# THE TOTAL $\{k\}$-DOMATIC NUMBER OF DIGRAPHS 

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#### Abstract

For a positive integer $k$, a total $\{k\}$-dominating function of a digraph $D$ is a function $f$ from the vertex set $V(D)$ to the set $\{0,1,2, \ldots, k\}$ such that for any vertex $v \in V(D)$, the condition $\sum_{u \in N^{-}(v)} f(u) \geq k$ is fulfilled, where $N^{-}(v)$ consists of all vertices of $D$ from which arcs go into $v$. A set $\left\{f_{1}, f_{2}, \ldots, f_{d}\right\}$ of total $\{k\}$-dominating functions of $D$ with the property that $\sum_{i=1}^{d} f_{i}(v) \leq k$ for each $v \in V(D)$, is called a total $\{k\}$-dominating family (of functions) on $D$. The maximum number of functions in a total $\{k\}$-dominating family on $D$ is the total $\{k\}$-domatic number of $D$, denoted by $d_{t}^{\{k\}}(D)$. Note that $d_{t}^{\{1\}}(D)$ is the classic total domatic number $d_{t}(D)$. In this paper we initiate the study of the total $\{k\}$-domatic number in digraphs, and we present some bounds for $d_{t}^{\{k\}}(D)$. Some of our results are extensions of well-know properties of the total domatic number of digraphs and the total $\{k\}$-domatic number of graphs.


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## 1. Introduction

In this paper, $D$ is a finite and simple digraph with vertex set $V=V(D)$ and arc set $A=A(D)$. The order $|V|$ of $D$ is denoted by $n=n(D)$. We write $d_{D}^{+}(v)=d^{+}(v)$ for the outdegree of a vertex $v$ and $d_{D}^{-}(v)=d^{-}(v)$ for its indegree. The minimum and maximum indegree are $\delta^{-}(D)$ and $\Delta^{-}(D)$. The sets $N^{+}(v)=$ $\{x \mid(v, x) \in A(D)\}$ and $N^{-}(v)=\{x \mid(x, v) \in A(D)\}$ are called the outset and inset of the vertex $v$. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by $X$. For an $\operatorname{arc}(x, y) \in A(D)$, the vertex $y$ is an outer neighbor of $x$ and $x$ is an inner neighbor of $y$. We write $K_{n}^{*}$ for the complete digraph of order $n$. Consult [5] for the notation and terminology which are not defined here.

For a positive integer $k$, a total $\{k\}$-dominating function ( $\mathrm{T}\{k\} \mathrm{DF}$ ) of a digraph $D$ with $\delta^{-}(D) \geq 1$ is a function $f$ from the vertex set $V(D)$ to the set $\{0,1,2, \ldots, k\}$ such that for any vertex $v \in V(D)$, the condition $\sum_{u \in N^{-}(v)} f(u) \geq$ $k$ is fulfilled. The weight of a $\mathrm{T}\{k\} \mathrm{DF} f$ is the value $\omega(f)=\sum_{v \in V(D)} f(v)$. The total $\{k\}$-domination number of a digraph $D$, denoted by $\gamma_{t}^{\{k\}}(D)$, is the minimum weight of a $\mathrm{T}\{k\} \mathrm{DF}$ of $D$. A $\gamma_{t}^{\{k\}}(D)$-function is a total $\{k\}$-dominating function of $D$ with weight $\gamma_{t}^{\{k\}}(D)$. Note that $\gamma_{t}^{\{1\}}(D)$ is the classical total domination number $\gamma_{t}(D)$. If $F$ is a minimum total dominating set of a digraph $D$ with $\delta^{-}(D) \geq 1$, then the function $f$ from $V(D)$ to $\{0,1,2, \ldots, k\}$ with $f(v)=k$ for $v \in F$ and $f(x)=0$ for $x \in V(D)-F$ is a total $\{k\}$-dominating function of $D$ and therefore

$$
\gamma_{t}^{\{k\}}(D) \leq k|F|=k \gamma_{t}(D)
$$

In this paper we always assume that $D$ is a digraph with $\delta^{-}(D) \geq 1$.
A set $\left\{f_{1}, f_{2}, \ldots, f_{d}\right\}$ of distinct total $\{k\}$-dominating functions of $D$ with the property that $\sum_{i=1}^{d} f_{i}(v) \leq k$ for each $v \in V(D)$, is called a total $\{k\}$-dominating family (of functions) on $D$. The maximum number of functions in a total $\{k\}$ dominating family ( $\mathrm{T}\{k\} \mathrm{D}$ family) on $D$ is the total $\{k\}$-domatic number of $D$, denoted by $d_{t}^{\{k\}}(D)$. The total $\{k\}$-domatic number is well-defined and

$$
\begin{equation*}
d_{t}^{\{k\}}(D) \geq 1, \text { for all digraphs } D \text { with } \delta^{-}(D) \geq 1 \tag{1}
\end{equation*}
$$

since the set consisting of the function $f: V(D) \rightarrow\{0,1,2, \ldots, k\}$ defined by $f(v)=k$ for each $v \in V(D)$, forms a $\mathrm{T}\{k\} \mathrm{D}$ family on $D$. The total domatic number of a digraph was introduced by Jacob and Arumugam in [6].

