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ON COMPOSITION OF SIGNED GRAPHS

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Abstract

A graph whose edges are labeled either as positive or negative is called a signed graph. In this article, we extend the notion of composition of (unsigned) graphs (also called lexicographic product) to signed graphs. We employ Kronecker product of matrices to express the adjacency matrix of this product of two signed graphs and hence find its eigenvalues when the second graph under composition is net-regular. A signed graph is said to be net-regular if every vertex has constant net-degree, namely, the difference of the number of positive and negative edges incident with a vertex. We also characterize balance in signed graph composition and have some results on the Laplacian matrices of this product.

Keywords: signed graph, eigenvalues, graph composition, regular graphs, net-regular signed graphs.

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1. Introduction

All graphs in this article are finite and simple. The objective of this paper is to extend the notion and some results available in unsigned graph theory associated with the lexicographic product of graphs to signed graphs. Moreover we deal with the balance of the lexicographic product of signed graph as the theory of balance is an important aspect in the case of signed graphs. For all definitions in (unsigned) graph theory used here, unless otherwise mentioned, reader may refer to [5, 6]. Much has been discussed in literature about the lexicographic product of graphs (for example, see [5, 6, 11, 12]). We denote by $G = (V, E)$, a simple (unsigned) graph with the vertex set V and the edge set E. If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs, their lexicographic product $G_1[G_2]$ is defined as the graph with the vertex set $V_1 \times V_2$ and the vertices $\mathbf{u} = (u_i, v_j)$ and $\mathbf{v} = (u_k, v_l)$ are adjacent whenever u_i is adjacent to u_k or when $u_i = u_k$ and v_j is adjacent to v_l. We shall extend this definition to signed graphs.

Signed graphs (also called sigraphs), with positive and negative labels on the edges, are much studied in the literature because of their use in modeling a variety of physical and socio-psychological processes (for example, see [2, 3]) and also because of their interesting connections with many classical mathematical systems (see [15]). Formally, a sigraph is an ordered pair $\Sigma = (G, \sigma)$ where $G = (V, E)$ is a graph called the *underlying graph* of Σ and $\sigma : E \to \{+1, -1\}$ called a *signing*, is a function (also called a *signature*) from the edge set E of G into the set $\{+1, -1\}$. We define the *lexicographic product* $\Sigma_1[\Sigma_2]$ (also called composition) of two signed graphs $\Sigma_1 = (V_1, E_1, \sigma_1)$ and $\Sigma_2 = (V_2, E_2, \sigma_2)$ as the signed graph $(V_1 \times V_2, E, \sigma)$ where the edge set E is that of the lexicographic product of underlying unsigned graphs and the signature function σ for the labeling of the edges is defined by

$$\sigma\big((u_i, v_j)(u_k, v_l)\big) = \begin{cases} \sigma_1\big(u_i u_k\big) & \text{if } i \neq k, \\ \sigma_2\big(v_j v_l\big) & \text{if } i = k. \end{cases} \tag{1}$$

A signed graph is *all-positive* (respectively, *all-negative*) if all of its edges are positive (respectively, negative); further, it is said to be *homogeneous* if it is either all-positive or all-negative and *heterogeneous* otherwise. Note that a graph can be considered to be a homogeneous signed graph. A signed graph Σ is said to be *balanced* or *cycle balanced* if all of its cycles are positive, where the sign of a cycle in a signed graph is the product of the signs of its edges.

To obtain certain results in lexicographic products of signed graphs, in the sequel, we deal mainly with the adjacency matrix and Laplacian matrix of a signed graph which are direct generalization of familiar matrices from ordinary, unsigned graph theory. If $\Sigma = (G, \sigma)$ is a signed graph where $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$ and if we denote an edge belonging to the edge set E of Σ as $e_{ij} = v_i v_j$, then its adjacency matrix $A(\Sigma) = (a_{ij})$ is defined as,

$$a_{ij} = \begin{cases} \sigma(v_i v_j) & \text{if } v_i v_j \in E, \\ 0 & \text{otherwise.} \end{cases}$$

The Laplacian matrix of a signed graph $\Sigma = (G, \sigma)$ is given by $L(\Sigma) = D(\Sigma) - A(\Sigma)$, where $D(\Sigma)$ is the diagonal matrix of degrees of vertices of Σ.

