*Discussiones Mathematicae*
*Graph Theory 32 (2012) 205–219*
*doi:10.7151/dmgt.1613*

# 3-TRANSITIVE DIGRAPHS

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**Abstract**

Let $D$ be a digraph, $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of $D$, respectively.

A digraph $D$ is 3-transitive if the existence of the directed path $(u, v, w, x)$ of length 3 in $D$ implies the existence of the arc $(u, x) \in A(D)$. In this article strong 3-transitive digraphs are characterized and the structure of non-strong 3-transitive digraphs is described. The results are used, e.g., to characterize 3-transitive digraphs that are transitive and to characterize 3-transitive digraphs with a kernel.

**Keywords:** digraph, kernel, transitive digraph, quasi-transitive digraph, 3-transitive digraph, 3-quasi-transitive digraph.

**2010 Mathematics Subject Classification:** 05C20.

## 1. Introduction

In this work, $D = (V(D), A(D))$ will denote a finite digraph without loops or multiple arcs in the same direction, with vertex set $V(D)$ and arc set $A(D)$. For general concepts and notation we refer the reader to [1, 4] and [7], particularly we will use the notation of [7] for walks, if $\mathscr{C} = (x_0, x_1, \ldots, x_n)$ is a walk and $i < j$ then $x_i \mathscr{C} x_j$ will denote the subwalk $(x_i, x_{i+1}, \ldots, x_{j-1}, x_j)$ of $\mathscr{C}$. Union of walks will be denoted by concatenation or with $\cup$. For a vertex $v \in V(D)$, we define the *out-neighborhood* of $v$ in $D$ as the set $N^+_D(v) = \{u \in V(D) \big| (v, u) \in A(D)\}$; when there is no possibility of confusion we will omit the subscript $D$. The elements of $N^+(v)$ are called the *out-neighbors* of $v$, and the *out-degree* of $v$, $d^+_D(v)$, is the number of out-neighbors of $v$. Definitions of in-neighborhood, in-neighbors and in-degree of $v$ are analogously given. We say that a vertex $u$ reaches a vertex $v$ in

$D$ if a directed $uv$-directed path (a path with initial vertex $u$ and terminal vertex $v$) exists in $D$. An arc $(u,v) \in A(D)$ is called *asymmetrical* (resp. *symmetrical*) if $(v,u) \notin A(D)$ (resp. $(v,u) \in A(D)$).

If $D$ is a digraph and $X, Y \subseteq V(D)$, an $XY$-arc is an arc with initial vertex in $X$ and terminal vertex in $Y$. If $X \cap Y = \emptyset$, $X \to Y$ will denote that $(x,y) \in A(D)$ for every $x \in X$ and $y \in Y$. Again, if $X$ and $Y$ are disjoint, $X \Rightarrow Y$ will denote that there are not $YX$-arcs in $D$. When $X \to Y$ and $X \Rightarrow Y$ we will simply write $X \mapsto Y$. If $D_1, D_2$ are subdigraphs of $D$, we will abuse notation to write $D_1 \to D_2$ or $D_1D_2$-arc, instead of $V(D_1) \to V(D_2)$ or $V(D_1)V(D_2)$-arc, respectively. Also, if $X = \{v\}$, we will abuse notation to write $v \to Y$ or $vY$-arc instead of $\{v\} \to Y$ or $\{v\}Y$-arc, respectively. Analogously if $Y = \{v\}$.

A digraph is *strongly connected* (or strong) if for every $u, v \in V(D)$, there exists a $uv$-directed path, i.e., a directed path with initial vertex $u$ and terminal vertex $v$. A *strong component* (or component) of $D$ is a maximal strong subdigraph of $D$. The *condensation* of $D$ is the digraph $D^\star$ with $V(D^\star)$ equal to the set of all strong components of $D$, and $(S,T) \in A(D^\star)$ if and only if there is an $ST$-arc in $D$. Clearly $D^\star$ is an acyclic digraph (a digraph without directed cycles), and thus, it has both vertices of out-degree equal to zero and vertices of in-degree equal to zero. A *terminal component* of $D$ is a strong component $T$ of $D$ such that $d^+_{D^\star}(T) = 0$. An *initial component* of $D$ is a strong component $S$ of $D$ such that $d^-_{D^\star}(S) = 0$.

A *biorientation* of the graph $G$ is a digraph $D$ obtained from $G$ by replacing each edge $\{x,y\} \in E(G)$ by either the arc $(x,y)$ or the arc $(y,x)$ or the pair of arcs $(x,y)$ and $(y,x)$. A *semicomplete* digraph is a biorientation of a complete graph. An *orientation* of a graph $G$ is an asymmetrical biorientation of $G$; thus, an *oriented* graph is an asymmetrical digraph. A tournament is an orientation of a complete graph. An orientation of a digraph $D$ is a maximal asymmetrical subdigraph of $D$. A *complete* digraph is a biorientation of a complete graph obtained by replacing each edge $\{x,y\}$ by the arcs $(x,y)$ and $(y,x)$.

Let $D$ be a digraph with vertex set $V(D) = \{v_1, v_2, \dots, v_n\}$ and $H_1, H_2, \dots, H_n$ a family of vertex disjoint digraphs. The *composition* of digraphs $D[H_1, H_2, \dots, H_n]$ is the digraph having $\bigcup_{i=1}^n V(H_i)$ as its vertex set and arc set $\bigcup_{i=1}^n A(H_i) \cup \big\{(u,v) \big| u \in V(H_i), v \in V(H_j), (v_i, v_j) \in A(D)\big\}$. The dual (or converse) of $D$, $\overleftarrow{D}$ is the digraph with vertex set $V(\overleftarrow{D}) = V(D)$ and such that $(u,v) \in A(\overleftarrow{D})$ if and only if $(v,u) \in A(D)$. The directed cycle of length 3 will be denoted, as usual, by $C_3$.