Our purpose in this paper is to initiate the study of the total $\{k\}$-domatic number in digraphs. We first study basic properties and bounds for the total $\{k\}$-domatic number of a digraph. In addition, we determine the total $\{k\}$ domatic number of some classes of digraphs. Some of our results are extensions of well-know properties of the total domatic number of digraphs and the total $\{k\}$-domatic number of graphs (see, for example, $[2,3,4,6,8]$ ).

We start with the following observation.
Observation 1. Let $k$ be an integer, and let $D$ be a digraph with $\delta^{-}(D) \geq 1$. Then $\gamma_{t}^{\{k\}}(D) \geq k+1$, with equality if and only if there exists a subset $S \subseteq V(D)$ of size $k+1$ such that $D[S]$ is a complete digraph, and each vertex $x \in V(D)-S$ has at least $k$ inner neighbors in $S$.
Proof. Let $f$ be a $\gamma_{t}^{\{k\}}(D)$-function, and let $v \in V(D)$ be an arbitrary vertex. The definition implies that $\sum_{x \in N^{-}(v)} f(x) \geq k$. If $\sum_{x \in N^{-}(v)} f(x) \geq k+1$, then $\gamma_{t}^{\{k\}}(D) \geq k+1$. If $\sum_{x \in N^{-}(v)} f(x)=k$, then let $u \in N^{-}(v)$ be a vertex such that $f(u) \geq 1$. Since $\sum_{x \in N^{-}(u)} f(x) \geq k$ and $u \notin N^{-}(u)$, we deduce that $\omega(f)=$ $\sum_{x \in V(D)} f(v) \geq \sum_{x \in\left(N^{-}(u) \cup\{u\}\right)} f(x) \geq k+1$ and therefore $\gamma_{t}^{\{k\}}(D) \geq k+1$.

Assume that $\gamma_{t}^{\{k\}}(D)=k+1$. Let $f$ be a $\gamma_{t}^{\{k\}}(D)$-function. If there exists a vertex $v$ such that $f(v) \geq 2$, then we obtain the contradiction $\sum_{x \in N^{-}(v)} f(x) \leq$ $k+1-2=k-1$. Hence $f(x)=1$ or $f(x)=0$ for each vertex $x \in V(D)$. Let $S \subseteq V(D)$ such that $f(x)=1$ for each $x \in S$. Then $|S|=k+1, D[S]$ is a complete digraph, and each vertex $x \in V(D)-S$ has at least $k$ inner neighbors in $S$.

Conversely, assume that there exists a subset $S \subseteq V(D)$ of size $k+1$ such that $D[S]$ is a complete digraph, and each vertex $x \in V(D)-S$ has at least $k$ inner neighbors in $S$. Define the function $f$ by $f(x)=1$ for $x \in S$ and $f(x)=0$ for $x \in V(D)-S$. Then $f$ is a total $\{k\}$-dominating function of $D$ such that $\omega(f)=k+1$. Since $\gamma_{t}^{\{k\}}(D) \geq k+1$, we deduce that $\gamma_{t}^{\{k\}}(D)=k+1$.

## 2. Properties of the $\{k\}$-Domatic Number

In this section we mainly present basic properties of $d_{t}^{\{k\}}(D)$ and bounds on the total $\{k\}$-domatic number of a digraph.
Theorem 2. If $D$ is a digraph of order $n$, then $\gamma_{t}^{\{k\}}(D) \cdot d_{t}^{\{k\}}(D) \leq k n$. Moreover, if $\gamma_{t}^{\{k\}}(D) \cdot d_{t}^{\{k\}}(D)=k n$, then for each $T\{k\} D$ family $\left\{f_{1}, f_{2}, \ldots, f_{d}\right\}$ on $D$ with $d=d_{t}^{\{k\}}(D)$, each function $f_{i}$ is a $\gamma_{t}^{\{k\}}(D)$-function and $\sum_{i=1}^{d} f_{i}(v)=k$ for all $v \in V(D)$.

Proof. Let $\left\{f_{1}, f_{2}, \ldots, f_{d}\right\}$ be a $\mathrm{T}\{k\} \mathrm{D}$ family on $D$ such that $d=d_{t}^{\{k\}}(D)$. Then

$$
\begin{aligned}
d \cdot \gamma_{t}^{\{k\}}(D) & =\sum_{i=1}^{d} \gamma_{t}^{\{k\}}(D) \leq \sum_{i=1}^{d} \sum_{v \in V(D)} f_{i}(v)=\sum_{v \in V(D)} \sum_{i=1}^{d} f_{i}(v) \\
& \leq \sum_{v \in V(D)} k=k n
\end{aligned}
$$

If $\gamma_{t}^{\{k\}}(D) \cdot d_{t}^{\{k\}}(D)=k n$, then the two inequalities occurring in the proof become equalities. Hence for the $\mathrm{T}\{k\} \mathrm{D}$ family $\left\{f_{1}, f_{2}, \ldots, f_{d}\right\}$ on $D$ and for each $i$,
$\sum_{v \in V(D)} f_{i}(v)=\gamma_{t}^{\{k\}}(D)$. Thus each function $f_{i}$ is a $\gamma_{t}^{\{k\}}(D)$-function, and $\sum_{i=1}^{d} f_{i}(v)=k$ for all $v \in V(D)$.

