}(D) \le \frac{kn}{d_t^{\{k\}}(D)} + d_t^{\{k\}}(D)$. Note that $d_t^{\{k\}}(G) \ge 1$, by inequality (1), and that Corollary 5 implies that $d_t^{\{k\}}(D) \le n$. Using these inequalities, and the fact that the function $g(x) = x + (kn)/x$ is decreasing for $1 \le x \le \sqrt{kn}$ and increasing for $\sqrt{kn} \le x \le n$, we obtain $\gamma_t^{\{k\}}(D) + d_t^{\{k\}}(D) \le \max\left\{kn+1, \frac{kn}{n} + n\right\} = nk+1$, and this is the desired bound. ■

If $C_n$ denotes a directed cycle on $n$ vertices, then the function $f : V(C_n) \to \{0, 1, \ldots, k\}$ defined by $f(x) = k$ for each $x \in V(C_n)$ is the unique total $\{k\}$-dominating function of $C_n$ and hence $\gamma_t^{\{k\}}(C_n) = nk$ and $d_t^{\{k\}}(C_n) = 1$. This demonstrates that Theorem 6 is sharp.

**Theorem 7.** *Let $D$ be a digraph of order $n \ge 3$, and let $k \ge 1$ be an integer. If $d_t^{\{k\}}(D) \ge 2$, then $\gamma_t^{\{k\}}(D) + d_t^{\{k\}}(D) \le \frac{kn}{2} + 2$.*

***Proof.*** Theorem 2 implies that $\gamma_t^{\{k\}}(D)+d_t^{\{k\}}(D) \le \gamma_t^{\{k\}}(D)+\frac{kn}{\gamma_t^{\{k\}}(D)}$. It follows from Observation 1 and Theorem 2 that $k+1 \le \gamma_t^{\{k\}}(D) \le kn/2$. Using these inequalities, and the fact that the function $g(x) = x + (kn)/x$ is decreasing for $k+1 \le x \le \sqrt{kn}$ and increasing for $\sqrt{kn} \le x \le kn/2$, we obtain

$$\gamma_t^{\{k\}}(G) + d_t^{\{k\}}(G) \le \max\left\{k+1+\frac{kn}{k+1}, \frac{kn}{2}+2\right\} = \frac{kn}{2} + 2,$$

and this is the desired bound. ■

**Theorem 8.** *If $D$ is a digraph and $k \ge 1$ an integer, then $d_t^{\{k\}}(D) \le \delta^-(D)$. Moreover, if $d_t^{\{k\}}(D) = \delta^-(D)$, then for each function of any T$\{k\}$D family $\{f_1, f_2, \ldots, f_d\}$ and for all vertices $v$ of indegree $\delta^-(D)$, $\sum_{u\in N^-(v)} f_i(u) = k$ and $\sum_{i=1}^{d} f_i(u) = k$ for every $u \in N^-(v)$.*

***Proof.*** Let $\{f_1, f_2, \ldots, f_d\}$ be a T$\{k\}$D family on $D$ such that $d = d_t^{\{k\}}(D)$, and let $v$ be a vertex of minimum indegree $\delta^-(D)$. Since $\sum_{u\in N^-(v)} f_i(u) \ge k$ for all $i \in \{1, 2, \ldots, d\}$, we obtain $kd \le \sum_{i=1}^{d}\sum_{u\in N^-(v)} f_i(u) = \sum_{u\in N^-(v)}\sum_{i=1}^{d} f_i(u) \le \sum_{u\in N^-(v)} k = k\delta^-(D)$, and this leads to the desired bound.

If $d_t^{\{k\}}(D) = \delta^-(D)$, then the two inequalities occurring in the proof become equalities, which leads to the two properties given in the statement. ■

The special case $k = 1$ in Theorem 8 can be found in [6].