A digraph is *transitive* if for every three distinct vertices $u, v, w \in V(D)$, $(u,v), (v,w) \in A(D)$ implies that $(u,w) \in A(D)$. Transitive digraphs have a lot of properties, many of which can be verified straightforward by using the following structural characterization, which can be found in [1] as an exercise.

**Theorem 1.** *Let $D$ be a digraph $D$ with strong components $S_1, S_2, \ldots, S_n$. Then $D$ is a transitive digraph if and only if $D = D^\star[S_1, S_2, \ldots, S_n]$, where $S_i$ is a complete digraph for $1 \leq i \leq n$.*

But, the structure of transitive digraphs is so rich that, working on this family, many problems become trivial or have a very simple solution. In view of this situation, some generalizations of transitive digraphs have been studied. Without doubt, the most studied generalization of transitive digraphs is the family of quasi-transitive digraphs. A digraph is quasi-transitive if for every three distinct vertices $u, v, w \in V(D)$, $(u, v), (v, w) \in A(D)$ implies that $(u, w) \in A(D)$ or $(w, u) \in A(D)$. Clearly, every semicomplete digraph is a quasi-transitive digraph, so, quasi-transitive digraphs generalize both, transitive and semicomplete digraphs. Quasi-transitive have been characterized by Bang Jensen and Huang in [2], and their structure is very similar to the structure of transitive digraphs. Once again, this structural characterization has been very helpful to solve a large number of problems over this family, e.g., characterization of quasi-transitive digraphs with 3-kings, Hamiltonicity in quasi-transitive digraphs, or the Laborde-Payan-Xuong Conjecture for quasi-transitive digraphs.

Quasi-transitive digraphs were generalized with 3-quasi-transitive digraphs. A digraph $D$ is 3-quasi-transitive if for every directed path, $(v_0, v_1, v_2, v_3)$, either $(v_0, v_3) \in A(D)$ or $(v_3, v_0) \in A(D)$. Let us notice that in the definition of 3-quasi-transitive digraphs, the subdigraph $(v_0, v_1, v_2, v_3)$ considered is a directed path, so it cannot happen that $v_0 = v_3$ and we can effectively work on digraphs without loops. The family of 3-quasi-transitive digraphs were introduced by Bang-Jensen in the context of arc-locally semicomplete digraphs, which generalize both, semicomplete digraphs and semicomplete bipartite digraphs. A digraph is arc-locally in-semicomplete if $(z, x), (x, y), (w, y) \in A(D)$ and $z \neq w$ implies that $(z, w) \in A(D)$ or $(w, z) \in A(D)$. A digraph is arc-locally out-semicomplete if $(x, z), (x, y), (y, w) \in A(D)$ and $z \neq w$ implies that $(x, w) \in A(D)$ or $(w, x) \in A(D)$. A digraph is arc-locally semicomplete if it is arc-locally in-semicomplete and arc-locally out-semicomplete. These families are defined to fulfill a property on some specific orientation of a path of length 3, in all of them, the existence of a (undirected) 4-cycle can be inferred from the existence of the specific orientation. There is one more orientation of a directed path of length 3 that induces the existence of a fourth family of digraphs. A digraph is often called of the type $\mathcal{H}_4$ if $(x, w), (x, y), (z, y) \in A(D)$ and $z \neq w$ implies that $(w, z) \in A(D)$ or $(z, w) \in A(D)$. The problem of finding structural characterizations of these four families of digraphs was proposed by Bang-Jensen. Besides transitive and quasi-transitive digraphs, also arc-locally semicomplete digraphs [8] and arc-locally in-semicomplete digraphs [13] have been characterized.

In [10], Galeana-Sánchez and the author introduce $k$-transitive and $k$-quasi-transitive digraphs. A digraph $D$ is $k$-transitive if the existence of a directed

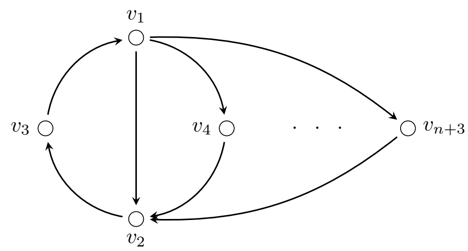


Figure 1. The family of digraphs $F_n$.

path $(v_0, v_1, \ldots, v_k)$ of length $k$ in $D$ implies that $(v_0, v_k) \in A(D)$. A digraph $D$ is $k$-quasi-transitive if the existence of a directed path $(v_0, v_1, \ldots, v_k)$ of length $k$ in $D$ implies that $(v_0, v_k) \in A(D)$ or $(v_k, v_0) \in A(D)$. Also in [10], some basic properties on the structure of both $k$-transitive and $k$-quasi-transitive are proved. These properties are used to prove the existence of $n$-kernels in both families.

The aim of this article is to characterize strong 3-transitive digraphs and give a thorough description of the structure of non-strong 3-transitive digraphs. We will use the following characterization of strong 3-quasi-transitive digraphs given by Galeana-Sánchez, Goldfeder and Urrutia in [9].

**Theorem 2** (Galeana-Sánchez, Goldfeder, Urrutia)**.** *Let $D$ be a strong 3-quasi-transitive digraph of order $n$. Then $D$ is either a semicomplete digraph, a semicomplete bipartite digraph or isomorphic to $F_n$ (Figure 1).*

Thus, Section 2 will be devoted to prove some basic results about 3-transitive digraphs. In Section 3 the characterization of strong 3-transitive digraphs and the structural description of non-strong 3-transitive digraphs are given. In Section 4, one application of the results of Section 3 is given: A characterization of 3-transitive digraphs having a kernel. Also, an interesting problem concerning underlying graphs of 3-transitive and 3-quasi-transitive digraphs is proposed.

