

ON THE TOTAL k -DOMINATION NUMBER OF GRAPHS

ADEL P. KAZEMI

Department of Mathematics
University of Mohaghegh Ardabili
P. O. Box 5919911367, Ardabil, Iran

e-mail: a.p.kazemi@uma.ac.ir
adelpkazemi@yahoo.com

Abstract

Let k be a positive integer and let $G = (V, E)$ be a simple graph. The k -tuple domination number $\gamma_{\times k}(G)$ of G is the minimum cardinality of a k -tuple dominating set S , a set that for every vertex $v \in V$, $|N_G[v] \cap S| \geq k$. Also the total k -domination number $\gamma_{\times k, t}(G)$ of G is the minimum cardinality of a total k -dominating set S , a set that for every vertex $v \in V$, $|N_G(v) \cap S| \geq k$. The k -transversal number $\tau_k(H)$ of a hypergraph H is the minimum size of a subset $S \subseteq V(H)$ such that $|S \cap e| \geq k$ for every edge $e \in E(H)$.

We know that for any graph G of order n with minimum degree at least k , $\gamma_{\times k}(G) \leq \gamma_{\times k, t}(G) \leq n$. Obviously for every k -regular graph, the upper bound n is sharp. Here, we give a sufficient condition for $\gamma_{\times k, t}(G) < n$. Then we characterize complete multipartite graphs G with $\gamma_{\times k}(G) = \gamma_{\times k, t}(G)$. We also state that the total k -domination number of a graph is the k -transversal number of its open neighborhood hypergraph, and also the domination number of a graph is the transversal number of its closed neighborhood hypergraph. Finally, we give an upper bound for the total k -domination number of the cross product graph $G \times H$ of two graphs G and H in terms on the similar numbers of G and H . Also, we show that this upper bound is strict for some graphs, when $k = 1$.

Keywords: total k -domination (k -tuple total domination) number, k -tuple domination number, k -transversal number.

2010 Mathematics Subject Classification: 05C69.

1. INTRODUCTION

Domination in graphs is now well studied in graph theory and the literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [3, 4].

A set $S \subseteq V$ is a *dominating set* if each vertex in $V \setminus S$ is adjacent to at least one vertex of S . The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set. A set $S \subseteq V$ is a *total dominating set* if each vertex in V is adjacent to at least one vertex of S , while the minimum cardinality of a total dominating set is the *total domination number* $\gamma_t(G)$ of G .

In [2] Harary and Haynes defined a generalization of domination as follow. Let k be a positive integer. A subset S of V is a *k -tuple dominating set*, abbreviated kDS, of G if for every vertex $v \in V$, either v is in S and has at least $k - 1$ neighbors in S or v is in $V - S$ and has at least k neighbors in S . The *k -tuple domination number* $\gamma_{\times k}(G)$ is the minimum cardinality of a k -tuple dominating set of G . Clearly, $\gamma(G) = \gamma_{\times 1}(G) \leq \gamma_{\times k}(G)$, while $\gamma_t(G) \leq \gamma_{\times 2}(G)$. For a graph to have a k -tuple dominating set, its minimum degree is at least $k - 1$. Hence for trees, $k \leq 2$. A k -tuple dominating set where $k = 2$ is called a *double dominating set* (DDS).

In [5], Henning and Kazemi started the studying of the k -tuple total domination (= total k -domination) number in graphs. Let k be a positive integer. A subset S of V is a *total k -dominating set* (or *k -tuple total dominating set*) of G , abbreviated kTDS, if every vertex in V has at least k neighbors in S . The *total k -domination* (or *k -tuple total domination*) number $\gamma_{\times k, t}(G)$ is the minimum cardinality of a kTDS of G . We remark that a total 1-domination number is the well-studied total domination number. Thus, $\gamma_t(G) = \gamma_{\times 1, t}(G)$, and we write TDS instead of 1TDS. For a graph to have a total k -dominating set, its minimum degree is at least k . Since every total $(k + 1)$ -dominating set is also a total k -dominating set, we note that $\gamma_{\times k, t}(G) \leq \gamma_{\times (k+1), t}(G)$ for all graphs with minimum degree at least $k + 1$. A kTDS of cardinality $\gamma_{\times k, t}(G)$ we call a $\gamma_{\times k, t}(G)$ -set. A total 2-dominating set is called a *double total dominating set*, abbreviated DTDS, and the total 2-domination number is called the *double total domination number*. The redundancy involved in total k -domination makes it useful in many applications.