The roots of the characteristic polynomial $\det(\lambda I_n - A(\Sigma))$ of the adjacency matrix $A(\Sigma)$ are called the *eigenvalues* of Σ. We denote the eigenvalues of a signed graph of order n by λ_j for $1 \leq j \leq n$ and Laplacian eigenvalues by λ_j^L. Eigenvalues of the adjacency and Laplacian matrices of a graph have been widely used to characterize properties of a graph and extract some useful information from its structure. When we have two signed graphs to deal with, say Σ_1 and Σ_2, the former is considered to be of order m and the latter to be of order n. Their eigenvalues are taken, respectively, as λ_i and μ_j. We denote by J_n the square matrix of order n with all ones and by $\mathbf{j}$ the column vector of all ones, and $+K_n$ denotes the all-positive complete graph of order n.

Kronecker product of an $m \times n$ matrix $A = (a_{ij})$ and a $p \times q$ matrix B is defined to be the $mp \times nq$ matrix $A \otimes B = (a_{ij}B)$.

Lemma 1 [16]**.** *If A and B are square matrices of order m and n respectively, then $A \otimes B$ is a square matrix of order mn. Also $(A \otimes B)(C \otimes D) = AC \otimes BD$, if the products AC and BD exist.*

Lemma 2 [16]**.** *If A and B are square matrices of order m and n respectively with eigenvalues λ_i $(1 \leq i \leq m)$ and μ_j $(1 \leq i \leq n)$, then the eigenvalues of $A \otimes B$ are $\lambda_i \mu_j$ and that of $A \otimes I_n + I_m \otimes B$ are $\lambda_i + \mu_j$.*

2. Preliminaries

For the definitions and the eigenvalues of the following graph products available for (unsigned) graph, one may refer to [5] and the same for signed graphs can be found in [8].

Given two signed graphs $\Sigma_1 = (V_1, E_1, \sigma_1)$ and $\Sigma_2 = (V_2, E_2, \sigma_2)$, their *Cartesian product* $\Sigma_1 \times \Sigma_2$ is defined as the signed graph $(V_1 \times V_2, E, \sigma)$ where the edge set E is that of the Cartesian product of underlying unsigned graphs and the signature function σ for the labeling of the edges is defined by

$$\sigma\big((u_i, v_j)(u_k, v_l)\big) = \begin{cases} \sigma_1\big(u_i u_k\big) & \text{if } j = l, \\ \sigma_2\big(v_j v_l\big) & \text{if } i = k. \end{cases}$$

The *strong product* $\Sigma_1 \boxtimes \Sigma_2$ of two signed graphs $\Sigma_1 = (V_1, E_1, \sigma_1)$ and $\Sigma_2 = (V_2, E_2, \sigma_2)$ is defined as the signed graph $(V_1 \times V_2, E, \sigma)$ where the edge set E is that of the strong product of the underlying unsigned graphs and the signature function σ for the labeling of the edges is defined by

$$\sigma\big((u_i, v_j)(u_k, v_l)\big) = \sigma_1\big(u_i u_k\big)\sigma_2\big(v_j v_l\big).$$

In the examples given in the main section, we use the following lemmas found in [7] and [8]. We follow the notation, $[r]$, for an integer r, such that $[r] = 0$ if r is even and $[r] = 1$ if it is odd. We denote by $P_n^{(r)}$, where $0 \leq r \leq n-1$, signed paths of order n and size $n-1$ with r negative edges where the underlying graph is the path P_n. Also $C_n^{(r)}$, for $0 \leq r \leq n$, denotes signed cycles with r negative edges.

Lemma 3 [7]. *The signed paths $P_n^{(r)}$, where $0 \leq r \leq n-1$, have the eigenvalues (independent of r) given by $\lambda_j = 2\cos\frac{\pi j}{n+1}$ for $j = 1, 2, \ldots, n$.*

Lemma 4 [8]. *The eigenvalues λ_j of $C_n^{(r)}$ for $j = 1, 2, \ldots, n$ and $0 \leq r \leq n$ are given by $\lambda_j = 2\cos\dfrac{(2j-[r])\pi}{n}$.*

The following three results on Cartesian product and strong product of signed graphs are found in [8].