The special case $k=1$ in Theorem 2 can be found in [6].
Corollary 3. Let $k, n$ be two positive integers. If $k+1$ is a divisor of $n$ and $\frac{n}{k+1} \geq 2$, then $d_{t}^{\{k\}}\left(K_{n}^{*}\right)=\frac{k n}{k+1}$.

Proof. Applying Observation 1 and Theorem 2, we see that $d_{t}^{\{k\}}\left(K_{n}^{*}\right) \leq \frac{k n}{k+1}$.
Now we consider a partition of $V\left(K_{n}^{*}\right)$ into $s=\frac{n}{k+1}$ sets $V_{1}, V_{2}, \ldots, V_{s}$ such that $\left|V_{i}\right|=k+1$ for each $i$. Let $V_{i}=\left\{v_{1}^{i}, v_{2}^{i}, \ldots, v_{k+1}^{i}\right\}$ for $1 \leq i \leq s$. Define, for $1 \leq i \leq s$ and $1 \leq j \leq k$,

$$
\begin{gathered}
f_{i}^{j}\left(v_{1}^{i}\right)=\cdots=f_{i}^{j}\left(v_{j}^{i}\right)=1, f_{i}^{j}\left(v_{j+1}^{i+1}\right)=\cdots=f_{i}^{j}\left(v_{k+1}^{i+1}\right)=1 \text { and } \\
f_{i}^{j}(x)=0 \text { otherwise, where the indices } i+1 \text { are taken modulo } s .
\end{gathered}
$$

It is easy see that $\left\{f_{i}^{j} \left\lvert\, 1 \leq i \leq \frac{n}{k+1}\right., 1 \leq j \leq k,\right\}$ is a $\mathrm{T}\{k\} \mathrm{D}$ family on $K_{n}^{*}$, and therefore $d_{t}^{\{k\}}\left(K_{n}^{*}\right) \geq \frac{k n}{k+1}$. Since $k+1$ is a divisor of $n$, the proof is complete.
A further consequence of Theorem 2 and Observation 1 now follows.
Corollary 4. If $k \geq 2$ is an integer, and $D$ is a digraph of order $k+1$, then $d_{t}^{\{k\}}(D) \leq k-1$.
Proof. Since $\gamma_{t}^{\{k\}}(D) \geq k+1$, it follows from Theorem 2 that $d_{t}^{\{k\}}(D) \leq k$. If $\gamma_{t}^{\{k\}}(D) \geq k+2$, then Theorem 2 implies $d_{t}^{\{k\}}(D) \leq k-1$ immediately. If $\gamma_{t}^{\{k\}}(D)=k+1$ and $d_{t}^{\{k\}}(D)=k$, then for the $\mathrm{T}\{k\} \mathrm{D}$ family $\left\{f_{1}, f_{2}, \ldots, f_{k}\right\}$ on $D$, each function $f_{i}$ is a $\gamma_{t}^{\{k\}}(D)$-function, and Observation 1 leads to the contradiction that $f_{1} \equiv f_{2} \equiv \cdots \equiv f_{k}$. This completes the proof.

Corollary 5. If $k$ is a positive integer, and $D$ is a digraph of order $n$, then $d_{t}^{\{k\}}(D) \leq \frac{k n}{k+1}$, with equality only if $k+1$ is a divisor of $n$ and $\frac{n}{k+1} \geq 2$ when $k \geq 2$.

Proof. Since $\gamma_{t}^{\{k\}}(D) \geq k+1$, it follows from Theorem 2 that $d_{t}^{\{k\}}(D) \leq$ $\frac{k n}{\gamma_{t}^{\frac{k}{}(k)}(D)} \leq \frac{k n}{k+1}$, and this is the desired inequality.

Assume that $d_{t}^{\{k\}}(D)=\frac{k n}{k+1}$. Since $(k, k+1)=1, k+1$ must be a divisor of $n$. If $k \geq 2$, then it follows from Corollary 4 that $\frac{n}{k+1} \geq 2$.

Corollary 3 demonstrates that Corollary 5 is sharp.
Theorem 6. If $D$ is a digraph of order $n$ and $k$ a positive integer, then

$$
\gamma_{t}^{\{k\}}(D)+d_{t}^{\{k\}}(D) \leq n k+1
$$

Proof. Applying Theorem 2, we obtain $\gamma_{t}^{\{k\}}(D)+d_{t}^{\{k\}}(D) \leq \frac{k n}{d_{t}^{\{k\}}(D)}+d_{t}^{\{k\}}(D)$. Note that $d_{t}^{\{k\}}(G) \geq 1$, by inequality (1), and that Corollary 5 implies that $d_{t}^{\{k\}}(D) \leq n$. Using these inequalities, and the fact that the function $g(x)=$ $x+(k n) / x$ is decreasing for $1 \leq x \leq \sqrt{k n}$ and increasing for $\sqrt{k n} \leq x \leq n$, we obtain $\gamma_{t}^{\{k\}}(D)+d_{t}^{\{k\}}(D) \leq \max \left\{k n+1, \frac{k n}{n}+n\right\}=n k+1$, and this is the desired bound.