## 2. Preliminary Results

We begin this section with a very simple remark that will be very useful through this work.

**Remark 3.** A digraph $D$ is a 3-transitive digraph if and only if $\overleftarrow{D}$ is 3-transitive.

The following is another simple, yet useful, property of $k$-transitive digraphs.

**Proposition 4.** *If $D$ is a $k$-transitive digraph with $k \geq 2$, then $D$ is $k + n(k-1)$-transitive for every $n \in \mathbb{N}$.*

***Proof.*** Let $D$ be a $k$-transitive digraph. We will proceed by induction on $n$.

For $n = 1$, consider $(v_0, v_1, \ldots, v_{k+(k-1)})$, a directed path of length $k+(k-1)$. From the $k$-transitivity of $D$ we have that $(v_0, v_k) \in A(D)$, so $(v_0, v_k, v_{k+1}, \ldots, v_{k+(k-1)})$ is a $v_0v_k$-directed path of length $k$, and by the $k$-transitivity of $D$, we have that $(v_0, v_{k+(k-1)}) \in A(D)$.

Let us assume the result valid for $n-1$ and let $(v_0, v_1, \ldots, v_{k+n(k-1)})$ be a directed path of length $k + n(k-1)$ in $D$. By the induction hypothesis $(v_0, v_{k+(n-1)(k-1)}) \in A(D)$, and clearly $(v_0, v_{k+(n-1)(k-1)}, \ldots, v_{k+n(k-1)})$ is a directed path of length $k$ in $D$.

It follows from the $k$-transitivity that $(v_0, v_{k+n(k-1)}) \in A(D)$. The result is now obtained by the Principle of Mathematical Induction. ■

As a particular case of Proposition 4, we can observe that a 3-transitive digraph is $n$-transitive for every odd integer $n$. We can state this observation as the following corollary.

**Corollary 5.** *Let $D$ be a 3-transitive digraph and $(v_0, v_1, \ldots, v_n)$ a directed path in $D$. Then $(v_0, v_i) \in A(D)$ for every odd integer $1 \le i \le n$.*

***Proof.*** It is straightforward from Proposition 4. ■

In [14], Wang and Wang proved some results describing the structure of non-strong 3-quasi-transitive digraphs. Since every 3-transitive digraph is also 3-quasi-transitive, the properties stated next hold also for 3-transitive digraphs.

**Proposition 6** [14]**.** *Let $D'$ be a non-trivial strong induced subdigraph of a 3-quasi-transitive digraph $D$ and let $s \in V(D) \setminus V(D')$ with at least one arc from $D'$ to $s$ and $D' \Rightarrow s$. Then each of the following holds:*

1. *If $D'$ is a bipartite digraph with bipartition $(X, Y)$ and there exists a vertex of $X$ which dominates $s$, then $X \mapsto s$.*
2. *If $D'$ is a non-bipartite digraph, then $D' \mapsto s$.*

In the case of 3-transitive digraphs, the condition $D' \Rightarrow s$ in Proposition 6 not necessary. The following proposition is some kind of analogous of Proposition 6 for 3-transitive digraphs, emphasizing the behavior of certain strong subdigraphs.

**Proposition 7.** *Let $D$ be a 3-transitive digraph and $v \in V(D)$. The following statements hold:*

1. *For every $C_3$ in $D$ such that there is a $C_3v$-arc in $D$, then $C_3 \to v$.*
2. *For every $C_3$ in $D$ such that there is a $vC_3$-arc in $D$, then $v \to C_3$.*
3. *For every $\overleftrightarrow{K_n}$ in $D$, $n \ge 3$, such that there is a $\overleftrightarrow{K_n}v$-arc in $D$, then $\overleftrightarrow{K_n} \to v$.*
4. *For every $\overleftrightarrow{K_n}$ in $D$, $n \ge 3$, such that there is a $v\overleftrightarrow{K_n}$-arc in $D$, then $v \to \overleftrightarrow{K_n}$.*

5. *For every* $\overleftrightarrow{K_{n,m}} = (X, Y)$ *in* $D$ *such that there is a* $Xv$*-arc in* $D$*, then* $X \to v$.
6. *For every* $\overleftrightarrow{K_{n,m}} = (X, Y)$ *in* $D$ *such that there is a* $vX$*-arc in* $D$*, then* $v \to X$.

***Proof.*** For 1. Let $C_3 = (x, y, z, x)$ be a cycle in $D$ and $(x, v) \in A(D)$. The existence of the directed path $(y, z, x, v)$ in $D$, implies that $(y, v) \in A(D)$. Finally, since $(z, x, y, v)$ is a directed path of length 3 in $D$, $(z, v) \in A(D)$. Thus $C_3 \to v$.

For 2. It suffices to dualize 1 using Remark 3.

For 3. Let $D[S]$, with $S = \{1, 2, \ldots, n\}$, be a complete subdigraph of $D$ and $(1, v) \in A(D)$. Let $i \in S \setminus \{1\}$ be an arbitrary vertex. Remember that $n \geq 3$, so there exists a vertex $j \in S \setminus \{1, i\}$. Now, since $D[S] = \overleftrightarrow{K_n}$, we have the existence of the directed path $(i, j, 1, v)$, which implies that $(i, v) \in A(D)$. But $i$ is an arbitrary vertex of $D[S]$, and then we can conclude that $D[S] \to v$.

For 4. It suffices to dualize 3 using Remark 3.