Lemma 5 [8]. *Given two signed graphs $\Sigma_1 = (V_1, E_1, \sigma_1)$ and $\Sigma_2 = (V_2, E_2, \sigma_2)$ where $|V_1| = m$ and $|V_2| = n$, then the adjacency matrix $A(\Sigma_1 \times \Sigma_2)$ of the Cartesian product $\Sigma_1 \times \Sigma_2$ is $A(\Sigma_1) \otimes I_n + I_m \otimes A(\Sigma_2)$. Hence eigenvalues of $\Sigma_1 \times \Sigma_2$ will be the sum of the eigenvalues of Σ_1 and Σ_2.*

Lemma 6 [8]. *The Cartesian product $\Sigma_1 \times \Sigma_2$, of the signed graphs Σ_1 and Σ_2, is balanced if and only if Σ_1 and Σ_2 are both balanced.*

Lemma 7 [8]. *The adjacency matrix of the strong product $\Sigma_1 \boxtimes \Sigma_2$ will be the Kronecker product of the adjacency matrices of Σ_1 and Σ_2 and its eigenvalues will be the product of the eigenvalues of Σ_1 and Σ_2.*

3. Adjacency Eigenvalues of Lexicographic Product

In this section, we generalize to signed graphs the expression for the adjacency matrix of the composition of two unsigned graphs given in [5]. This expression provides a way to calculate their eigenvalues in the sequel.

Theorem 8. *If $\Sigma_1 = (V_1, E_1, \sigma_1)$ and $\Sigma_2 = (V_2, E_2, \sigma_2)$ are two signed graphs where $|V_1| = m$ and $|V_2| = n$, then the adjacency matrix $A(\Sigma_1[\Sigma_2])$ of the lexicographic product $\Sigma_1[\Sigma_2]$ is $A(\Sigma_1) \otimes J_n + I_m \otimes A(\Sigma_2)$.*

Proof. A direct computation shows that $A(\Sigma_1) \otimes J_n + I_m \otimes A(\Sigma_2)$

$$= \begin{bmatrix} 0J_n & \sigma_1(u_1u_2)J_n & \sigma_1(u_1u_3)J_n & \ldots & \sigma_1(u_1u_m)J_n \\ \sigma_1(u_2u_1)J_n & 0J_n & \sigma_1(u_2u_3)J_n & \ldots & \sigma_1(u_2u_m)J_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \sigma_1(u_mu_1)J_n & \sigma_1(u_mu_2)J_n & \sigma_1(u_mu_3)J_n & \ldots & 0J_n \end{bmatrix}$$

$$+\begin{bmatrix} A(\Sigma_2) & 0 & 0 & \dots & 0 \\ 0 & A(\Sigma_2) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & A(\Sigma_2) \end{bmatrix}$$

$$= \begin{bmatrix} A(\Sigma_2) & \sigma_1(u_1u_2)J_n & \sigma_1(u_1u_3)J_n & \dots & \sigma_1(u_1u_m)J_n \\ \sigma_1(u_2u_1)J_n & A(\Sigma_2) & \sigma_1(u_2u_3)J_n & \dots & \sigma_1(u_2u_m)J_n \\ \dots & \dots & \dots & \dots & \dots \\ \sigma_1(u_mu_1)J_n & \sigma_1(u_mu_2)J_n & \sigma_1(u_mu_3)J_n & \dots & A(\Sigma_2) \end{bmatrix}.$$

Now let us examine the $(u_i, v_j)(u_k, v_l)$ elements of this matrix. We find them as

$$\begin{cases} \sigma_1(u_iu_k) & \text{if } u_i \neq u_k \text{ and } u_iu_k \in E_1, \\ \sigma_2(v_jv_l) & \text{if } v_jv_l \in E_2 \text{ and } u_i = u_k, \\ 0 & \text{otherwise.} \end{cases}$$

They are exactly what appear in the $(u_i, v_j)(u_k, v_l)$ positions of $A(\Sigma_1[\Sigma_2])$. ■

Corollary 9. *The adjacency matrix of* $\Sigma_1[\Sigma_2]$ *can also be expressed as*

$$A(\Sigma_1[\Sigma_2]) = A(\Sigma_1 \times \Sigma_2) + A(\Sigma_1 \boxtimes (+K_n)). \tag{2}$$