Hypergraphs are systems of sets which are conceived as natural extensions of graphs. A *hypergraph* $H = (V, E)$ is a finite set V of elements, called *vertices*, together with a finite multiset E of arbitrary subsets of V , called *edges*. A *k -uniform hypergraph* is a hypergraph in which every edge has size k . Every simple graph is a 2-uniform hypergraph. Thus graphs are special hypergraphs. Let k be a positive integer. A *k -transversal* in a hypergraph H is a subset $S \subseteq V$ such that $|S \cap e| \geq k$ for every edge $e \in E$. The *k -transversal number* $\tau_k(H)$ of a hypergraph H is the minimum size of a k -transversal in H . The 1-transversal and 1-transversal number $\tau_1(H)$ we call, respectively, *transversal* and *transversal number* $\tau(H)$.

For a graph $G = (V, E)$, the *open neighborhood hypergraph* $H_G = (V, C)$, abbreviated by ONH, of G is the hypergraph with the vertex set $V(H_G) = V$

and the edge set $E(H_G) = C$ consisting of the open neighborhoods of vertices of V in G . Now we define a new hypergraph. For a graph $G = (V, E)$, the *closed neighborhood hypergraph* $H^G = (V, C)$, abbreviated by CNH, of G is the hypergraph with vertex set $V(H^G) = V$ and with edge set $E(H^G) = C$ consisting of the closed neighborhoods of vertices of V in G .

The *cross* (or *categorical*) *product* $G \times H$ of two graphs G and H is the graph with $V(G \times H) = V(G) \times V(H)$ and $(u, v)(u', v') \in E(G \times H)$ if and only if $uu' \in E(G)$ and $vv' \in E(H)$. For each vertex $v \in V(H)$, we denote the set of vertices $\{(u, v) : u \in V(G)\}$ by G_v and we call G_v the G -*level* of $G \times H$ corresponding to the vertex v . Similarly, for each vertex $u \in V(G)$, we denote the set of vertices $\{(u, v) : v \in V(H)\}$ by H_u and we call H_u the H -*level* of $G \times H$ corresponding to the vertex u . We note that G_v and H_u are independent sets in $G \times H$.

The k -*join* of a graph G to a graph H of order at least k is the graph obtained from the disjoint union of G and H by joining each vertex of G to at least k vertices of H . We denote this graph by $G \circ_k H$.

The notation we use is as follows. Let G be a simple graph with *vertex set* $V = V(G)$ and *edge set* $E = E(G)$. The *order* $|V|$ and *size* $|E|$ of G are denoted by $n = n(G)$ and $m = m(G)$, respectively. For every vertex $v \in V$, the *open neighborhood* $N_G(v)$ is the set $\{u \in V : uv \in E\}$ and the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. If every vertex of G has degree k , then G is said to be k -*regular*. We write K_n and K_{n_1, \dots, n_p} for the *complete graph* of order n , and the *complete p -partite graph*, respectively.

We know that for any graph G of order n with $\delta(G) \geq k$,

$$(1) \quad \gamma_{\times k}(G) \leq \gamma_{\times k, t}(G) \leq n.$$

Obviously for every k -regular graph, the upper bound n is sharp. Here, we give a sufficient condition for $\gamma_{\times k, t}(G) < n$. Then we characterize complete multipartite graphs G satisfy $\gamma_{\times k}(G) = \gamma_{\times k, t}(G)$. We also state that the total k -domination number of a graph is the k -transversal number of its open neighborhood hypergraph, and also the domination number of a graph is the transversal number of its closed neighborhood hypergraph. Finally, we give an upper bound for the total k -domination number of the cross product graph $G \times H$ of two graphs G and H in terms on the total k -domination numbers of G and H . Also, we show that this upper bound is strict for some graphs, when $k = 1$.

The following theorems are useful in the context.