**Observation 9.** *Let $D$ be a digraph with the property that the underlying graph is connected and bipartite. If $k \ge 1$ is an integer, then $\gamma_t^{\{k\}}(D) \ge 2k$.*

***Proof.*** Let $f$ be a $\gamma_t^{\{k\}}(D)$-function, and let $V_1$ and $V_2$ be the partite sets of the underlying graph. If $w_i \in V_i$, then the definition implies that $\sum_{x\in N^-(w_i)} f(x) \geq k$ for $i = 1, 2$. It follows that $w(f) = \sum_{x\in V(D)} f(x) = \sum_{x\in V(D)-V_1} f(x) + \sum_{x\in V(D)-V_2} f(x) \geq \sum_{x\in N^-(w_2)} f(x) + \sum_{x\in N^-(w_1)} f(x) \geq 2k$, thus $\gamma_t^{\{k\}}(D) \geq 2k$. ■

**Corollary 10.** *If $K^*_{p,p}$ is the complete bipartite digraph and $k \geq 1$ an integer, then $d_t^{\{k\}}(K^*_{p,p}) = p$.*

***Proof.*** Theorem 2 and Observation 9 show that $d_t^{\{k\}}(K^*_{p,p}) \leq p$.

Now let $\{u_1, u_2, \ldots, u_p\}$ and $\{v_1, v_2, \ldots, v_p\}$ be the partite sets of the complete bipartite digraph. Define $f_i(u_i) = f_i(v_i) = k$ and $f_i(x) = 0$ for each vertex $x \in V(D) - \{u_i, v_i\}$ and each $i \in \{1, 2, \ldots, p\}$. Then we observe that $f_i$ is a T$\{k\}$DF of $K^*_{p,p}$ for each $i \in \{1, 2, \ldots, p\}$. Therefore $\{f_1, f_2, \ldots, f_p\}$ is a T$\{k\}$D family on $K^*_{p,p}$. Consequently, $d_t^{\{k\}}(K^*_{p,p}) \geq p$ and so $d_t^{\{k\}}(K^*_{p,p}) = p$. ■

Corollary 10 demonstrates that Theorem 8 is sharp.

**Theorem 11.** *Let $k \geq 1$ be an integer and $D$ a digraph of order $n$ with $\delta^-(D) \geq 1$. If $\delta^-(D) \mid k$, then $d_t^{\{k\}}(D) \geq \delta^-(D) - 1$.*

***Proof.*** If $\delta^-(D) = 1$, then the result is immediate.

Let $\delta^-(D) \geq 2$ and let $V(D) = \{v_1, v_2, \ldots, v_n\}$. Define $f_i : V(D) \to \{0, 1, \ldots, k\}$ by

$$f_i(v_j) = \begin{cases} \frac{k}{\delta^-(D)} + 1 & \text{if } j = i, \\ \frac{k}{\delta^-(D)} & \text{if } j \neq i, \end{cases} \quad \text{for every } 1 \leq i \leq \delta^-(D) - 1 \text{ and } 1 \leq j \leq n.$$

Then for each $v \in V(D)$ and each $1 \leq i \leq \delta^-(D) - 1$,

$$\textstyle\sum_{u\in N^-(v)} f_i(u) \geq \sum_{u\in N^-(v)} \frac{k}{\delta^-(D)} \geq \frac{k}{\delta^-(D)}\delta^-(D) = k.$$

Hence $f_i$ is a T$\{k\}$DF of $D$ for each $1 \leq i \leq \delta^-(D) - 1$. Now, since $\delta^-(D) \mid k$, we have

$$\textstyle\sum_{i=1}^{\delta(D)-1} f_i(v) \leq \frac{k}{\delta^-(D)}(\delta^-(D) - 2) + \left(\frac{k}{\delta^-(D)} + 1\right) = k + \left(1 - \frac{k}{\delta^-(D)}\right) \leq k$$

for each $v \in V(D)$. Thus $\{f_1, f_2, \ldots, f_{\delta^-(D)-1}\}$ is a T$\{k\}$D family on $D$, and the proof is complete. ■

**Theorem 12.** *Let $k \geq 1$ be an integer and $D$ a digraph of order $n$. If $\delta^-(D) \nmid k$, then $d_t^{\{k\}}(D) \geq \left\lfloor \frac{k}{\lceil k/\delta^-(D)\rceil} \right\rfloor$.*