Proof. We have $J_n = I_n + A(+K_n)$. Applying this in the expression for the adjacency matrix of $\Sigma_1[\Sigma_2]$ given in Theorem 8, we have the result. ■

Before we proceed further, we need some more definitions and notations. The *net-degree* $d^{\pm}_{\Sigma}(v)$ of a vertex v of a signed graph Σ is defined as $d^{\pm}_{\Sigma}(v) = d^{+}_{\Sigma}(v) - d^{-}_{\Sigma}(v)$, where $d^{+}_{\Sigma}(v)$ and $d^{-}_{\Sigma}(v)$ denote, respectively, the number of positive edges and the number of negative edges incident with v. If no confusion arises, we may omit the suffix and write them as $d^+(v)$ and $d^-(v)$. Also as usual, $d(v)$ denotes the total number of edges incident at v and of course, $d(v) = d^+(v) + d^-(v)$. Properties of the degree sequence of a signed graph can be seen in [4, 9, 10]. A signed graph Σ is called *net-regular* if every vertex has the same net-degree and in that case, we write the common value of net-degree as $d^{\pm}(\Sigma)$. We define, a signed graph $\Sigma = (G, \sigma)$ to be *co-regular*, if the underlying graph G is r-regular for some positive integer r and Σ is net-regular with net-degree k for some integer k. In this case we also define the *co-regularity pair* to be the ordered pair (r, k). For example, the alternately signed cycle $C^{(n)}_{2n}$ is a co-regular signed graph with co-regularity pair $(2, 0)$.

In general, though we cannot come up with a precise formula for the calculation of the eigenvalues of the lexicographic product of two signed graphs, which is the case even with (unsigned) graphs (see [5]), we have Theorem 12, using Lemma 10 and Lemma 11, which generalizes a similar result given in [5] for (unsigned) graph composition.

Lemma 10 [14]**.** *If Σ is a net-regular signed graph, then $d^{\pm}(\Sigma)$ is an eigenvalue of Σ with $\mathbf{j}$ as an eigenvector.*

Also, since the adjacency matrix of a signed graph is real and symmetric, from the general spectral theory, we know that other eigenvectors will be orthogonal to $\mathbf{j}$. For emphasis, we state the result in the following lemma to fit it suitable for the proof of Theorem 12.

Lemma 11. *If Σ is a net-regular graph, then the eigenvector $Y_j = [y_1, y_2, \ldots, y_n]^T$ corresponding to the eigenvalue $\mu_j \neq d^{\pm}(\Sigma)$ satisfies $\sum_{k=1}^{n} y_k = 0$.*

Theorem 12. *Let Σ_1 and Σ_2 be two signed graphs such that the latter is net-regular. If the eigenvalues of Σ_1 and Σ_2 are, respectively, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ and $\mu_1 = d^{\pm}(\Sigma_2), \mu_2, \ldots, \mu_n$, then $\Sigma_1[\Sigma_2]$ has eigenvalues $\lambda_1 n + d^{\pm}(\Sigma_2), \lambda_2 n + d^{\pm}(\Sigma_2), \ldots, \lambda_m n + d^{\pm}(\Sigma_2)$ (each of multiplicity one) and $\mu_2, \mu_3, \ldots, \mu_n$ (each of multiplicity m).*

Proof. Let $\mathbf{X}_i$ be the eigenvector corresponding to the eigenvalue λ_i of $A(\Sigma_1)$ and $\mathbf{Y}_j$ be the eigenvector corresponding to the eigenvalue μ_j of $A(\Sigma_2)$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. By Lemma 10, we have $\mathbf{Y}_1 = \mathbf{j}$ which is the eigenvector corresponding to $\mu_1 = d^{\pm}(\Sigma_2)$. Since J_n has rank 1, there is only one non-zero eigenvalue which will be its trace$= n$. That is, J_n has one non-zero eigenvalue n (with multiplicity 1) with the eigenvector $\mathbf{j}$ and 0 (with multiplicity $n-1$) as the other eigenvalues. Then,