For 5. Let $\overleftrightarrow{K_{n,m}} = (X, Y)$ be a complete subdigraph of $D$ and $x \in X$. If $|X| = 1$, then we are done. If not, let $z \in X$ be a vertex such that $z \neq x$. Since $Y \neq \emptyset$, there is a vertex $y \in Y$. Also, $(z, y), (y, x) \in A(D)$, because $D[X \cup Y]$ is a complete bipartite digraph. So $(z, y, x, v)$ is a directed path of length 3 in $D$ and hence, $(z, v) \in A(D)$. Thus, $X \to v$.

For 6. It suffices to dualize 5 using Remark 3. ■

The following proposition is also due to Wang and Wang.

**Proposition 8** [14]**.** *Let* $D'$ *be a non-trivial strong subdigraph of a* 3*-quasi-transitive digraph* $D$*. For any* $s \in V(D) \setminus V(D')$*, if there exists a directed path between* $s$ *and* $D'$*, then* $s$ *and* $D'$ *are adjacent.*

In the case of 3-transitive digraphs we can be a little more specific. The proof of the following proposition will be omitted since it is almost the same as the one given by Wang and Wang in [14].

**Proposition 9.** *Let* $D'$ *be a non-trivial strong subdigraph of a* 3*-transitive digraph* $D$ *and* $s \in V(D) \setminus V(D')$*. Then each of the following holds:*

1. *If there exists an* $sD'$*-directed path in* $D$*, then an* $sD'$*-arc exists.*
2. *If there exists a* $D's$*-directed path in* $D$*, then a* $D's$*-arc exists.*

The following couple of propositions will be used later to characterize strong 3-transitive digraphs.

**Proposition 10.** *Let* $D$ *be a strong* 3*-transitive digraph of order* $n \geq 4$*. If* $D$ *is semicomplete, then* $D$ *is complete.*

***Proof.*** For any $(x, y) \in A(D)$, let $P = (y_0, y_1, \ldots, y_s)$ be a shortest path from $y$ to $x$. If $s \geq 3$, then by Corollary 5 we can find a shorter path than $P$ from $y$ to

$x$. Suppose that $s = 2$, then $(x, y, y_1, x)$ is a 3-cycle in $D$. Let $D' = D[\{x, y, y_1\}]$. Since the order of $D$ is $n \geq 4$, there exists $v \in V(D) \setminus V(D')$. Also, $D$ is strong, so a $D's$-directed path and an $sD'$-directed path exist in $D$. It follows from Propositions 7 (1 and 2) and 9 that $(y_1, v), (v, x) \in A(D)$. So $(y, y_1, v, x)$ is a directed path of length 3 in $D$ and hence, $(y, x) \in A(D)$. This contradicts that $s = 2$. Thus, $(y, x) \in A(D)$. ■

**Proposition 11.** *Let $D$ be a strong 3-transitive digraph. If $D$ is semicomplete bipartite, then $D$ is complete bipartite.*

***Proof.*** Let $(X, Y)$ be the bipartition of $D$. It suffices to prove that for any $(v, u) \in A(D)$, $(u, v) \in A(D)$. Since $D$ is strong, there exists a path $P$ from $u$ to $v$ of length $n$. Again, since $D$ is bipartite and $u$ and $v$ belong to the different partite, $n$ must be odd. By Corollary 5, $(u, v) \in A(D)$. ■

## 3. The Structure of 3-transitive Digraphs

Let $C_3^*$ and $C_3^{**}$ be directed triangles with one and two symmetrical arcs, respectively. Digraphs $C_3, C_3^*$ and $C_3^{**}$ are shown in Figure 2.

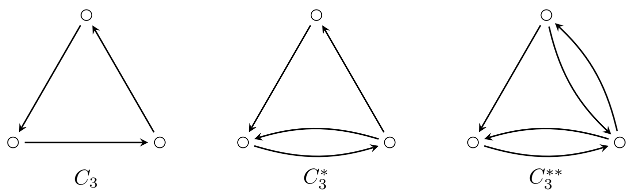


Figure 2. The digraphs $C_3, C_3^*$ and $C_3^{**}$.

The characterization of strong 3-transitive digraphs is now proved.

**Proposition 12.** *A strong digraph $D$ of order $n$ is 3-transitive if and only if it is one of the following:*

1. *A complete digraph,*
2. *A complete bipartite digraph,*
3. $C_3, C_3^*$ *or* $C_3^{**}$.

***Proof.*** Since every 3-transitive digraph is 3-quasi-transitive, in virtue of Theorem 2, a strong 3-transitive digraph must be either semicomplete, semicomplete bipartite or isomorphic to $F_n$. But $F_n$ is not 3-transitive, so a strong 3-transitive digraph must be either semicomplete or semicomplete bipartite. It is clear that every strong digraph of order less than or equal to 3 is either complete, complete bipartite or one of the digraphs $C_3, C_3^*$ or $C_3^{**}$. If $D$ has order greater than or equal to 4, and it is a semicomplete digraph, it follows from Proposition 10 that $D$ is complete. Finally, if $D$ is semicomplete bipartite, it follows from Proposition 11 that $D$ is complete bipartite. ■

As immediate corollary from Proposition 12, we get the following result.

**Corollary 13.** *Let $D$ be a 3-transitive digraph. Then $D$ is Hamiltonian if and only if $D$ is strong and it is not bipartite or it is regular.*

Let us recall that Proposition 7 describes the interaction of a single vertex with some subdigraphs of a 3-transitive digraph $D$. This covers the case when a strong component of $D$ consists of a single vertex. In [14], the following proposition is proved.