Theorem 1 [5, 6]. *Let G be a graph with $\delta(G) \geq k$. Then for any integer $m \geq k+1$, $\gamma_{\times k, t}(G) = m$ if and only if $G = K'_m$ or $G = F \circ_k K'_m$, for some graph F and some spanning subgraph K'_m of K_m such that K'_m has minimum degree at least k and m is minimum in the set $\{t : G = F \circ_k K'_t, \text{ for some graph } F \text{ and some spanning subgraph } K'_t \text{ of } K_t \text{ with } \delta(K'_t) \geq k\}$.*

Theorem 2 [6]. *If G and H are graphs satisfying $\delta(G) \geq k \geq 1$ and $\delta(H) \geq \ell \geq 1$, then $\gamma_{\times k\ell, t}(G \times H) \leq \gamma_{\times k, t}(G) \cdot \gamma_{\times \ell, t}(H)$.*

Theorem 3 [5]. *Let $p \geq 2$ be an integer and let $G = K_{n_1, n_2, \dots, n_p}$ be a complete p -partite graph where $n_1 \leq n_2 \leq \dots \leq n_p$.*

- (i) *If $k < p$, then $\gamma_{\times k, t}(G) = k + 1$.*
- (ii) *If $k = p$ and $\sum_{i=1}^{k-1} n_i \geq k$, then $\gamma_{\times k, t}(G) = k + 2$.*
- (iii) *If $2 \leq p < k$ and $\lceil k/(p-1) \rceil \leq n_1 \leq n_2 \leq \dots \leq n_p$, then $\gamma_{\times k, t}(G) = \lceil kp/(p-1) \rceil$.*

2. ON THE SHARPNESS OF BOUNDS IN THE CHAIN (1)

Before calculating total k -domination number of the complete multipartite graphs, we state the following theorem that gives a sufficient condition for $\gamma_{\times k, t}(G) < n$.

Theorem 4. *Let k be an integer and G be a graph of order n . If $\delta(G) > k$, then $\gamma_{\times k, t}(G) < n$.*

Proof. Let $v \in V(G)$. Then $S = V(G) - \{v\}$ is a k TDS of G , and so $\gamma_{\times k, t}(G) < n$. ■

We want now to characterize complete multipartite graphs G satisfy $\gamma_{\times k}(G) = \gamma_{\times k, t}(G)$. For this aim, we first find $\gamma_{\times k}(G)$. The proof of the next lemma is easy and it is left to the reader.

Lemma 5. *Let $p \geq 2$ and $k \geq 1$ be integers and let $G = K_{n_1, n_2, \dots, n_p}$ be a complete p -partite graph with $1 \leq n_1 \leq n_2 \leq \dots \leq n_p$ and $\delta(G) \geq k-1$. If $p \geq k$, then $\gamma_{\times k}(G) = k$ if and only if $n_1 = n_2 = \dots = n_k = 1$.*

Lemma 5 implies $\gamma_{\times k}(G) = k+1$ if $p \geq k+1$ and $t = |\{i \mid n_i = 1\}| < k$. Because with choosing one vertex from each of the $k+1$ parts of the parts of its vertex partition, we may obtain a k D with the minimum cardinal. The next theorem presents this number for $p = k$.

Theorem 6. Let $k \geq 2$ be integers and let $G = K_{n_1, n_2, \dots, n_k}$ be a complete k -partite graph of order n with $\delta(G) \geq k - 1$ and $1 \leq n_1 \leq n_2 \leq \dots \leq n_p$. Let $t = |\{i : n_i = 1\}|$. Then

$$\gamma_{\times k}(G) = \begin{cases} k & \text{if } t = k, \\ n & \text{if } t = k - 1, \\ k + 1 & \text{if } t < k - 1 \text{ and } n_{t+1} = 2, \\ k + 2 & \text{if } t < k - 1 \text{ and } n_{t+1} > 2. \end{cases}$$

