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THE LAPLACIAN SPECTRUM OF SOME DIGRAPHS OBTAINED FROM THE WHEEL

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Abstract

The problem of distinguishing, in terms of graph topology, digraphs with real and partially non-real Laplacian spectra is important for applications. Motivated by the question posed in [R. Agaev, P. Chebotarev, Which digraphs with rings structure are essentially cyclic?, Adv. in Appl. Math. 45 (2010), 232–251], in this paper we completely list the Laplacian eigenvalues of some digraphs obtained from the wheel digraph by deleting some arcs.

Keywords: digraph, Laplacian matrix, eigenvalue, wheel.

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1. Introduction

Since the Laplacian matrices of digraphs are not necessarily symmetric, as distinct from those of the undirected graphs, the Laplacian eigenvalues of digraphs need not be all real. The problem of characterizing all digraphs that have completely real Laplacian spectra is difficult and yet unsolved. In general, it is not an easy problem to distinguish, in terms of graph topology, digraphs with real Laplacian spectra from those having some non-real Laplacian eigenvalues. It is easy to see that the digraphs of the latter type are guaranteed to have cycles and are called

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essentially cyclic (see [1]), otherwise, the Laplacian of them will have a triangular form whose eigenvalues are the diagonal elements. The problem is important for some applications of graph theory, especially in decentralized control of multi-agent systems [6, 10].

R. Agaev and P. Chebotarev [1] have pointed out that a rational approach to attacking this difficult problem is studying various classes of cyclic digraphs. Digraph with ring structure means a digraph whose arc set only contains a collection of arcs forming a Hamiltonian cycle and an arbitrary number of arcs that belong to the inverse Hamiltonian cycle. In [1] a necessary and sufficient condition of essential cyclicity for the digraphs with ring structure was obtained.

Let $\vec{W}_n$ be the digraph with vertex set $\{1, 2, \ldots, n\}$ and arc set $\{(2, 3), (3, 4), \ldots, (n-1, n), (n, 2)\} \cup \{(i, 1), (1, i) : i = 2, 3, \ldots, n\}$, which will be called as the *wheel digraph* herein, as depicted in Figure 1 where the undirected edge means one pair of counter-directional arcs. In Section 2, we first consider three mutually related operations on digraphs and determine the Laplacian spectra and characteristic polynomials of digraphs derived from digraphs with known characteristic polynomials by these operations. By using these results, in Section 3 we completely list Laplacian eigenvalues of some digraphs as follows.

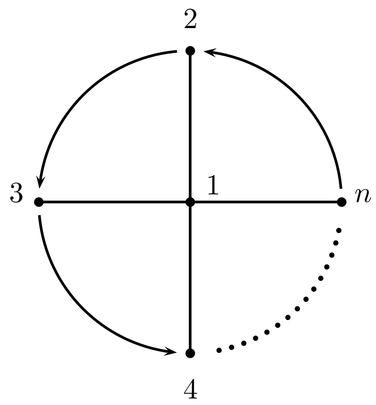

Figure 1. $\vec{W}_n$

- Let $\vec{W}_n'$ be the digraph obtained from $\vec{W}_n$ $(n > 3)$ by deleting l $(1 \le l \le n-1)$ arbitrary arcs in the cycle $\{(2, 3), (3, 4), \ldots, (n-1, n), (n, 2)\}$, then the Laplacian eigenvalues of $\vec{W}_n'$ are $0, n$, 1 (multiplicity $l-1$) and 2 (multiplicity $n-l-1$).
- Let $\vec{W}_n''$ be the digraph obtained from $\vec{W}_n$ $(n > 3)$ by deleting l $(1 \le l \le n-1)$ arbitrary arcs $\{(1, a_i) : i = 1, 2, \ldots, l\}$, where $a_i \in \{2, 3, \ldots, n\}$, then the Laplacian spectrum of $\vec{W}_n''$ is $\{0, n-l, 1 + 2\sin^2 \frac{k\pi}{n-1} + i \sin \frac{2k\pi}{n-1} : k = 1, 2, \ldots, n-2\}$.

2. NOTATIONS AND PRELIMINARIES

Let Γ denote a simple digraph with vertex set $V(\Gamma) = \{1, 2, \ldots, n\}$ and arc set $E(\Gamma)$. First we shall consider three mutually related operations on digraphs. The *complement* $\bar{\Gamma}$ of a digraph Γ is the digraph with the same vertex set, with an arc $(x, y) \in E(\bar{\Gamma})$ if and only if there exists no arc from x to y in Γ. The *direct sum* $\Gamma_1 \dot{+} \Gamma_2$ of digraphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ ($V_1 \cap V_2 = \emptyset$) is the digraph $\Gamma = (V, E)$ for which $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. The *complete product* $\Gamma_1 \bigtriangledown \Gamma_2$ of digraphs Γ_1 and Γ_2 is the digraph obtained from $\Gamma_1 \dot{+} \Gamma_2$ by joining every vertex of Γ_1 with every vertex of Γ_2 by one pair of counter-directional arcs. It is quite easy to see that the relation $\overline{\Gamma_1 \bigtriangledown \Gamma_2} = \bar{\Gamma}_1 \dot{+} \bar{\Gamma}_2$ holds.