**Proposition 14.** *Let $D_1$ and $D_2$ be two distinct non-trivial strong components of a 3-quasi-transitive digraph with at least one $D_1D_2$-arc. Then either $D_1 \mapsto D_2$ or the digraph induced by $D_1 \cup D_2$ is a semicomplete bipartite digraph.*

As it was noted before, every 3-transitive digraph is a 3-quasi-transitive digraph, so Proposition 14 is also valid for 3-transitive digraphs. In an attempt to be more explicit with the interaction between non-trivial strong components of a 3-transitive digraph, we state the following proposition. Nonetheless, we omit the proof, since it is very similar to the proof of Proposition 14.

**Proposition 15.** *Let $D$ be a 3-transitive digraph and $S_1, S_2$ be distinct strong components of $D$ such that there exists an $S_1S_2$-arc. The following statements hold:*

1. *If $S_1$ contains a subdigraph isomorphic to $C_3$, then $S_1 \to S_2$.*
2. *If $S_2$ contains a subdigraph isomorphic to $C_3$, then $S_1 \to S_2$.*
3. *If $S_i$ is a complete bipartite digraph with bipartition $(X_i, Y_i)$ for $i \in \{1, 2\}$ and if the $S_1S_2$-arc is an $X_1X_2$-arc, then $X_1 \to X_2$.*
4. *If $S_i$ is a complete bipartite digraph with bipartition $(X_i, Y_i)$ for $i \in \{1, 2\}$ and there exist an $X_1X_2$-arc and a $Y_1X_2$-arc, then $S_1 \to S_2$.*
5. *If $S_i$ is a complete bipartite digraph with bipartition $(X_i, Y_i)$ for $i \in \{1, 2\}$ and there exist an $X_1X_2$-arc and an $X_1Y_2$-arc, then $S_1 \to S_2$.*

As a direct consequence of Propositions 9 and 15, we have the following corollary.

**Corollary 16.** *Let $D$ be a 3-transitive digraph and $S_1$ a strong component of $D$ which contains a subdigraph isomorphic to $C_3$. If $S_1 \to v$ for some vertex $v \in V$, then $S_1 \to u$ for every vertex $u \in V$ that can be reached from $v$. Dually, if $v \to S_1$ for some vertex $v \in V$, then $u \to S_1$ for every vertex $u \in V$ that reaches $v$.*

We have already proved that the structure of 3-transitive digraphs is very similar to the structure of transitive digraphs. The following results are devoted to a deeper exploration of the similarities between these families of digraphs. A structural characterization of 3-transitive digraphs that are transitive is given.

**Theorem 17.** *Let $D$ be a non-strong 3-transitive digraph with strong components $S_1, S_2, \ldots, S_p$. Then $D = D^\star[S_1, S_2, \ldots, S_p]$ if and only if, for every pair of strong components $S_i, S_j$ of $D$, such that an $S_iS_j$-arc exists in $D$, then:*

1. *If $S_i, S_j$ are complete bipartite digraphs, then $D[S_i \cup S_j]$ is not bipartite.*
2. *If one of $S_i$ and $S_j$ is a complete bipartite digraph and the other consists of a single vertex, then $D[S_i \cup S_j]$ is not bipartite.*

***Proof.*** The necessity is trivial. In order to prove the sufficient, let $S_i$ and $S_j$ be two distinct strong components of $D$ such that there is an $S_iS_j$-arc. If both $S_i$ and $S_j$ are both non-trivial digraphs, then by 1 of the theorem and Proposition 14, we have that $S_i \to S_j$. Since the converse of a 3-transitive digraph is still a 3-transitive digraph, we assume, without loss of generality, that $S_i$ is a non-trivial complete bipartite digraph with bipartition $(X_i, Y_i)$ and $S_j = \{v\}$. Since $D[S_i \cup S_j]$ is not a bipartite digraph, then there is a vertex $x \in X_i$ such that $x \to v$ and there is a vertex $y \in Y_i$ such that $y \to v$. By Proposition 6.1, we have that $S_i \to v$. ■

**Theorem 18.** *Let $D$ be a 3-transitive digraph. Then $D^\star$ is a transitive digraph if and only if for every triplet of strong components $S_1, S_2, S_3$ of $D$, such that: $S_i$ consists of a single vertex $v_i$, $i \in \{1, 3\}$; $S_2$ is either a single vertex $v_2$ or a complete bipartite digraph with bipartition $(X, Y)$ and $v_1 \to v_2 \to v_3$ or $v_1 \to X \to v_3$ but there are neither $v_1Y$-arcs nor $Yv_3$-arcs in $D$, respectively, then $(v_1, v_3) \in A(D)$.*

***Proof.*** Let $D$ be a 3-transitive digraph. If $D^\star$ is a transitive digraph, then for every triplet of strong components $S_1, S_2$ and $S_3$ of $D$, such that there is an $S_1S_2$-arc in $D$ and an $S_2S_3$-arc in $D$, then there is an $S_1S_3$-arc in $D$. In particular, if $S_1$ and $S_3$ consist of single vertices $v_1$ and $v_3$ respectively, then $(v_1, v_3) \in A(D)$.

For the converse, let $D$ be a 3-transitive digraph and $S_1, S_2$ and $S_3$ strong components of $D$, such that there is an $S_1S_2$-arc in $D$ and an $S_2S_3$-arc in $D$. We will prove that there is an $S_1S_3$-arc in $D$. If $S_1$ contains an isomorphic copy of $C_3$, then, by Corollary 16, we have that $S_1 \to S_3$ in $D$. If $S_3$ contains an isomorphic copy of $C_3$, again, by Corollary 16, we have that $S_1 \to S_3$. So, let us assume that neither $S_1$ nor $S_3$ contains an isomorphic copy of $C_3$.