Proof. Let $G = K_{n_1, n_2, \dots, n_k}$ be a complete k -partite graph with the partition sets X_1, X_2, \dots, X_k and $\delta(G) \geq k - 1 \geq 1$. If $t = k$, then $\gamma_{\times k}(G) = k$, by Lemma 5. If $t = k - 1$, then $\gamma_{\times k}(G) = n$. Because $V(G)$ is the unique k DS of G . Now let $t < k - 1$ and $n_{t+1} = 2$. In this case, let S be a vertex set such that $|S \cap X_i| = 1$ if and only if $i \neq t$ and $|S \cap X_t| = 2$. Since S is a k DS of G of cardinality $k + 1$, Lemma 5 implies $\gamma_{\times k}(G) = k + 1$. If $t < k - 1$ and $n_{t+1} > 2$, then obviously $\gamma_{\times k}(G) \geq k + 2$. In this case, let S be a vertex set such that $|S \cap X_i| = 1$ if and only if $i \neq t + 1, t + 2$ and $|S \cap X_{t+1}| = |S \cap X_{t+2}| = 2$. Since S is a k DS of G with cardinal $k + 2$, we get $\gamma_{\times k}(G) = k + 2$. ■

Lemma 7. Let G be a complete p -partite graph with the partition sets X_1, X_2, \dots, X_p and $\delta(G) \geq k$. Then $\gamma_{\times k, t}(G) = \gamma_{\times k}(G)$ if and only if there exists a $\gamma_{\times k}(G)$ -set S such that for each $1 \leq i \leq p$, $|S - X_i| \geq k$.

Proof. If there exists a $\gamma_{\times k}(G)$ -set S such that for each $1 \leq i \leq p$, $|S - X_i| \geq k$, then S is obviously a k TDS of G and so $\gamma_{\times k, t}(G) \leq |S| = \gamma_{\times k}(G)$. With this fact that for each graph G , $\gamma_{\times k}(G) \leq \gamma_{\times k, t}(G)$, we conclude $\gamma_{\times k, t}(G) = \gamma_{\times k}(G)$.

Conversely, let $\gamma_{\times k, t}(G) = \gamma_{\times k}(G)$. Let also S be a $\gamma_{\times k, t}(G)$ -set. Then S is also a k DS of G and for each $1 \leq i \leq p$, $|S - X_i| \geq k$. Since $|S| = \gamma_{\times k}(G)$, it implies that S is a $\gamma_{\times k}(G)$ -set such that for each $1 \leq i \leq p$, $|S - X_i| \geq k$. ■

By Theorems 3, 6 and Lemmas 5, 7, we have the next result.

Theorem 8. Let $p \geq 2$ and $k \geq 1$ be integers and let $G = K_{n_1, n_2, \dots, n_p}$ be a complete p -partite graph of order n with $\delta(G) \geq k$ and $1 \leq n_1 \leq n_2 \leq \dots \leq n_p$. If $p \geq k$ and $t = |\{i : n_i = 1\}|$, then $\gamma_{\times k, t}(G) = \gamma_{\times k}(G)$ if and only if either $t < k < p$ or $t = k - 1 < k = p$ and $n = k + 2$ or $t < k - 1 < k = p$ and $n_{t+1} \geq 3$.

By Theorem 3, $\gamma_{\times k, t}(K_{n_1, n_2, \dots, n_p})$ is not calculated, yet, when $3 \leq p < k$ and $n_1 \leq \dots \leq n_m < \lceil k/(p - 1) \rceil \leq n_{m+1} \leq \dots \leq n_p$, for some $1 \leq m < p$, and $(p - 1) \sum_{1 \leq i \leq p} n_i \geq pk$. We calculate this number for $p = 3$.

Theorem 9. Let $3 < k$ and let K_{n_1, n_2, n_3} be a complete 3-partite graph such that $1 \leq n_1 < \lfloor k/2 \rfloor \leq n_2 \leq n_3$. Then $\gamma_{\times k, t}(K_{n_1, n_2, n_3}) = 2k - n_1$.