The *adjacency matrix* $A = (a_{ij})$ of a digraph Γ is the $n \times n$ matrix whose rows and columns are indexed by the vertices, and where the ij-entry $a_{ij} = 1$ if $(i, j) \in E(\Gamma)$ and zero otherwise. The *out-degree matrix* D is the $n \times n$ matrix whose ii-entry is the sum of the entries of the i-th row of A and non-diagonal entries are zero. The matrix $L = D - A$ is sometimes referred to as the *Laplacian* (or *row Laplacian*) matrix of Γ (see [2, 4, 5]). If, in addition, L is symmetric, then L may be viewed as the Laplacian matrix of an undirected graph. Let J and I denote the matrix with all entries 1 and the identity matrix of appropriate order. Now it is easy to verify that the Laplacian of the complement $\bar{\Gamma}$ is $((n-1)I - D) - (J - I - A)$ whenever the Laplacian of Γ is $L = D - A$.

Many published works have been concerned with the operations on undirected graphs and the resulting Laplacian spectra. For the classical results on the subject, we refer to [3, 7] and an excellent survey [9] and the references wherein. Along the similar line, we investigate the operations on digraphs and the resulting Laplacian spectra and get the corresponding results on digraph. Let $\mu_\Gamma(\lambda) = |\lambda I - D + A|$ denote the characteristic polynomial of the Laplacian $L = D - A$. If $\Gamma = \Gamma_1 \dot{+} \Gamma_2$, then

$$\mu_\Gamma(\lambda) = \begin{vmatrix} \lambda I_1 - D_1 + A_1 & O \\ O & \lambda I_2 - D_2 + A_2 \end{vmatrix},$$

where $D_1 - A_1, D_2 - A_2$ are the Lapalcian matrices of digraphs Γ_1, Γ_2, respectively. Therefore,

$$\mu_{\Gamma_1 \dot{+} \Gamma_2}(\lambda) = \mu_{\Gamma_1}(\lambda)\mu_{\Gamma_2}(\lambda). \tag{1}$$

Next we shall present a relation (see (6)) between the Laplacian characteristic polynomial of the complement and that of a digraph. Although (6) can be obtained from Theorem 2 in [2], we keep it here for self-contained.

For the complement $\bar{\Gamma}$ of digraph Γ with n vertices, we have

$$\mu_{\bar{\Gamma}}(\lambda) = |\lambda I - ((n-1)I - D) + (J - I - A)| = |(\lambda - n)I + D - A + J|. \tag{2}$$

After adding all remaining columns to the first column of this determinant, every element of the first column becomes λ. Taking this factor out and then subtracting the first column from all the other columns, we get that

$$\mu_{\bar{\Gamma}}(\lambda) = \lambda|((\lambda - n)I + D - A)^*|, \tag{3}$$

where X^* denotes the matrix obtained from X by replacing all elements of the first column by 1's. Consider the determinant

$$|(\lambda - n)I + D - A| = (-1)^n \mu_\Gamma(n - \lambda). \tag{4}$$

If we add all the other columns to the first column, the entries of the first column all become $\lambda - n$. Therefore,

$$|(\lambda - n)I + D - A| = (\lambda - n)|((\lambda - n)I + D - A)^*|. \tag{5}$$

According to (3),(4) and (5) we have that

$$\mu_{\bar{\Gamma}}(\lambda) = (-1)^n \frac{\lambda}{\lambda - n}\mu_\Gamma(n - \lambda). \tag{6}$$

An analogous result for the undirected case may be traced back to Kelmans [8]. By (6) an expression of the Laplacian characteristic polynomial of the complete product of two digraphs can be obtained as follows. For the complete product $\Gamma = \Gamma_1 \bigtriangledown \Gamma_2$ of digraphs Γ_1 and Γ_2 with n_1 and n_2 $(n_1 + n_2 = n)$ vertices, respectively, by means of (1) and (6), we have

$$\begin{aligned} \mu_{\Gamma_1 \bigtriangledown \Gamma_2}(\lambda) &= \mu_{\overline{\bar{\Gamma_1} \dot{+} \bar{\Gamma_2}}}(\lambda) = (-1)^n \frac{\lambda}{\lambda - n}\mu_{\bar{\Gamma_1} \dot{+} \bar{\Gamma_2}}(n - \lambda) \\ &= (-1)^n \frac{\lambda}{\lambda - n}\mu_{\bar{\Gamma_1}}(n - \lambda)\mu_{\bar{\Gamma_2}}(n - \lambda) \\ &= \frac{\lambda(\lambda - n)}{(\lambda - n_1)(\lambda - n_2)}\mu_{\Gamma_1}(\lambda - n_2)\mu_{\Gamma_2}(\lambda - n_1). \end{aligned} \tag{7}$$