***Proof.*** Let $V(D)=\{v_1,v_2,\ldots,v_n\}$. Define $f_i: V(D)\to\{0,1,\ldots,k\}$ by

$$f_i(v_j)=\begin{cases}\lfloor\frac{k}{\delta^-(D)}\rfloor & \text{if } \ j=i,\\ \lceil\frac{k}{\delta^-(D)}\rceil & \text{if } \ j\neq i,\end{cases}\quad \text{for every } 1\le i\le\left\lfloor\frac{k}{\lceil k/\delta^-(D)\rceil}\right\rfloor \text{ and } 1\le j\le n.$$

Then for each $v\in V(D)$ and each $1\le i\le\left\lfloor\frac{k}{\lceil k/\delta^-(D)\rceil}\right\rfloor$,

$$\sum_{u\in N^-(v)} f_i(u)\ge\left\lfloor\frac{k}{\delta^-(D)}\right\rfloor+\left\lceil\frac{k}{\delta^-(D)}\right\rceil(\delta^-(D)-1)\ge\left\lceil\frac{k}{\delta^-(D)}\right\rceil\delta^-(D)-1\ge k.$$

Hence $f_i$ is a T$\{k\}$DF of $D$ for each $i$. Since $\delta^-(D)\nmid k$, we have

$$\sum_{i=1}^{\left\lfloor\frac{k}{\lceil k/\delta^-(D)\rceil}\right\rfloor} f_i(v)\le\left\lceil\frac{k}{\delta^-(D)}\right\rceil\cdot\left\lfloor\frac{k}{\lceil\frac{k}{\delta^-(D)}\rceil}\right\rfloor\le\left\lceil\frac{k}{\delta^-(D)}\right\rceil\cdot\frac{k}{\lceil\frac{k}{\delta^-(D)}\rceil}=k$$

for each $v\in V(D)$. Thus $\{f_1,f_2,\ldots,f_{\left\lfloor\frac{k}{\lceil k/\delta^-(D)\rceil}\right\rfloor}\}$ is a T$\{k\}$D family on $D$, and the proof is complete. ∎

Using Theorems 2, 8, 11 and 12, we will improve Theorem 6 considerably for some cases.

**Corollary 13.** *Let $k\ge 1$ be an integer, and let $D$ be a digraph of order $n$. If $\delta^-(D)>k$, then $\gamma_t^{\{k\}}(D)+d_t^{\{k\}}(D)\le n+k$.*

***Proof.*** Since $\delta^-(D)>k$, it follows from Theorem 12 that

$$d_t^{\{k\}}(D)\ge\left\lfloor\frac{k}{\lceil\frac{k}{\delta^-(D)}\rceil}\right\rfloor=k.$$

In addition, Theorem 8 implies that $d_t^{\{k\}}(D)\le\delta^-(D)\le n$. Using these two inequalities, and the fact that the function $g(x)=x+(kn)/x$ is decreasing for $k\le x\le\sqrt{kn}$ and increasing for $\sqrt{kn}\le x\le n$, Theorem 2 leads to

$$\gamma_t^{\{k\}}(D)+d_t^{\{k\}}(D)\le\frac{kn}{d_t\{k\}(D)}+d_t^{\{k\}}(D)\le\max\left\{\frac{kn}{k}+k,\frac{kn}{n}+n\right\}=n+k.$$

This is the desired bound, and the proof is complete. ∎

**Corollary 14.** *Let $k\ge1$ be an integer, and let $D$ be a digraph of order $n$ with $\delta^-(D)\ge2$. If $\delta^-(D)\mid k$, then $\gamma_t^{\{k\}}(D)+d_t^{\{k\}}(D)\le\frac{kn}{\delta^-(D)-1}+\delta^-(D)-1$.*