$$\begin{aligned} &A(\Sigma_1[\Sigma_2])(\mathbf{X}_i \otimes \mathbf{j}) = (A(\Sigma_1) \otimes J_n + I_m \otimes A(\Sigma_2))(\mathbf{X}_i \otimes \mathbf{j}) \\ &= A(\Sigma_1)\mathbf{X}_i \otimes J_n\mathbf{j} + I_m\mathbf{X}_i \otimes A(\Sigma_2)\mathbf{j} = \lambda_i\mathbf{X}_i \otimes n\mathbf{j} + \mathbf{X}_i \otimes d^{\pm}(\Sigma_2)\mathbf{j} \\ &= (\lambda_i n + d^{\pm}(\Sigma_2))(\mathbf{X}_i \otimes \mathbf{j}) \end{aligned}$$

showing that $\lambda_i n + d^{\pm}(\Sigma_2)$ is an eigenvalue of $A(\Sigma_1[\Sigma_2])$, for $1 \leq i \leq m$. Again when $j \neq 1$,

$$\begin{aligned} &A(\Sigma_1[\Sigma_2])(\mathbf{X}_i \otimes \mathbf{Y}_j) = (A(\Sigma_1) \otimes J_n + I_m \otimes A(\Sigma_2))(\mathbf{X}_i \otimes \mathbf{Y}_j) \\ &= A(\Sigma_1)\mathbf{X}_i \otimes J_n\mathbf{Y}_j + I_m\mathbf{X}_i \otimes A(\Sigma_2)\mathbf{Y}_j = \lambda_i\mathbf{X}_i \otimes (\textstyle\sum_{k=1}^{n} y_k)\mathbf{j} + \mathbf{X}_i \otimes \mu_j\mathbf{Y}_j \\ &= \lambda_i\mathbf{X}_i \otimes 0\mathbf{j} + \mathbf{X}_i \otimes \mu_j\mathbf{Y}_j = \mu_j(\mathbf{X}_i \otimes \mathbf{Y}_j) \end{aligned}$$

which gives the result that μ_j for $2 \leq j \leq n$ are the eigenvalues of $A(\Sigma_1[\Sigma_2])$. ■

4. Balance of the Lexicographic Product of Two Signed Graphs

Before we prove the criterion for the balance of a lexicographic product of two signed graphs, we need an important notion called switching of signed graphs (for more details refer to [14]). If $\theta : V \to \{+1, -1\}$ is a function called *switching function*, then *switching* of the signed graph $\Sigma = (G, \sigma)$ by θ means changing σ to σ^θ defined by:

$$\sigma^\theta(uv) = \theta(u)\sigma(uv)\theta(v).$$

The switched graph denoted by Σ^θ, is the signed graph $\Sigma^\theta = (G, \sigma^\theta)$. We call two signed graphs $\Sigma_1 = (G, \sigma_1)$ and $\Sigma_2 = (G, \sigma_2)$ to be *switching equivalent*, if there exists a switching function $\theta : V \to \{+1, -1\}$ such that $\Sigma_1 = \Sigma_2^\theta$. It can be seen that switching preserves many features of the two signed graphs including the eigenvalues [14]. Indeed, the following is a very important result.

Lemma 13 [14]**.** *A signed graph is balanced if and only if it can be switched to an all-positive signed graph.*

Theorem 14. *If Σ_1 and Σ_2 are two signed graphs with at least one edge for each, then their lexicographic product or composition $\Sigma_1[\Sigma_2]$, is balanced if and only if Σ_1 is balanced and Σ_2 is all-positive.*

Proof. If Σ_1 is balanced and Σ_2 is all-positive, then by Lemma 13, it is possible to switch Σ_1 to an all-positive signed graph Σ_1', say and let $\theta : V(\Sigma_1) \to \{+1, -1\}$ be the corresponding switching function. Define $\theta_1 : V(\Sigma_1[\Sigma_2]) \to \{+1, -1\}$ by $\theta_1(u_i, v_j) = \theta(u_i)$. Then we claim that $\Sigma_1[\Sigma_2]$ is switching equivalent to the all-positive signed graph $\Sigma_1'[\Sigma_2]$. To see this, let the signatures of Σ_1, Σ_2 and $\Sigma_1[\Sigma_2]$, respectively, be σ_1, σ_2 and σ_c. As Σ_1 is switching equivalent to Σ_1', we have

$$\sigma_1{}^\theta(u_i u_k) = \theta(u_i)\sigma_1(u_i u_k)\theta(u_k)$$

which implies that $\sigma_1(u_i u_k) = \theta(u_i)\theta(u_k)$, since $\sigma_1{}^\theta$ is the all-positive signature. Also, for $\mathbf{uv} = (u_i, v_j)(u_k, v_l) \in E(\Sigma_1'[\Sigma_2])$

$$\sigma_c^{\theta_1}(\mathbf{uv}) = \theta_1(\mathbf{u})\sigma_c(\mathbf{uv})\theta_1(\mathbf{v}) = \theta(u_i)\sigma_c(\mathbf{uv})\theta(u_k).$$