It follows from Proposition 12 that $S_1$ and $S_3$ are either a single vertex or complete bipartite digraphs. If $S_1$ is not a single vertex, then it is a complete bipartite digraph with bipartition $(X_1, Y_1)$. Let us assume without loss of generality that the $S_1S_2$-arc is an $X_1S_2$-arc. Let $(x_1, u)$ be the $S_1S_2$-arc in $D$. Since

$S_2$ is a strong component of $D$, we have, by Propositions 12 and 15, two cases. The first case is that a vertex $s_3 \in V(S_3)$ exists, such that $(u, s_3) \in A(D)$. In this case is clear that, for any vertex $y_1 \in Y_1$ (recall that $Y_1 \neq \emptyset$), $(y_1, x_1, u, s_3)$ is a directed path of length 3 in $D$. By the 3-transitivity of $D$, we have that $(y_1, s_3) \in A(D)$, the desired $S_1S_3$-arc. The second case is that vertices $v \in V(S_2)$ and $s_3 \in V(S_3)$ exist, such that $(u, v), (v, s_3) \in A(D)$. Again, it is clear that $(x_1, u, v, s_3)$ is a directed path of length 3 and thus, $(x_1, s_3) \in A(D)$, the desired $S_1S_3$-arc. The case when $S_3$ is a complete bipartite digraph can be obtained dualizing the previous argument using Remark 3.

So, the remaining cases are when $S_1$ and $S_3$ consist of single vertices. We have again two cases. First, when $S_2$ contains a subdigraph isomorphic to $C_3$, then $S_2 \to S_3$. So, there exist vertices $s_1 \in V(S_1), u, v \in V(S_2), s_3 \in V(S_3)$ such that $(s_1, u), (u, v), (v, s_3) \in A(D)$. Thus, $(s_1, u, v, s_3)$ is a directed path of length 3 in $D$. By the 3-transitivity of $D$, $(s_1, s_3) \in A(D)$ is the desired $S_1S_3$-arc. If $S_2$ does not contain a subdigraph isomorphic to $C_3$, then $S_2$ is a single vertex or complete bipartite. If $S_2$ is a single vertex $v_2$ or a complete bipartite digraph with bipartition $(X, Y)$ such that $v_1 \to v_2 \to v_3$ or $v_1 \to X \to v_3$ but there are neither $v_1Y$-arcs nor $Yv_3$-arcs in $D$, respectively, then, by hypothesis $(v_1, v_3) \in A(D)$. Hence, we have the existence of an $S_1S_3$-arc. The remaining case is that $S_2$ is a complete bipartite digraph with bipartition $(X, Y)$ such that $v_1 \to X \to v_3$, and either a $v_1Y$-arc or a $Yv_3$-arc exists. In the first case we have by Proposition 15 that $v_1 \to S_2$, and thus, vertices $u \in X$, $v \in Y$ exist such that $(v_1, v), (u, v_3) \in A(D)$. So, $(v_1, v, u, v_3)$ is a directed path of length 3 in $D$. For the second case, again by Proposition 15, it follows that $S_2 \to v_3$. Then, vertices $u \in X$ and $v \in Y$ exist such that $(v_1, u), (v, v_3) \in A(D)$. Therefore, $(v_1, u, v, v_3)$ is a directed path of length 3 in $D$. In either case, it follows by the 3-transitivity of $D$ that $(v_1, v_3) \in A(D)$. So an $S_1S_3$-arc exists.

Since the cases are exhaustive, we have that $D^\star$ is transitive. ■

**Corollary 19.** *Let $D$ be a 3-transitive digraph. Then $D$ is a transitive digraph if and only if every strong component of $D$ is a complete digraph and, for every triplet of strong components $S_1, S_2, S_3$ of $D$, such that: $S_i$ consists of a single vertex $v_i$, $i \in \{1, 3\}$; $S_2$ is either a single vertex $v_2$ or a symmetrical arc $(v_2, v_2') \in A(D)$ and $v_1 \to v_2 \to v_3$ but $(v_1, v_2'), (v_2', v_3) \notin A(D)$, then $(v_1, v_3) \in A(D)$.*

***Proof.*** It is clear from Theorems 1, 17 and 18. ■

**Corollary 20.** *Let $D$ be a 3-transitive digraph. If every strong component of $D$ is a complete digraph of order greater than or equal to 3, then $D$ is transitive.*

***Proof.*** Let $D$ be a 3-transitive digraph such that every strong component of $D$ is a complete digraph of order greater than or equal to 3. Then, by Theorem 18, it is clear that $D^\star$ is transitive. Also, in virtue of Theorem 15, we can observe

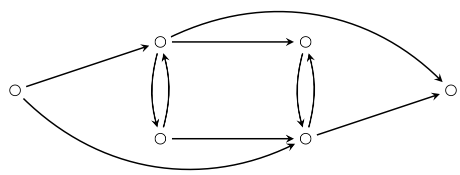

Figure 3. A 3-transitive digraph without 3-transitive condensation.

that $S_i \to S_j$ for every pair of strong components $S_i, S_j$ of $D$ such that there exists an $S_iS_j$-arc in $D$. Thus, $D = D^\star[S_1, S_2, \ldots, S_n]$, where $\{S_1, S_2, \ldots, S_n\}$ is the set of strong components of $D$ and $D^\star$ is transitive. So, by Theorem 1, $D$ is transitive. ■

As we have already shown, the structure of 3-transitive digraphs is very similar to the structure of transitive digraphs. We know that the condensation of a transitive digraph is again transitive. A characterization of 3-transitive digraphs with a transitive condensation has been already given, but a natural question arises. Is the condensation of a 3-transitive digraph 3-transitive again? Sadly, the answer is no, Figure 3 shows a counterexample to this fact.