***Proof.*** Since $\delta^-(D)\mid k$, Theorem 11 shows that $d_t^{\{k\}}(D)\ge\delta^-(D)-1$, and Theorem 8 implies that $d_t^{\{k\}}(D)\le\delta^-(D)$. Using these two inequalities and Theorem 2, we obtain the desired bound as follows $\gamma_t^{\{k\}}(D)+d_t^{\{k\}}(D)\le\frac{kn}{d_t^{\{k\}}(D)}+d_t^{\{k\}}(D)\le\max\left\{\frac{kn}{\delta^-(D)-1}+\delta^-(D)-1,\frac{kn}{\delta^-(D)}+\delta^-(D)\right\}=\frac{kn}{\delta^-(D)-1}+\delta^-(D)-1.$ ∎

Let $D$ be a digraph. By $D^{-1}$ we denote the digraph obtained by reversing all arcs of $D$. A digraph without directed cycles of length 2 is called an *oriented graph.* An oriented graph $D$ is a *tournament* when either $(x,y) \in A(D)$ or $(y,x) \in A(D)$ for each pair of distinct vertices $x, y \in V(D)$.

**Theorem 15.** *For every oriented graph $D$ with $\delta^-(D) \geq 1$ and $\delta^-(D^{-1}) \geq 1$, $d_t^{\{k\}}(D) + d_t^{\{k\}}(D^{-1}) \leq n-1$. If $d_t^{\{k\}}(D) + d_t^{\{k\}}(D^{-1}) = n-1$, then $D$ is a regular tournament.*

***Proof.*** Since $\delta^-(D) + \delta^-(D^{-1}) \leq n-1$, Theorem 8 leads to

$$d_t^{\{k\}}(D) + d_t^{\{k\}}(D^{-1}) \leq \delta^-(D) + \delta^-(D^{-1}) \leq n-1.$$

If $D$ is not a tournament or $D$ is a non-regular tournament, then $\delta^-(D) + \delta^-(D^{-1}) \leq n-2$, and hence we deduce from Theorem 8 that

$$d_t^{\{k\}}(D) + d_t^{\{k\}}(D^{-1}) \leq \delta^-(D) + \delta^-(D^{-1}) \leq n-2.$$

■

Now we present further lower bounds on the total $\{k\}$-domatic number.

**Theorem 16.** *Let $k \geq 1$ be an integer, and $D$ a digraph with $\delta^-(D) = \delta^- \geq 1$.*

(i) *If $k < \delta^-$, then $d_t^{\{k\}}(D) \geq k$.*

(ii) *If $k = p\delta^-$ with an integer $p \geq 1$, then $d_t^{\{k\}}(D) \geq \delta^- - 1$.*

(iii) *If $k = p\delta^- + r$ with integers $p, r \geq 1$ and $r \leq \delta^- - 1$, then $d_t^{\{k\}}(D) \geq \left\lceil \frac{p(\delta^- - 1)+1}{p+1} \right\rceil$.*

***Proof.*** (i) If $k < \delta^-$, then Theorem 12 implies immediately $d_t^{\{k\}}(D) \geq k$.

(ii) If $k = p\delta^-$, then Theorem 11 implies immediately $d_t^{\{k\}}(D) \geq \delta^- - 1$.

(iii) If $k = p\delta^- + r$ with integers $p, r \geq 1$ and $r \leq \delta^- - 1$, then $\lceil \frac{k}{\delta^-} \rceil = p+1$ and therefore we deduce from Theorem 12 that

$$d_t^{\{k\}}(D) \geq \left\lfloor \frac{k}{\lceil k/\delta^- \rceil} \right\rfloor = \left\lfloor \frac{k}{p+1} \right\rfloor = \left\lfloor \frac{p\delta^- + r}{p+1} \right\rfloor \geq \frac{p\delta^- + r}{p+1} - \frac{p}{p+1} \geq \frac{p(\delta^- - 1)+1}{p+1}.$$

This leads to the desired bound, and the proof is complete. ■

**Corollary 17.** *If $k \geq 1$ is an integer and $D$ a digraph with $\delta^-(D) \geq 1$, then $d_t^{\{k\}}(D) \geq \min\left\{k, \frac{\delta^-(D)}{2}\right\}$.*

The *complement* $\overline{D}$ of a digraph $D$ is that digraph with vertex set $V(D)$ such that for two arbitrary different vertices $u$ and $v$ the arc $(u,v)$ belongs to $\overline{D}$ if and only if $(u,v)$ does not belong to $D$.