It is observed from (7) immediately that the Laplacian spectrum of the join of two digraphs with real Laplacian spectrum is real.

3. MAIN RESULTS

In this section, we investigate the Laplacian spectra of some digraphs obtained from the wheel $\vec{W}_n$ by deleting certain arcs. First we shall give the Laplacian spectra of the wheel $\vec{W}_n$ by (7).

For the path digraph $\vec{P}_n$, since it has no cycle, its Laplacian has the triangular form and so its eigenvalues are the diagonal elements. Then we have the following results immediately.

Lemma 3.1. *The Laplacian spectrum of* $\vec{P}_n$ *consists of* 0*, and* 1 *with multiplicity* $n-1$.

Lemma 3.2 [1]**.** *The Laplacian spectrum of* $\vec{C}_n$ *is* $\{2\sin^2\frac{k\pi}{n} + i\sin\frac{2k\pi}{n} : k = 1, 2, \ldots, n\}$.

Theorem 3.3. *The Laplacian spectrum of the wheel* $\vec{W}_n$ *is* $\{0, n, 1+ 2\sin^2\frac{k\pi}{n-1} + i\sin\frac{2k\pi}{n-1} : k = 1, 2, \ldots, n-2\}$.

Proof. By the definition of the wheel and (7), we have that

$$\mu_{\vec{W}_n}(\lambda) = \mu_{K_1 \triangledown \vec{C}_{n-1}}(\lambda) = \frac{\lambda(\lambda-n)}{(\lambda-1)(\lambda-n+1)}\mu_{K_1}(\lambda-n+1)\mu_{\vec{C}_{n-1}}(\lambda-1).$$

Since $\mu_{K_1}(\lambda) = \lambda$, we have $\mu_{\vec{W}_n}(\lambda) = \frac{\lambda(\lambda-n)}{(\lambda-1)}\mu_{\vec{C}_{n-1}}(\lambda-1)$. The result follows immediately by Lemma 3.2. ∎

Theorem 3.4. *Let* $\vec{W}_n'$ *be the digraph obtained from* $\vec{W}_n$ $(n > 3)$ *by deleting* l $(1 \le l \le n-1)$ *arbitrary arcs in the cycle* $\{(2,3), (3,4), \ldots, (n-1, n), (n, 2)\}$, *then the Laplacian eigenvalues of* $\vec{W}_n'$ *are* $0, n$, 1 *with multiplicity* $l-1$ *and* 2 *with multiplicity* $n-l-1$.

Proof. Note that when l arcs are deleted from the cycle $\{(2,3), (3,4), \ldots, (n-1, n), (n, 2)\}$ the resulting graph is made up of l disjoint paths, say, $\vec{P}_{k_1}, \vec{P}_{k_2}, \ldots, \vec{P}_{k_l}$, where $k_1, \ldots, k_l \ge 1$, $k_1 + k_2 + \cdots + k_l = n-1$. Then $\vec{W}_n' = K_1 \triangledown (\vec{P}_{k_1} \dot{+} \vec{P}_{k_2} \dot{+} \cdots \dot{+} \vec{P}_{k_l})$. By Lemma 3.1, the Laplacian eigenvalues of the digraph $\vec{P}_{k_1} \dot{+} \vec{P}_{k_2} \dot{+} \cdots \dot{+} \vec{P}_{k_l}$ are 0 with multiplicity l and 1 with multiplicity $n-l-1$. Then the result follows by (7). ∎

Now reconsider the Laplacian matrix $L_n \in \mathbb{R}^{n\times n}$ of the wheel $\vec{W}_n$:

$$L_n = \begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ -1 & 0 & 2 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ -1 & 0 & & 0 & 2 & -1 \\ -1 & -1 & 0 & \cdots & 0 & 2 \end{pmatrix}. \tag{8}$$