Following similar ideas to those used to characterize the 3-transitive digraphs with a transitive condensation in Theorem 18, we can characterize 3-transitive digraphs with a 3-transitive condensation. The 'bad' configurations, preventing the condensation of a 3-transitive digraph to be 3-transitive, are pointed out in the following theorem.

**Theorem 21.** *Let $D$ be a 3-transitive digraph. Then $D^\star$ is a 3-transitive digraph if and only if for every 4-set, $\{S_1, S_2, S_3, S_4\}$, of strong components of $D$ such that: $S_i$ consists of a single vertex $v_i$, $i \in \{1, 4\}$ and one of the following conditions is fulfilled:*

1. *$S_2$ consists of single vertex $v_2$ and $S_3$ is a complete bipartite digraph with bipartition $(X, Y)$, such that $v_1 \to v_2 \to X$ and $Y \to v_4$, but there are neither $v_2Y$-arcs nor $Xv_4$-arcs in $D$;*
2. *$S_2$ is a complete bipartite digraph with bipartition $(X, Y)$ and $S_3$ consists of single vertex $v_3$, such that $v_1 \to X$ and $Y \to v_3 \to v_4$, but there are neither $v_1Y$-arcs nor $Xv_3$-arcs in $D$;*
3. *$S_j$ is a complete bipartite digraph with bipartition $(X_j, Y_j)$, $j \in \{2, 3\}$, such that $v_1 \to X_2 \to X_3$ and $Y_3 \to v_4$, but there are neither $v_1Y_2$-arcs, $v_1X_3$-arcs, $Y_2v_4$-arcs, nor $X_3v_4$-arcs, and $D[V(S_2) \cup V(S_3)]$ is a semicomplete bipartite digraph,*

*then $(v_1, v_4) \in A(D)$.*

***Proof.*** Let $D$ be a 3-transitive digraph. If $D^\star$ is a 3-transitive digraph, then for every 4-set of strong components $\{S_1, S_2, S_3, S_4\}$ of $D$, such that there is an $S_iS_{i+1}$-arc in $D$, $i \in \{1,2,3\}$, then there is an $S_1S_3$-arc in $D$. In particular, if $S_1$ and $S_4$ consist of single vertices $v_1$ and $v_4$ respectively, then $(v_1, v_4) \in A(D)$.

Conversely, let $\{S_1, S_2, S_3, S_4\}$ be a 4-set of strong components of $D$ such that there is an $S_iS_{i+1}$-arc in $D$, $i \in \{1,2,3\}$. If $S_1$ or $S_4$ are non-trivial, then by Proposition 9, there exists an $S_1S_4$-arc in $D$. So let us assume without loss of generality that $S_i$ consists of a single vertex $S_i$, $i \in \{1,4\}$. Suppose that $S_2$ or $S_3$ contains $C_3$ as a subdigraph. It can be easily derived from Corollary 16 the existence of an $S_1S_4$-arc in $D$. So, we have 3 cases.

Before the analysis of the cases, let us recall that, by Proposition 7, if $S = (X, Y)$ is a bipartite strong component of $D$ and $v \in V(D) \setminus V(S)$ such that a $vX$-arc exists, then $v \to X$; and if an $Xv$-arc exists, then $X \to v$.

The first case is when $S_2$ consists of single vertex $v_2$ and $S_3$ is a complete bipartite digraph with bipartition $(X, Y)$. Clearly, if a $v_2X$-arc, and an $Xv_4$-arc exist, then $v_2 \to X \to v_4$. Thus, a $v_1v_4$-directed path of length 3 exists and $(v_1, v_4) \in A(D)$ by the 3-transitivity of $D$. Analogously, if a $v_2Y$-arc and a $Yv_4$-arc exist in $D$, clearly $(v_1, v_4) \in A(D)$. So, we can assume without loss of generality that $v_2 \to X, Y \to v_4$ and there are neither $v_2Y$-arcs nor $Xv_4$-arcs in $D$. Then, by hypothesis, $(v_1, v_4) \in A(D)$.

The second case is when $S_2$ is a complete bipartite digraph with bipartition $(X, Y)$ and $S_3$ consists of single vertex $v_3$. But this case is just the dual of the first case, so, using Remark 3, it can be easily shown that $(v_1, v_4) \in A(D)$.

The third case is when $S_j$ is a complete bipartite digraph with bipartition $(X_j, Y_j)$, $j \in \{2,3\}$. Let us assume without loss of generality that $v_1 \to X_2$ and $Y_3 \to v_4$. If $X_2 \to Y_3$, then $v_1 \to X_2 \to Y_3 \to v_4$ and clearly $(v_1, v_4) \in A(D)$. If $Y_2 \to X_3$, it is easy to observe that $X_2 \to Y_3$. So, we can suppose that $X_2 \to X_3$ (thus $Y_2 \to Y_3$) and that there are neither $X_2Y_3$-arcs nor $Y_2X_3$-arcs. Thus, $D[V(S_2) \cup V(S_3)]$ is semicomplete bipartite. If $v_1 \to Y_2$, then $v_1 \to Y_2 \to Y_3 \to v_4$ and we are done. If $v_1 \to X_3$, then $v_1 \to X_3 \to Y_3 \to v_4$ and $(v_1, v_4) \in A(D)$. Symmetrically, if $Y_2 \to v_4$ or $X_3 \to v_4$ we can conclude that $(v_1, v_4) \in A(D)$. Hence, we can suppose that there are neither $v_1Y_2$-arcs, $v_1X_3$-arcs, $Y_2v_4$-arcs, nor $X_3v_4$-arcs in $D$. By hypothesis $(v_1, v_4) \in A(D)$.