Using the definition of the composition of two signed graphs, see Equation (1), this gives

$$\sigma_c^{\theta_1}(\mathbf{uv}) = \begin{cases} \theta(u_i)\sigma_1(u_i u_k)\theta(u_k) = (\theta(u_i)\theta(u_k))^2 = 1 & \text{if } i \neq k, \\ \theta(u_i)\sigma_2(v_j v_l)\theta(u_i) = (\theta(u_i))^2 = 1 & \text{if } i = k, \end{cases}$$

since σ_2 is the all-positive signature which thus leads to $\sigma_c^{\theta_1}(\mathbf{uv}) = 1$ for all $\mathbf{uv} \in E(\Sigma_1'[\Sigma_2]) = E(\Sigma_1[\Sigma_2])$, as required. Conversely, assuming that $\Sigma_1[\Sigma_2]$ is balanced, we have Cartesian product $\Sigma_1 \times \Sigma_2$ as a subgraph of $\Sigma_1[\Sigma_2]$. So Lemma 6 is applicable and hence Σ_1 and Σ_2 must be at least balanced. Now, we claim that Σ_2 cannot have any negative edge. On the contrary, if we assume that Σ_2 contains a negative edge, say $v_j v_l$, then we claim that it would result in an unbalanced triangle in $\Sigma_1[\Sigma_2]$, as per the definition of signed graph composition, leading to a contradiction. To prove this claim consider the following cases.

Case 1. If there is a negative edge $u_i u_k$ in Σ_1. In this case the required negative triangle, for example, is $(u_i, v_1)(u_k, v_j), (u_k, v_j)(u_k, v_l), (u_k, v_l)(u_i, v_1)$, with three edges being negative.

Case 2. If the edge $u_i u_k$ is positive, then the same triangle in Case 1 will be negative with one negative edge, giving an unbalanced triangle, as required. ■

Example 15. Consider the lexicographic product $P_m^{(r_1)}[C_{2n}^{(n)}]$ of the signed path $P_m^{(r_1)}$ and the co-regular signed cycle $C_{2n}^{(n)}$ such that $d^{\pm}(C_{2n}^{(n)}) = 0$. Then the eigenvalues of this signed graph product are: $4n\cos(\frac{\pi i}{m+1})$ for $1 \leq i \leq m$ with multiplicity one and $2\cos(\frac{(2j-[n])\pi}{2n})$ each of multiplicity m for $1 \leq j \leq 2n$ such that $j \neq \frac{n+[n]}{2}$. Also, by Theorem 14, $P_m^{(r_1)}[C_{2n}^{(n)}]$ is unbalanced, since $C_{2n}^{(n)}$ contains negative edge, though $P_m^{(r_1)}$ is balanced.

On the other hand, if the cycle $-C_n = C_n^{(n)}$ is the all-negative cycle, where $n \geq 3$ so that the eigenvalues of $-C_n$ are $-2 = d^{\pm}(-C_n)$ and $-2\cos(\frac{2\pi j}{n})$ for $1 \leq j \leq n-1$, then the eigenvalues of $P_m^{(r_1)}[-C_n]$, by applying the result in Theorem 12, are $n(2\cos(\frac{\pi i}{m+1})) + (-2) = 2(n\cos(\frac{\pi i}{m+1}) - 1)$ for $1 \leq i \leq m$ with multiplicity one and $-2\cos(\frac{2\pi j}{n})$ each of multiplicity m for $1 \leq j \leq n-1$. Here also, as all the edges of $-C_n$ are negative, by Theorem 14, $P_m^{(r_1)}[-C_n]$ is unbalanced.