Let $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ denote the columns of the matrix obtained from the characteristic matrix $\lambda I - L_n$ by deleting its first row. Let $\mathbf{1}$ denote the column vector with all entries 1 of appropriate order. We see that $\alpha_0 = \mathbf{1}$. Expanding the determinant of $|\lambda I - L_n|$ along the first row, we have

$$\begin{aligned} \det(\lambda I - L_n) &= (\lambda - n + 1)\det[\alpha_1, \alpha_2, \ldots, \alpha_{n-1}] \\ &\quad + \sum_{j=1}^{n-1}(-1)^j \det[\mathbf{1}, \alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_{n-1}]. \end{aligned} \tag{9}$$

Consider the $(n-1)\times(n-1)$ minor $\det[\alpha_1,\alpha_2,\dots,\alpha_{n-1}]$. Adding all other columns to the j-th column in this determinant, for $1\le j\le n-1$, every element of the j-th column becomes $\lambda-1$. Taking this factor out and then by column-switching transformations, after $j-1$ steps by swapping column j and its previous column consecutively, we have

$$\begin{aligned}&\det[\alpha_1,\alpha_2,\dots,\alpha_{n-1}]\\ &=(-1)^{j-1}(\lambda-1)\det[\mathbf{1},\alpha_1,\dots,\alpha_{j-1},\alpha_{j+1},\dots,\alpha_{n-1}].\end{aligned}\tag{10}$$

Note that $\det[\mathbf{1},\alpha_1,\dots,\alpha_{j-1},\alpha_{j+1},\dots,\alpha_{n-1}]$ in the case $j=1$ is just $\det[\mathbf{1},\alpha_2,\alpha_3,\dots,\alpha_{n-1}]$.

Denote by $f(\lambda)$, $g(\lambda)$ the polynomials $\det[\alpha_1,\alpha_2,\dots,\alpha_{n-1}]$ and $\det[\mathbf{1},\alpha_2,\alpha_3,\dots,\alpha_{n-1}]$, respectively. Then by (10), we have for any $1\le j\le n-1$,

$$(-1)^j\det[\mathbf{1},\alpha_1,\dots,\alpha_{j-1},\alpha_{j+1},\dots,\alpha_{n-1}]=\frac{1}{1-\lambda}f(\lambda).\tag{11}$$

So by (9), we have

$$\begin{aligned}\det(\lambda I-L_n)&=(\lambda-n+1)f(\lambda)+\frac{n-1}{1-\lambda}f(\lambda)\\ &=(\lambda-n+1)(\lambda-1)g(\lambda)-(n-1)g(\lambda)\\ &=\lambda(\lambda-n)g(\lambda).\end{aligned}\tag{12}$$

Let $\vec{W}_n''$ be the digraph obtained from $\vec{W}_n$ $(n>3)$ by deleting l $(1\le l\le n-1)$ arbitrary spokes $\{(1,a_i): i=1,2,\dots,l\}$. Denote by L_n'' the Laplacian of $\vec{W}_n''$. Note that the characteristic matrices $\lambda I-L_n''$ and $\lambda I-L_n$ are different only in the first row. For $j=2,3,\dots,n$, by (11) we find that the cofactor of the $(1,j)$-th element of $\lambda I-L_n$, and hence also that of $\lambda I-L_n''$, is equal to $-g(\lambda)$. Expanding the determinant of $\lambda I-L_n''$ along the first row, we have

$$\begin{aligned}\det(\lambda I-L_n'')&=(\lambda-(n-1-l))(\lambda-1)g(\lambda)-(n-1-l)g(\lambda)\\ &=\lambda(\lambda-(n-l))g(\lambda).\end{aligned}$$

By Theorem 3.3 and (12), we know that the roots of $\det(\lambda I-L_n)=\lambda(\lambda-n)g(\lambda)$ are $\{0,n,1+2\sin^2\frac{k\pi}{n-1}+i\sin\frac{2k\pi}{n-1}: k=1,2,\dots,n-2\}$ and so the roots of the polynomial $g(\lambda)$ are $\{1+2\sin^2\frac{k\pi}{n-1}+i\sin\frac{2k\pi}{n-1}: k=1,2,\dots,n-2\}$. Therefore we arrive at the following result:

Theorem 3.5. *Let $\vec{W}_n''$ be the digraph obtained from $\vec{W}_n$ $(n>3)$ by deleting l $(1\le l\le n-1)$ arbitrary arcs $\{(1,a_i): i=1,2,\dots,l\}$, where $a_i\in\{2,3,\dots,n\}$, then the Laplacian spectrum of $\vec{W}_n''$ is $\{0,n-l,1+2\sin^2\frac{k\pi}{n-1}+i\sin\frac{2k\pi}{n-1}: k=1,2,\dots,n-2\}$.*

It is an interesting fact from above that the Laplacian spectrum of $\vec{W}_n''$ is the same as that of $\vec{W}_n$, except that the eigenvalue n of L_n now becomes $n-l$.

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