Since the cases are exhaustive, we have that $D^\star$ is 3-transitive. ■

## 4. Consequences

### 4.1. Existence of kernels

Let $D$ be a digraph and $N \subseteq V(D)$. We say that $N$ is $l$-absorbent if for every vertex $u \in V(D) \setminus N$, there is a vertex $v \in N$ such that $d(u, v) \leq l$ in $D$. The set

$N$ is $k$-independent if for every $u, v \in N$, we have that $d(u,v), d(v,u) \geq N$. We call $N$ a $(k,l)$-kernel of $D$ if $D$ is $k$-independent and $l$-absorbent. A $(k, k-1)$-kernel is a $k$-kernel and a 2-kernel is simply a kernel. In [12], von Neumann and Morgenstern introduce the concept of kernel of a digraph in the context of Game Theory. Since then, kernels have been largely studied for their applications within many branches of Mathematics, we can find in [5] a very good survey on the subject. Also, in [6] is proved that the problem of determining if a given digraph has a kernel is $NP$-complete, so, finding sufficient conditions for a digraph to have a kernel or finding large families of digraphs with a kernel is a very valuable progress.

**Theorem 22.** *Let $D$ be a 3-transitive digraph. Then $D$ has a kernel if and only if it has no terminal strong component isomorphic to $C_3$.*

***Proof.*** The 'only if' part will be proved by contrapositive. Let $D$ be a 3- transitive digraph such that a terminal strong component $S$ is isomorphic to $C_3$. Let $V(S) = \{v_0, v_1, v_2\}$ and $A(S) = \{(v_i, v_{i+1})\}_{i=0}^{2}$ (mod 3). Since $S$ is terminal, we have that $d^+(v) = 1$ for every $v \in V(S)$. Thus, the only out- neighbor of $v_i$ is $v_{i+1}$ (mod 3). It is clear that $S$ has no kernel and vertices in $S$ cannot be absorbed by any other vertex in $D$, thus, $D$ has no kernel.

The 'if' implication will be proved by induction on the number of strong components of $D$. Let us assume that $D$ is strong. It can be directly verified that the digraphs mentioned in Proposition 12, except for $C_3$ have a kernel. So, let us assume that every 3-transitive digraph such that no terminal strong component isomorphic to $C_3$ and with $n$ strong components has a kernel. Let $D$ be a 3-transitive digraph such that no terminal strong component isomorphic to $C_3$ and with $n+1$ strong components. Let us recall that $D^\star$ is an acyclic digraph, so, we can consider an initial strong component $S$ of $D$. By induction hypothesis, $D - S$ has a kernel $N$. If $S$ is not a complete bipartite digraph, then, either $S$ consists of a single vertex or contains a subdigraph isomorphic to $C_3$. If $S$ consists of a single vertex $v$, and $v$ is absorbed by $N$, we are done. If $v$ is not absorbed by $N$, since $S$ is initial, $N \cup \{v\}$ is independent and thus a kernel of $D$. If $D$ contains a subdigraph isomorphic to $C_3$, we can use Corollary 16 to prove that $S \mapsto S_t$ for some terminal strong component $S_t$ of $D$. But since $S_t$ is terminal, at least one vertex of $S_t$ must belong to $N$, and thus $S$ is absorbed by $N$. So, $N$ is a kernel of $D$. If $S$ is a complete bipartite digraph, we must consider three cases. Let $(X, Y)$ be the bipartition of $S$. If neither $X$ nor $Y$ is absorbed by $N$, then we consider $N \cup X$. Since $S$ is an initial component, every arc between $X$ and $N$ must be an $XN$-arc. But if such arc exists, we would have by Proposition 7.5 that $X \to N$, contradicting our assumption. So $N \cup X$ is an independent set, and $Y \to X$ because $S$ is a complete bipartite digraph. Thus, $N \cup X$ is a kernel for $D$. If some vertex of $X$ is absorbed by $N$, then by Proposition 7.5 $X$ is absorbed by $N$. So let us assume that $Y$ is not absorbed by $N$. Once again, since $S$ is an initial component, every arc between $N$ and $Y$ must be a $YN$-arc, but no such arc can exist. So, $N \cup Y$ is an independent absorbent set of $D$, and hence a kernel of $D$. The case when $Y$ is absorbed but $X$ is not is analogous. Finally, if $S$ is absorbed by $N$, we have that $N$ is the desired kernel of $D$.

Since in every case $D$ has a kernel, the result follows from the Principle of Mathematical Induction. ■

In [10], Galeana-Sánchez and the author proved that a $k$-transitive digraph $D$ has a $n$-kernel for every $n \geq k$. Thus, Theorem 22 completes the study of existence of $k$-kernels in 3-transitive digraphs.

### 4.2. One further problem

A graph $G$ is a comparability graph if it can be oriented as an asymmetrical transitive digraph. In [11], Ghouila-Houri proved that the underlying graphs of asymmetrical quasi-transitive digraphs are comparability graphs. That is to say, a graph $G$ can receive a transitive orientation if and only if $G$ can receive a quasi-transitive orientation. In view of this result, and considering the great similarity between the structure of transitive and 3-transitive digraphs, we propose the following conjecture.

**Conjecture 23.** *Let $D$ be an asymmetrical 3-quasi-transitive digraph, then the underlying graph of $D$, $UG(D)$, admit a 3-transitive asymmetrical orientation.*


### Acknowledgements

The author wishes to express his deepest gratitude to an Anonymous Referee for her/his very careful examination of the present work. Also the author is very thankful for many accurate and pertinent suggestions (including the rewriting of some fragments of the text) that helped to improve substantially the elegance of the proofs and the overall quality of the present paper.


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Received 16 February 2011
Revised 02 April 2011
Accepted 04 April 2011