If $C_{n}$ denotes a directed cycle on $n$ vertices, then the function $f: V\left(C_{n}\right) \rightarrow$ $\{0,1, \ldots, k\}$ defined by $f(x)=k$ for each $x \in V\left(C_{n}\right)$ is the unique total $\{k\}$ dominating function of $C_{n}$ and hence $\gamma_{t}^{\{k\}}\left(C_{n}\right)=n k$ and $d_{t}^{\{k\}}\left(C_{n}\right)=1$. This demonstrates that Theorem 6 is sharp.

Theorem 7. Let $D$ be a digraph of order $n \geq 3$, and let $k \geq 1$ be an integer. If $d_{t}^{\{k\}}(D) \geq 2$, then $\gamma_{t}^{\{k\}}(D)+d_{t}^{\{k\}}(D) \leq \frac{k n}{2}+2$.

Proof. Theorem 2 implies that $\gamma_{t}^{\{k\}}(D)+d_{t}^{\{k\}}(D) \leq \gamma_{t}^{\{k\}}(D)+\frac{k n}{\gamma_{t}^{\{k\}}(D)}$. It follows from Observation 1 and Theorem 2 that $k+1 \leq \gamma_{t}^{\{k\}}(D) \leq k n / 2$. Using these inequalities, and the fact that the function $g(x)=x+(k n) / x$ is decreasing for $k+1 \leq x \leq \sqrt{k n}$ and increasing for $\sqrt{k n} \leq x \leq k n / 2$, we obtain

$$
\gamma_{t}^{\{k\}}(G)+d_{t}^{\{k\}}(G) \leq \max \left\{k+1+\frac{k n}{k+1}, \frac{k n}{2}+2\right\}=\frac{k n}{2}+2
$$

and this is the desired bound.
Theorem 8. If $D$ is a digraph and $k \geq 1$ an integer, then $d_{t}^{\{k\}}(D) \leq \delta^{-}(D)$. Moreover, if $d_{t}^{\{k\}}(D)=\delta^{-}(D)$, then for each function of any $T\{k\} D$ family $\left\{f_{1}, f_{2}, \ldots, f_{d}\right\}$ and for all vertices $v$ of indegree $\delta^{-}(D), \sum_{u \in N^{-}(v)} f_{i}(u)=k$ and $\sum_{i=1}^{d} f_{i}(u)=k$ for every $u \in N^{-}(v)$.

Proof. Let $\left\{f_{1}, f_{2}, \ldots, f_{d}\right\}$ be a $\mathrm{T}\{k\}$ D family on $D$ such that $d=d_{t}^{\{k\}}(D)$, and let $v$ be a vertex of minimum indegree $\delta^{-}(D)$. Since $\sum_{u \in N^{-}(v)} f_{i}(u) \geq k$ for all $i \in\{1,2, \ldots, d\}$, we obtain $k d \leq \sum_{i=1}^{d} \sum_{u \in N^{-}(v)} f_{i}(u)=\sum_{u \in N^{-}(v)} \sum_{i=1}^{d} f_{i}(u) \leq$ $\sum_{u \in N^{-}(v)} k=k \delta^{-}(D)$, and this leads to the desired bound.

If $d_{t}^{\{k\}}(D)=\delta^{-}(D)$, then the two inequalities occurring in the proof become equalities, which leads to the two properties given in the statement.

The special case $k=1$ in Theorem 8 can be found in [6].
Observation 9. Let $D$ be a digraph with the property that the underlying graph is connected and bipartite. If $k \geq 1$ is an integer, then $\gamma_{t}^{\{k\}}(D) \geq 2 k$.

Proof. Let $f$ be a $\gamma_{t}^{\{k\}}(D)$-function, and let $V_{1}$ and $V_{2}$ be the partite sets of the underlying graph. If $w_{i} \in V_{i}$, then the definition implies that $\sum_{x \in N^{-}\left(w_{i}\right)} f(x) \geq$ $k$ for $i=1,2$. It follows that $w(f)=\sum_{x \in V(D)} f(x)=\sum_{x \in V(D)-V_{1}} f(x)+$ $\sum_{x \in V(D)-V_{2}} f(x) \geq \sum_{x \in N^{-}\left(w_{2}\right)} f(x)+\sum_{x \in N^{-}\left(w_{1}\right)} f(x) \geq 2 k$, thus $\gamma_{t}^{\{k\}}(D) \geq 2 k$.

Corollary 10. If $K_{p, p}^{*}$ is the complete bipartite digraph and $k \geq 1$ an integer, then $d_{t}^{\{k\}}\left(K_{p, p}^{*}\right)=p$.