Proof. Let $S = S_1 \cup S_2 \cup S_3$ be a kTDS of the graph, where $S_i = S \cap X_i$ and $|S \cap X_i| = s_i$, for $1 \leq i \leq 3$. Since must $s_1 + s_2 = s_1 + s_3 = k$, we have $s_2 + s_3 = 2k - 2s_1$, and so $s_2 = s_3 = k - s_1$. This implies $\gamma_{\times k,t}(K_{n_1,n_2,n_3}) = \min_{s_1,s_2,s_3} s_1 + s_2 + s_3 = \min_{1 \leq s_1 \leq n_1} 2k - s_1 = 2k - n_1$. ■

3. k -TRANSVERSALS IN HYPERGRAPHS

In this section, we state that the k -transversal number of the open neighborhood hypergraph and the transversal number of the closed neighborhood hypergraph of a graph are the total k -domination number and the domination number of the graph, respectively.

Theorem 10. *If G is a graph with $\delta(G) \geq k$ and H_G is the open neighborhood hypergraph of G , then $\gamma_{\times k,t}(G) = \tau_k(H_G)$.*

Proof. On the one hand, every kTDS of G contains at least k vertices from the open neighborhood of each vertex in G , and is therefore a k -transversal of H_G . In particular, if S is a $\gamma_{\times k,t}(G)$ -set, then S is a k -transversal of H_G , and so $\tau_k(H_G) \leq |S| = \gamma_{\times k,t}(G)$. On the other hand, every k -transversal of H_G contains at least k vertices from the open neighborhood of each vertex of G , and is therefore a kTDS of G . In particular, if T is a k -transversal of H_G of cardinality $\tau_k(H_G)$, then T is a kTDS of G , and so $\gamma_{\times k,t}(G) \leq |T| = \tau_k(H_G)$. Therefore $\gamma_{\times k,t}(G) = \tau_k(H_G)$. ■

There is a similar relation between the domination number of a graph and the transversal number of its closed neighborhood hypergraph.

Theorem 11. *For any simple graph G , $\gamma(G) = \tau(H^G)$.*

Proof. Let S be a γ -set of G , and let D be an edge in H^G . Then for some $x \in V(G)$, we have $D = N_G[x]$. But since S is $\gamma(G)$ -set, we have $S \cap N_G[x] \neq \emptyset$, and hence $S \cap D \neq \emptyset$. Thus $\tau(H^G) \leq |S| = \gamma(G)$. For the converse, let S be a transversal of H^G of cardinality $\tau(H^G)$, and we choose an arbitrary vertex $x \in V(G)$. Then $S \cap N_G[x] \neq \emptyset$, and so S is a dominating set of G . Hence $\gamma(G) \leq |S| = \tau(H^G)$, and this implies $\gamma(G) = \tau(H^G)$. ■

4. TOTAL k -DOMINATION IN CROSS PRODUCT OF GRAPHS

Here we discuss on total k -domination number of the cross product of two graphs. First we state and prove the next theorem.

Theorem 12. *If G is a graph with $\delta(G) \geq k + \ell$ where $k, \ell \geq 1$, then $\gamma_{\times k,t}(G) \leq \gamma_{\times (k+\ell),t}(G) - \ell$.*

Proof. Let G is a graph with $\delta(G) \geq k + \ell$ where $k, \ell \geq 1$. Since removing any ℓ vertices from a $(k + \ell)$ TDS of G produces a k TDS of G , we get $\gamma_{\times k, t}(G) \leq \gamma_{\times(k+\ell), t}(G) - \ell$. ■

As a consequence of Theorems 2 and 12, we have the following result.

Theorem 13. *If G and H are graphs satisfying $\delta(G) \geq k \geq 1$ and $\delta(H) \geq \ell \geq 1$, then*

- (a) $\gamma_{\times k, t}(G \times H) \leq \gamma_{\times k, t}(G) \cdot \gamma_{\times \ell, t}(H) - k(\ell - 1)$,
- (b) $\gamma_{\times \ell, t}(G \times H) \leq \gamma_{\times k, t}(G) \cdot \gamma_{\times \ell, t}(H) - \ell(k - 1)$.

Proof. We note that $\delta(G \times H) \geq \delta(G) \cdot \delta(H) = k\ell = k + k(\ell - 1)$. Hence $\gamma_{\times k, t}(G \times H) \leq \gamma_{\times k\ell, t}(G \times H) - k(\ell - 1)$, by Theorem 12. Theorem 2 implies $\gamma_{\times k\ell, t}(G \times H) \leq \gamma_{\times k, t}(G) \gamma_{\times \ell, t}(H)$.