5. Laplacian Eigenvalues of Lexicographic Product

Lemma 16. *For a vertex* $\mathbf{u} = (u_i, v_j)$ *in* $\Sigma_1[\Sigma_2]$, $d^{\pm}(\mathbf{u}) = nd^{\pm}(u_i) + d^{\pm}(v_j)$ *and* $d(\mathbf{u}) = nd(u_i) + d(v_j)$.

Proof. From the definition of $\Sigma_1[\Sigma_2]$, the number of edges (positive or negative) adjacent with $\mathbf{u} = (u_i, v_j)$ can be counted by first taking into account the number of edges $(u_i, v_j)(u_k, v_l)$, incident with (u_i, v_j) when $i = k$, which is either $d^{\pm}(v_j)$ or $d(v_j)$, as the case may be, and then as the edges of Σ_1 has a major role in the composition, we count the edges originating from (u_i, v_j) and incident with (u_k, v_l) when $i \neq k$, which is $nd^{\pm}(u_i)$ or $nd(u_i)$, as the case may be. By using the phrase 'as the case may be', we mean that the counting strategy may be to count the positive and negative edges adjacent to the vertices separately or the edge as such without considering the label on it. Thus we have the results in the lemma. ■

Corollary 17. *If* Σ_1 *and* Σ_2 *are net-regular signed graphs with net-degrees, respectively,* $d^{\pm}(\Sigma_1)$ *and* $d^{\pm}(\Sigma_2)$, *then* $\Sigma_1[\Sigma_2]$ *is a net-regular signed graph with net-degree* $d^{\pm}(\Sigma_1[\Sigma_2]) = nd^{\pm}(\Sigma_1) + d^{\pm}(\Sigma_2)$.

Proof. From Lemma 16, we have for a vertex $\mathbf{u} = (u_i, v_j)$ in $\Sigma_1[\Sigma_2]$, $d^{\pm}(\mathbf{u}) = nd^{\pm}(u_i) + d^{\pm}(v_j)$. By assumption, $d^{\pm}(u_i) = d^{\pm}(\Sigma_1)$ for any vertex u_i in Σ_1 and $d^{\pm}(v_j) = d^{\pm}(\Sigma_2)$ for any vertex v_j in Σ_2. Therefore, $d^{\pm}(\mathbf{u}) = nd^{\pm}(\Sigma_1) + d^{\pm}(\Sigma_2)$ which is a constant for any vertex $\mathbf{u} = (u_i, v_j)$ in $\Sigma_1[\Sigma_2]$. So, this constant value is the net-degree of $\Sigma_1[\Sigma_2]$, making it a net-regular signed graph. ■

Theorem 18.

$$D(\Sigma_1[\Sigma_2]) = nD(\Sigma_1) \otimes I_n + I_m \otimes D(\Sigma_2) \quad \text{and} \tag{3}$$

$$L(\Sigma_1[\Sigma_2]) = nD(\Sigma_1) \otimes I_n - A(\Sigma_1) \otimes J_n + I_m \otimes L(\Sigma_2) \tag{4}$$

Proof. We have from Lemma 16, $d(\mathbf{u}) = nd(u_i) + d(v_j)$ for a vertex $\mathbf{u} = (u_i, v_j)$ in $\Sigma_1[\Sigma_2]$. As such, noting the fact that $D(\Sigma_1[\Sigma_2])$ is a diagonal matrix of order mn, we first get that it can be written as a block partitioned diagonal matrix with diagonal entries $nd(u_i)I_n + D(\Sigma_2)$ for $1 \leq i \leq m$. This when simplified leads to Equation (3). To get Equation (4), apply Equation (3) and the expression for the adjacency matrix $A(\Sigma_1[\Sigma_2])$ given in Theorem 8 and note also the fact that $L(\Sigma_1[\Sigma_2]) = D(\Sigma_1[\Sigma_2]) - A(\Sigma_1[\Sigma_2])$. ■

Theorem 19. *If the underlying graph of Σ_1 is regular of degree r_1 and Σ_2 is a co-regular signed graph with the co-regularity pair $(r_2, d^{\pm}(\Sigma_2))$, then $L(\Sigma_1[\Sigma_2]) = (nr_1 + r_2)I_{mn} - A(\Sigma_1[\Sigma_2])$ and the Laplacian eigenvalues of $\Sigma_1[\Sigma_2]$ are*