Proof. Theorem 2 and Observation 9 show that $d_{t}^{\{k\}}\left(K_{p, p}^{*}\right) \leq p$.
Now let $\left\{u_{1}, u_{2}, \ldots, u_{p}\right\}$ and $\left\{v_{1}, v_{2}, \ldots, v_{p}\right\}$ be the partite sets of the complete bipartite digraph. Define $f_{i}\left(u_{i}\right)=f_{i}\left(v_{i}\right)=k$ and $f_{i}(x)=0$ for each vertex $x \in V(D)-\left\{u_{i}, v_{i}\right\}$ and each $i \in\{1,2, \ldots, p\}$. Then we observe that $f_{i}$ is a $\mathrm{T}\{k\} \mathrm{DF}$ of $K_{p, p}^{*}$ for each $i \in\{1,2, \ldots, p\}$. Therefore $\left\{f_{1}, f_{2}, \ldots, f_{p}\right\}$ is a $\mathrm{T}\{k\} \mathrm{D}$ family on $K_{p, p}^{*}$. Consequently, $d_{t}^{\{k\}}\left(K_{p, p}^{*}\right) \geq p$ and so $d_{t}^{\{k\}}\left(K_{p, p}^{*}\right)=p$.

Corollary 10 demonstrates that Theorem 8 is sharp.
Theorem 11. Let $k \geq 1$ be an integer and $D$ a digraph of order $n$ with $\delta^{-}(D) \geq 1$. If $\delta^{-}(D) \mid k$, then $d_{t}^{\{k\}}(D) \geq \delta^{-}(D)-1$.

Proof. If $\delta^{-}(D)=1$, then the result is immediate.
Let $\delta^{-}(D) \geq 2$ and let $V(D)=\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$. Define $f_{i}: V(D) \rightarrow$ $\{0,1, \ldots, k\}$ by
$f_{i}\left(v_{j}\right)=\left\{\begin{array}{ll}\frac{k}{\delta^{-}(D)}+1 & \text { if } \quad j=i, \\ \frac{k}{\delta-(D)} & \text { if } j \neq i,\end{array}\right.$ for every $1 \leq i \leq \delta^{-}(D)-1$ and $1 \leq j \leq n$.
Then for each $v \in V(D)$ and each $1 \leq i \leq \delta^{-}(D)-1$,

$$
\sum_{u \in N^{-}(v)} f_{i}(u) \geq \sum_{u \in N^{-}(v)} \frac{k}{\delta^{-}(D)} \geq \frac{k}{\delta^{-}(D)} \delta^{-}(D)=k .
$$

Hence $f_{i}$ is a $\mathrm{T}\{k\} \mathrm{DF}$ of $D$ for each $1 \leq i \leq \delta^{-}(D)-1$. Now, since $\delta^{-}(D) \mid k$, we have

$$
\sum_{i=1}^{\delta(D)-1} f_{i}(v) \leq \frac{k}{\delta^{-}(D)}\left(\delta^{-}(D)-2\right)+\left(\frac{k}{\delta^{-}(D)}+1\right)=k+\left(1-\frac{k}{\delta^{-}(D)}\right) \leq k
$$

for each $v \in V(D)$. Thus $\left\{f_{1}, f_{2}, \ldots, f_{\delta-(D)-1}\right\}$ is a $T\{k\} \mathrm{D}$ family on $D$, and the proof is complete.

Theorem 12. Let $k \geq 1$ be an integer and $D$ a digraph of order $n$. If $\delta^{-}(D) \nmid k$, then $d_{t}^{\{k\}}(D) \geq\left\lfloor\frac{k}{\left\lceil k / \delta^{-}(D)\right\rceil}\right\rfloor$.

Proof. Let $V(D)=\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$. Define $f_{i}: V(D) \rightarrow\{0,1, \ldots, k\}$ by $f_{i}\left(v_{j}\right)=\left\{\begin{array}{lll}\left\lfloor\frac{k}{\delta-(D)}\right\rfloor & \text { if } \quad j=i, \\ \left\lceil\frac{k}{\delta-(D)}\right\rceil & \text { if } \quad j \neq i,\end{array}\right.$ for every $1 \leq i \leq\left\lfloor\frac{k}{\left\lceil k / \delta^{-(D)\rceil}\right\rfloor}\right\rfloor$ and $1 \leq j \leq n$.

Then for each $v \in V(D)$ and each $1 \leq i \leq\left\lfloor\frac{k}{\left\lceil k / \delta^{-(D)\rceil}\right\rfloor}\right\rfloor$,

$$
\sum_{u \in N^{-}(v)} f_{i}(u) \geq\left\lfloor\frac{k}{\delta^{-( }(D)}\right\rfloor+\left\lceil\frac{k}{\delta^{-( }(D)}\right\rceil\left(\delta^{-}(D)-1\right) \geq\left\lceil\frac{k}{\delta^{-( }(D)}\right\rceil \delta^{-}(D)-1 \geq k .
$$

Hence $f_{i}$ is a T $\{k\} \mathrm{DF}$ of $D$ for each $i$. Since $\delta^{-}(D) \nmid k$, we have

$$
\sum_{i=1}^{\left\lfloor\frac{k}{\left\lceil k / \delta^{-(D)\rceil}\right.}\right\rfloor} f_{i}(v) \leq\left\lceil\frac{k}{\delta^{-}(D)}\right\rceil \cdot\left\lfloor\frac{k}{\left\lceil\frac{k}{\delta-(D)}\right\rceil}\right\rfloor \leq\left\lceil\frac{k}{\delta-(D)}\right\rceil \cdot \frac{k}{\left\lceil\frac{k}{\left.\delta^{-(D)}\right\rceil}\right.}=k
$$

for each $v \in V(D)$. Thus $\left\{f_{1}, f_{2}, \ldots, f_{\left\lfloor\frac{k}{I k / \delta-(D)]}\right\rfloor}\right\}$ is a $\mathrm{T}\{k\} \mathrm{D}$ family on $D$, and the proof is complete.