Hence, $\gamma_{\times k, t}(G \times H) \leq \gamma_{\times k, t}(G) \cdot \gamma_{\times \ell, t}(H) - k(\ell - 1)$. This establishes part (a). The proof of part (b) is similar. ■

The next corollary is an immediately result of Theorem 13.

Corollary 14 [1]. *If G and H are two graphs without isolated vertices, then $\gamma_t(G \times H) \leq \gamma_t(G) \gamma_t(H)$.*

Now we show that the inequality in Corollary 14 can be strict.

Theorem 15. *Let G and H be two respective complete m -partite and complete n -partite graphs and $n \leq m$. If $3 \leq n \leq m$, then $\gamma_t(G \times H) < \gamma_t(G) \gamma_t(H)$.*

Proof. Let $V(G) = X_1 \cup X_2 \cup \dots \cup X_m$ and $V(H) = Y_1 \cup Y_2 \cup \dots \cup Y_n$ be two corresponding partitions of $V(G)$ and $V(H)$. Then $\bigcup_{1 \leq i \leq m, 1 \leq j \leq n} (X_i \times Y_j)$ is a partition of $V(G \times H)$. We note that every vertex of $X_i \times Y_j$ is adjacent to all vertices of $X_t \times Y_l$ if and only if $t \neq i$ and $j \neq l$. We make a TDS of $G \times H$ with minimum cardinal. Let S be a minimum TDS of $G \times H$ that we will make it. Let $a \in S$. Without loss of generality, let $a \in X_1 \times Y_1$. Then a dominates all vertices of $V(G \times H) - V_1$ where $V_1 = (\bigcup_{2 \leq j \leq n} X_1 \times Y_j) \cup (\bigcup_{2 \leq i \leq m} X_i \times Y_1) \cup (X_1 \times Y_1 - \{a\})$. Since $\gamma_t(G \times H) \geq 2$, let $b \in S - \{a\}$ such that b is adjacent to a . Hence $b \in V(G \times H) - V_1$. Without loss of generality, let $b \in X_m \times Y_n$. Since $\{a, b\}$ dominates no vertices of $(X_m \times Y_1) \cup (X_1 \times Y_n)$, we have $|S| \geq 3$. The condition $3 \leq n \leq m$ implies that $S = \{a, b, c\}$ is a TDS of $G \times H$, when $c \in V(X_i \times Y_j)$ for some $2 \leq i \leq m - 1$ and some $2 \leq j \leq n - 1$. Hence $\gamma_t(G \times H) = 3$ that is less than $\gamma_t(G) \gamma_t(H) = 2 \times 2 = 4$. ■

Corollary 16. *If K_n and K_m are complete graphs of order at least 3, then $\gamma_t(K_n \times K_m) < \gamma_t(K_n) \gamma_t(K_m)$.*

REFERENCES

- [1] M. El-Zahar, S. Gravier and A. Klobucar, *On the total domination number of cross products of graphs*, Discrete Math. **308** (2008) 2025–2029.
doi:10.1016/j.disc.2007.04.034
- [2] F. Harary and T.W. Haynes, *Double domination in graphs*, Ars Combin. **55** (2000) 201–213.
- [3] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs* (Marcel Dekker, New York, 1998).
- [4] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Domination in Graphs; Advanced Topics* (Marcel Dekker, New York, 1998).
- [5] M.A. Henning and A.P. Kazemi, *k-tuple total domination in graphs*, Discrete Appl. Math. **158** (2010) 1006–1011.
doi:10.1016/j.dam.2010.01.009
- [6] M.A. Henning and A.P. Kazemi, *k-tuple total domination in cross products of graphs*, J. Comb. Optim. (2011).
doi:10.1007/s10878-011-9389-z
- [7] V. Chvátal and C. Mc Diarmid, *Small transversals in hypergraphs*, Combinatorica **12** (1992) 19–26.
doi:10.1007/BF01191201

Received 9 March 2011

Revised 23 July 2011

Accepted 25 July 2011