$\lambda_{i1}^L = n\lambda_i^L + \mu_1^L = n\lambda_i^L + r_2 - d^{\pm}(\Sigma_2)$ *with multiplicity one, for $1 \leq i \leq m$ and*

$\lambda_{1j}^L = nr_1 + \mu_j^L$ *with multiplicity m, for $2 \leq j \leq n$.*

Proof. As Σ_1 is regular of degree r_1, its Laplacian eigenvalues and adjacency eigenvalues are related by the equation $\lambda_i^L = r_1 - \lambda_i$. Similarly for Σ_2, we have $\mu_j^L = r_2 - \mu_j$ and $\mu_1^L = r_2 - d^{\pm}(\Sigma_2)$. Moreover the underlying graph of $\Sigma_1[\Sigma_2]$ is regular of degree $nr_1 + r_2$. Therefore, by the definition of the Laplacian matrix, $L(\Sigma_1[\Sigma_2]) = (nr_1 + r_2)I_{mn} - A(\Sigma_1[\Sigma_2])$. As such, the remaining results follow from Theorem 12 and Lemma 16. ■

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References

[1] B.D. Acharya, *Spectral criterion for cycle balance in networks*, J. Graph Theory **4** (1980) 1–11.
doi:10.1002/jgt.3190040102

[2] F. Barahona, *On the computational complexity of Ising spin glass models*, J. Phys. (A) **15** (1982) 3241–3253.
doi:10.1088/0305-4470/15/10/028

[3] D. Cartwright and F. Harary, *Structural balance: a generalization of Heiders theory*, Psychological Rev. **63** (1956) 277–293.
doi:10.1037/h0046049

[4] G. Chartrand, H. Gavlas, F. Harary and M. Schultz, *On signed degrees in signed graphs*, Czechoslovak Math. J. **44** (1994) 677–680.

[5] D.M. Cvetković, M. Doob and H. Sachs, Spectra of Graphs (third ed., Johann Abrosius Barth Verlag, 1995).

[6] F. Harary, Graph Theory (Addison Wesley, Reading, MA, 1972).

[7] K.A. Germina and K. Shahul Hameed, *On signed paths, signed cycles and their energies*, Applied Math. Sci. **70** (2010) 3455–3466.
doi:10.1016/j.laa.2010.10.026

[8] K.A. Germina, K. Shahul Hameed and T. Zaslavsky, *On product and line graphs of signed graphs, their eigenvalues and energy*, Linear Algebra Appl. **435** (2011) 2432–2450.
doi:10.1007/s10587-007-0079-z

[9] S. Pirzada, T.A. Naikoo and F.A. Dar, *Signed degree sets in signed graphs*, Czechoslovak Math. J. **57** (2007) 843–848.
doi:10.1215/S0012-7094-59-02667-5

[10] S. Pirzada, T.A. Naikoo and F.A. Dar, *Signed degree sets in signed bipartite graphs*, AKCE Int. J. Graphs and Combin. **4(3)** (2007) 301–312.

[11] G. Sabidussi, *The composition of graphs*, Duke Math. J. **26** (1959) 693–696.
doi:10.1215/S0012-7094-59-02667-5

[12] G. Sabidussi, *The lexicographic product of graphs*, Duke Math. J. **28** (1961) 573–578.
doi:10.1215/S0012-7094-61-02857-5

[13] T. Zaslavsky, *Signed graphs*, Discrete Appl. Math. **4** (1982) 47–74, Erratum: Discrete Appl. Math. **5** (1983) 248–248.
doi:10.1016/0166-218X(83)90047-1

[14] T. Zaslavsky, *Matrices in the theory of signed simple graphs*, Advances in Discrete Mathematics and Applications, Proc. Int. Conf. Discrete Math., Mysore, India, 2008, ed(s) B.D. Acharya, G.O.H. Katona, and J. Nešetřil, (Ramanujan Math. Soc. Lect. Notes Mysore, India, 2010) **13** 207–229.

[15] T. Zaslavsky, *A mathematical bibliography of signed and gain graphs and allied areas, VII Edition*, Electronic J. Combin. **8** (1998), Dynamic Surveys: #8.

[16] F. Zhang, Matrix Theory: Basic Theory and Techniques (Springer-Verlag, 1999).

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