Using Theorems 2, 8, 11 and 12, we will improve Theorem 6 considerably for some cases.

Corollary 13. Let $k \geq 1$ be an integer, and let $D$ be a digraph of order $n$. If $\delta^{-}(D)>k$, then $\gamma_{t}^{\{k\}}(D)+d_{t}^{\{k\}}(D) \leq n+k$.

Proof. Since $\delta^{-}(D)>k$, it follows from Theorem 12 that

$$
d_{t}^{\{k\}}(D) \geq\left\lfloor\frac{k}{\left|\frac{k}{\delta-(D)}\right|}\right\rfloor=k .
$$

In addition, Theorem 8 implies that $d_{t}^{\{k\}}(D) \leq \delta^{-}(D) \leq n$. Using these two inequalities, and the fact that the function $g(x)=x+(k n) / x$ is decreasing for $k \leq x \leq \sqrt{k n}$ and increasing for $\sqrt{k n} \leq x \leq n$, Theorem 2 leads to

$$
\gamma_{t}^{\{k\}}(D)+d_{t}^{\{k\}}(D) \leq \frac{k n}{d_{t}\{k\}(D)}+d_{t}^{\{k\}}(D) \leq \max \left\{\frac{k n}{k}+k, \frac{k n}{n}+n\right\}=n+k .
$$

This is the desired bound, and the proof is complete.
Corollary 14. Let $k \geq 1$ be an integer, and let $D$ be a digraph of order $n$ with $\delta^{-}(D) \geq 2$. If $\delta^{-}(D) \mid k$, then $\gamma_{t}^{\{k\}}(D)+d_{t}^{\{k\}}(D) \leq \frac{k n}{\delta^{-}(D)-1}+\delta^{-}(D)-1$.

Proof. Since $\delta^{-}(D) \mid k$, Theorem 11 shows that $d_{t}^{\{k\}}(D) \geq \delta^{-}(D)-1$, and Theorem 8 implies that $d_{t}^{\{k\}}(D) \leq \delta^{-}(D)$. Using these two inequalities and Theorem 2, we obtain the desired bound as follows $\gamma_{t}^{\{k\}}(D)+d_{t}^{\{k\}}(D) \leq \frac{k n}{d_{t}^{\{k\}}(D)}+$ $d_{t}^{\{k\}}(D) \leq \max \left\{\frac{k n}{\delta^{-}(D)-1}+\delta^{-}(D)-1, \frac{k n}{\delta^{-}(D)}+\delta^{-}(D)\right\}=\frac{k n}{\delta^{-}(D)-1}+\delta^{-}(D)-1$.

Let $D$ be a digraph. By $D^{-1}$ we denote the digraph obtained by reversing all arcs of $D$. A digraph without directed cycles of length 2 is called an oriented graph. An oriented graph $D$ is a tournament when either $(x, y) \in A(D)$ or $(y, x) \in A(D)$ for each pair of distinct vertices $x, y \in V(D)$.

Theorem 15. For every oriented graph $D$ with $\delta^{-}(D) \geq 1$ and $\delta^{-}\left(D^{-1}\right) \geq 1$, $d_{t}^{\{k\}}(D)+d_{t}^{\{k\}}\left(D^{-1}\right) \leq n-1$. If $d_{t}^{\{k\}}(D)+d_{t}^{\{k\}}\left(D^{-1}\right)=n-1$, then $D$ is a regular tournament.

Proof. Since $\delta^{-}(D)+\delta^{-}\left(D^{-1}\right) \leq n-1$, Theorem 8 leads to

$$
d_{t}^{\{k\}}(D)+d_{t}^{\{k\}}\left(D^{-1}\right) \leq \delta^{-}(D)+\delta^{-}\left(D^{-1}\right) \leq n-1 .
$$

If $D$ is not a tournament or $D$ is a non-regular tournament, then $\delta^{-}(D)+$ $\delta^{-}\left(D^{-1}\right) \leq n-2$, and hence we deduce from Theorem 8 that

$$
d_{t}^{\{k\}}(D)+d_{t}^{\{k\}}\left(D^{-1}\right) \leq \delta^{-}(D)+\delta^{-}\left(D^{-1}\right) \leq n-2 .
$$

Now we present further lower bounds on the total $\{k\}$-domatic number.
Theorem 16. Let $k \geq 1$ be an integer, and $D$ a digraph with $\delta^{-}(D)=\delta^{-} \geq 1$.
(i) If $k<\delta^{-}$, then $d_{t}^{\{k\}}(D) \geq k$.
(ii) If $k=p \delta^{-}$with an integer $p \geq 1$, then $d_{t}^{\{k\}}(D) \geq \delta^{-}-1$.
(iii) If $k=p \delta^{-}+r$ with integers $p, r \geq 1$ and $r \leq \delta^{-}-1$, then $d_{t}^{\{k\}}(D) \geq\left\lceil\frac{p\left(\delta^{-}-1\right)+1}{p+1}\right\rceil$.