**Theorem 18.** *Let $k \geq 1$ be an integer, and let $D$ be an $r$-diregular digraph of order $n \geq 3$ with $1 \leq r \leq n-2$. Then* $d_t^{\{k\}}(D) + d_t^{\{k\}}(\overline{D}) \geq \min\left\{k+1, \lceil \frac{n-1}{2} \rceil\right\}$.

***Proof.*** Assume first that $k < \delta^-(D)$. Then it follows from Theorem 16 (i) that $d_t^{\{k\}}(D) \geq k$ and thus $d_t^{\{k\}}(D) + d_t^{\{k\}}(\overline{D}) \geq k+1$.

Assume next that $k \geq \delta^-(D)$ and $k < \delta^-(\overline{D})$. Then Theorem 16 (i) implies $d_t^{\{k\}}(\overline{D}) \geq k$ and so $d_t^{\{k\}}(D) + d_t^{\{k\}}(\overline{D}) \geq k+1$.

Finally assume that $k \geq \delta^-(D)$ and $k \geq \delta^-(\overline{D})$. Applying Theorem 16 (ii) and (iii), we observe that $d_t^{\{k\}}(D) \geq \delta^-(D)/2$ and $d_t^{\{k\}}(\overline{D}) \geq \delta^-(\overline{D})/2$, and hence we deduce that

$$d_t^{\{k\}}(D) + d_t^{\{k\}}(\overline{D}) \geq \frac{\delta^-(D)}{2} + \frac{\delta^-(\overline{D})}{2} = \frac{n-1}{2}.$$

Combining these inequalities, we obtain the desired bound. ∎

**Theorem 19.** *For every digraph $D$ of order $n$,* $d_t^{\{k\}}(D) \geq \left\lfloor \frac{n}{n-\delta^-(D)} \right\rfloor$.

***Proof.*** Let $S$ be any subset of $V(D)$ with $|S| \geq n - \delta^-(D)$. If $v \in V(D) - S$, then there exists at least one vertex $u \in S$ such that $(u,v) \in A(D)$. Let $S_1, S_2, \ldots, S_{\left\lfloor \frac{n}{n-\delta^-(D)} \right\rfloor}$ be disjoint subsets of $V(D)$ each of cardinality $n - \delta^-(D)$. Define $f_i : V(G) \to \{0,1,\ldots,k\}$ by

$$f_i(v) = \begin{cases} k & \text{if } v \in S_i, \\ 0 & \text{otherwise,} \end{cases}$$

for each $1 \leq i \leq \left\lfloor \frac{n}{n-\delta^-(D)} \right\rfloor$.

Since $|S_i| = n - \delta^-(D)$, it is clear that $f_i$ is a total $\{k\}$-dominating function of $D$ for each $i$. Since also $S_i$ are disjoint subsets of $V(D)$, then for every $v \in V(D)$ $\sum_{i=1}^{\left\lfloor \frac{n}{n-\delta^-(D)} \right\rfloor} f_i(v) \leq k$ . Thus $\{f_1, f_2, \ldots, f_{\left\lfloor \frac{n}{n-\delta^-(D)} \right\rfloor}\}$ is a T$\{k\}$D family on $D$, and the proof is complete. ∎

The special case $k = 1$ in Theorems 15 and 19 can be found in [6].

## 3. The Total $\{k\}$-domatic Number of Graphs

The *total $\{k\}$-dominating function* of a graph $G$ is defined in [7] as a function $f : V(G) \longrightarrow \{0,1,2,\ldots,k\}$ such that $\sum_{x \in N_G(v)} f(x) \geq k$ for all $v \in V(G)$. The sum $\sum_{x \in V(G)} f(x)$ is the weight $w(f)$ of $f$. The minimum of weights $w(f)$, taken over all total $\{k\}$-dominating functions $f$ on $G$ is called the *total $\{k\}$-domination number* of $G$, denoted by $\gamma_t^{\{k\}}(G)$. In the special case $k = 1$, $\gamma_t^{\{k\}}(G)$ is the classical total domination number $\gamma_t(G)$.