Proof. (i) If $k<\delta^{-}$, then Theorem 12 implies immediately $d_{t}^{\{k\}}(D) \geq k$.
(ii) If $k=p \delta^{-}$, then Theorem 11 implies immediately $d_{t}^{\{k\}}(D) \geq \delta^{-}-1$.
(iii) If $k=p \delta^{-}+r$ with integers $p, r \geq 1$ and $r \leq \delta^{-}-1$, then $\left\lceil\frac{k}{\delta^{-}}\right\rceil=p+1$ and therefore we deduce from Theorem 12 that

$$
d_{t}^{\{k\}}(D) \geq\left\lfloor\frac{k}{\left\lceil k / \delta^{-}\right\rceil}\right\rfloor=\left\lfloor\frac{k}{p+1}\right\rfloor=\left\lfloor\frac{p \delta^{-}+r}{p+1}\right\rfloor \geq \frac{p \delta^{-}+r}{p+1}-\frac{p}{p+1} \geq \frac{p\left(\delta^{-}-1\right)+1}{p+1} .
$$

This leads to the desired bound, and the proof is complete.
Corollary 17. If $k \geq 1$ is an integer and $D$ a digraph with $\delta^{-}(D) \geq 1$, then $d_{t}^{\{k\}}(D) \geq \min \left\{k, \frac{\delta^{-}(D)}{2}\right\}$.

The complement $\bar{D}$ of a digraph $D$ is that digraph with vertex set $V(D)$ such that for two arbitrary different vertices $u$ and $v$ the arc $(u, v)$ belongs to $\bar{D}$ if and only if $(u, v)$ does not belong to $D$.

Theorem 18. Let $k \geq 1$ be an integer, and let $D$ be an r-diregular digraph of order $n \geq 3$ with $1 \leq r \leq n-2$. Then $d_{t}^{\{k\}}(D)+d_{t}^{\{k\}}(\bar{D}) \geq \min \left\{k+1,\left\lceil\frac{n-1}{2}\right\rceil\right\}$.

Proof. Assume first that $k<\delta^{-}(D)$. Then it follows from Theorem 16 (i) that $d_{t}^{\{k\}}(D) \geq k$ and thus $d_{t}^{\{k\}}(D)+d_{t}^{\{k\}}(\bar{D}) \geq k+1$.

Assume next that $k \geq \delta^{-}(D)$ and $k<\delta^{-}(\bar{D})$. Then Theorem 16 (i) implies $d_{t}^{\{k\}}(\bar{D}) \geq k$ and so $d_{t}^{\{k\}}(D)+d_{t}^{\{k\}}(\bar{D}) \geq k+1$.

Finally assume that $k \geq \delta^{-}(D)$ and $k \geq \delta^{-}(\bar{D})$. Applying Theorem 16 (ii) and (iii), we observe that $d_{t}^{\{\bar{k}\}}(D) \geq \delta^{-}(D) / 2$ and $d_{t}^{\{k\}}(\bar{D}) \geq \delta^{-}(\bar{D}) / 2$, and hence we deduce that

$$
d_{t}^{\{k\}}(D)+d_{t}^{\{k\}}(\bar{D}) \geq \frac{\delta^{-}(D)}{2}+\frac{\delta^{-}(\bar{D})}{2}=\frac{n-1}{2}
$$

Combining these inequalities, we obtain the desired bound.
Theorem 19. For every digraph $D$ of order $n, d_{t}^{\{k\}}(D) \geq\left\lfloor\frac{n}{n-\delta^{-}(D)}\right\rfloor$.
Proof. Let $S$ be any subset of $V(D)$ with $|S| \geq n-\delta^{-}(D)$. If $v \in V(D)-$ $S$, then there exists at least one vertex $u \in S$ such that $(u, v) \in A(D)$. Let $S_{1}, S_{2}, \ldots, S_{\left\lfloor\frac{n}{n-\delta^{-}(D)}\right\rfloor}$ be disjoint subsets of $V(D)$ each of cardinality $n-\delta^{-}(D)$. Define $f_{i}: V(G) \rightarrow\{0,1, \ldots, k\}$ by

$$
f_{i}(v)= \begin{cases}k & \text { if } v \in S_{i} \\ 0 & \text { otherwise }\end{cases}
$$

for each $1 \leq i \leq\left\lfloor\frac{n}{n-\delta^{-}(D)}\right\rfloor$.
Since $\left|S_{i}\right|=n-\delta^{-}(D)$, it is clear that $f_{i}$ is a total $\{k\}$-dominating function of $D$ for each $i$. Since also $S_{i}$ are disjoint subsets of $V(D)$, then for every $v \in V(D)$ $\sum_{i=1}^{\left\lfloor\frac{n}{n-\delta^{-(D)}}\right\rfloor} f_{i}(v) \leq k$. Thus $\left\{f_{1}, f_{2}, \ldots, f_{\left\lfloor\frac{n}{n-\delta^{-(D)}}\right\rfloor}\right\}$ is a $\mathrm{T}\{k\} \mathrm{D}$ family on $D$, and the proof is complete.

The special case $k=1$ in Theorems 15 and 19 can be found in [6].

## 3. The Total $\{k\}$-domatic Number of Graphs

The total $\{k\}$-dominating function of a graph $G$ is defined in [7] as a function $f: V(G) \longrightarrow\{0,1,2, \ldots, k\}$ such that $\sum_{x \in N_{G}(v)} f(x) \geq k$ for all $v \in V(G)$. The sum $\sum_{x \in V(G)} f(x)$ is the weight $w(f)$ of $f$. The minimum of weights $w(f)$, taken over all total $\{k\}$-dominating functions $f$ on $G$ is called the total $\{k\}$-domination number of $G$, denoted by $\gamma_{t}^{\{k\}}(G)$. In the special case $k=1, \gamma_{t}^{\{k\}}(G)$ is the classical total domination number $\gamma_{t}(G)$.

A set $\left\{f_{1}, f_{2}, \ldots, f_{d}\right\}$ of distinct total $\{k\}$-dominating functions on $G$ such that $\sum_{i=1}^{d} f_{i}(v) \leq k$ for each $v \in V(G)$, is called a total $\{k\}$-dominating family on $G$. The maximum number of functions in a total $\{k\}$-dominating family on $G$ is the total $\{k\}$-domatic number of $G$, denoted by $d_{t}^{\{k\}}(G)$. This parameter was introduced by Sheikholeslami and Volkmann in [8] and has been studied in [1]. In the case $k=1$, we write $d_{t}(G)$ instead of $d_{t}^{\{1\}}(G)$ which was introduced by Cockayne, Dawes and Hedetniemi [3], and has been studied in many articles.

The associated digraph $D(G)$ of a graph $G$ is the digraph obtained from $G$ when each edge $e$ of $G$ is replaced by two oppositely oriented arcs with the same ends as $e$. Since $N_{D(G)}^{-}(v)=N_{G}(v)$ for each vertex $v \in V(G)=V(D(G))$, the following useful observation is valid.

Observation 20. If $D(G)$ is the associated digraph of a graph $G$, then

$$
\gamma_{t}^{\{k\}}(D(G))=\gamma_{t}^{\{k\}}(D) \text { and } d_{t}^{\{k\}}(D(G))=d_{t}^{\{k\}}(D)
$$

There are a lot of interesting applications of Observation 20. Using Theorems 2 and 6 , we obtain the next results immediately.

Corollary 21 [8]. If $k \geq 1$ is an integer and $G$ a graph of order $n$ without isolated vertices, then $\gamma_{t}^{\{k\}}(G) \cdot d_{t}^{\{k\}}(G) \leq k n$.
The case $k=1$ in Corollary 21 leads to the well-known inequality $\gamma_{t}(G) \cdot d_{t}(G) \leq$ $n$, given by Cockayne, Dawes and Hedetniemi [3] in 1980.

Corollary 22 [8]. If $k \geq 1$ is an integer and $G$ a graph of order $n$ without isolated vertices, then $\gamma_{t}^{\{k\}}(G)+d_{t}^{\{k\}}(G) \leq n k+1$.
Corollary 23 [3]. If $G$ is graph of order $n$ without isolated vertices, then $\gamma_{t}(G)+$ $d_{t}(G) \leq n+1$.

Theorem 7 and Observation 20 lead to the following bound.
Corollary 24 [8]. Let $k \geq 1$ be an integer and $G$ a graph of order $n$ without isolated vertices. If $d_{t}^{\{k\}}(G) \geq 2$, then $\gamma_{t}^{\{k\}}(G)+d_{t}^{\{k\}}(G) \leq \frac{k n}{2}+2$.
Corollary 25 [4]. If $G$ is a graph of order $n$ without isolated vertices and if $d_{t}(G) \geq 2$, then $\gamma_{t}(G)+d_{t}(G) \leq \frac{n}{2}+2$.
Since $\delta^{-}(D(G))=\delta(G)$, the next result follows from Observation 20 and Theorem 8.

Corollary 26 [8]. If $k \geq 1$ is an integer and $G$ a graph without isolated vertices, then $d_{t}^{\{k\}}(G) \leq \delta(G)$.
The case $k=1$ in Corollary 26 can be found in [3]. Theorem 11 and Observation 20 imply the next result.

Corollary 27 [2]. Let $k \geq 1$ be an integer and $G$ a graph of order $n$ without isolated vertices. If $\delta(G) \mid k$, then $d_{t}^{\{k\}}(G) \geq \delta(G)-1$.

Finally, the next theorem follows from Theorem 18 and Observation 20.
Corollary 28 [1]. For every $\delta$-regular graph of order $n \geq 5$ in which neither $G$ nor $\bar{G}$ have isolated vertices, $d_{t}^{\{k\}}(G)+d_{t}^{\{k\}}(\bar{G}) \geq \min \left\{k+1,\left\lceil\frac{n-2}{2}\right\rceil\right\}$.

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