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct total $\{k\}$-dominating functions on $G$ such that $\sum_{i=1}^{d} f_i(v) \leq k$ for each $v \in V(G)$, is called a *total* $\{k\}$*-dominating family* on $G$. The maximum number of functions in a total $\{k\}$-dominating family on $G$ is the *total* $\{k\}$*-domatic number* of $G$, denoted by $d_t^{\{k\}}(G)$. This parameter was introduced by Sheikholeslami and Volkmann in [8] and has been studied in [1]. In the case $k = 1$, we write $d_t(G)$ instead of $d_t^{\{1\}}(G)$ which was introduced by Cockayne, Dawes and Hedetniemi [3], and has been studied in many articles.

The *associated digraph* $D(G)$ of a graph $G$ is the digraph obtained from $G$ when each edge $e$ of $G$ is replaced by two oppositely oriented arcs with the same ends as $e$. Since $N^-_{D(G)}(v) = N_G(v)$ for each vertex $v \in V(G) = V(D(G))$, the following useful observation is valid.

**Observation 20.** *If $D(G)$ is the associated digraph of a graph $G$, then*
$$\gamma_t^{\{k\}}(D(G)) = \gamma_t^{\{k\}}(D) \text{ and } d_t^{\{k\}}(D(G)) = d_t^{\{k\}}(D).$$

There are a lot of interesting applications of Observation 20. Using Theorems 2 and 6, we obtain the next results immediately.

**Corollary 21** [8]**.** *If $k \geq 1$ is an integer and $G$ a graph of order $n$ without isolated vertices, then $\gamma_t^{\{k\}}(G) \cdot d_t^{\{k\}}(G) \leq kn$.*

The case $k = 1$ in Corollary 21 leads to the well-known inequality $\gamma_t(G) \cdot d_t(G) \leq n$, given by Cockayne, Dawes and Hedetniemi [3] in 1980.

**Corollary 22** [8]**.** *If $k \geq 1$ is an integer and $G$ a graph of order $n$ without isolated vertices, then $\gamma_t^{\{k\}}(G) + d_t^{\{k\}}(G) \leq nk + 1$.*

**Corollary 23** [3]**.** *If $G$ is graph of order $n$ without isolated vertices, then $\gamma_t(G) + d_t(G) \leq n + 1$.*

Theorem 7 and Observation 20 lead to the following bound.

**Corollary 24** [8]**.** *Let $k \geq 1$ be an integer and $G$ a graph of order $n$ without isolated vertices. If $d_t^{\{k\}}(G) \geq 2$, then $\gamma_t^{\{k\}}(G) + d_t^{\{k\}}(G) \leq \frac{kn}{2} + 2$.*

**Corollary 25** [4]**.** *If $G$ is a graph of order $n$ without isolated vertices and if $d_t(G) \geq 2$, then $\gamma_t(G) + d_t(G) \leq \frac{n}{2} + 2$.*

Since $\delta^-(D(G)) = \delta(G)$, the next result follows from Observation 20 and Theorem 8.

**Corollary 26** [8]**.** *If $k \geq 1$ is an integer and $G$ a graph without isolated vertices, then $d_t^{\{k\}}(G) \leq \delta(G)$.*

The case $k = 1$ in Corollary 26 can be found in [3]. Theorem 11 and Observation 20 imply the next result.

**Corollary 27** [2]**.** *Let $k \geq 1$ be an integer and $G$ a graph of order $n$ without isolated vertices. If $\delta(G) \mid k$, then $d_t^{\{k\}}(G) \geq \delta(G) - 1$.*

Finally, the next theorem follows from Theorem 18 and Observation 20.

**Corollary 28** [1]**.** *For every $\delta$-regular graph of order $n \geq 5$ in which neither $G$ nor $\overline{G}$ have isolated vertices, $d_t^{\{k\}}(G) + d_t^{\{k\}}(\overline{G}) \geq \min\left\{k+1, \left\lceil \frac{n-2}{2} \right\rceil\right\}$.*

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