Discussiones Mathematicae
Graph Theory 32 (2012) 449–459
doi:10.7151/dmgt.1609

FRACTIONAL DISTANCE DOMINATION IN GRAPHS

S. Arumugam[1,2], Varughese Mathew[3] and K. Karuppasamy[1]

[1]*National Centre for Advanced Research in Discrete Mathematics (n-CARDMATH)*
Kalasalingam University, Anand Nagar, Krishnankoil-626 126, India
[2]*School of Electrical Engineering and Computer Science*
The University of Newcastle, NSW 2308, Australia
[3]*Department of Mathematics, Mar Thoma College, Tiruvalla-689 103, India*

e-mail: s.arumugam.klu@gmail.com
{k_karuppasamy,varughese_m1}@yahoo.co.in

Abstract

Let $G = (V, E)$ be a connected graph and let k be a positive integer with $k \leq rad(G)$. A subset $D \subseteq V$ is called a distance k-dominating set of G if for every $v \in V - D$, there exists a vertex $u \in D$ such that $d(u, v) \leq k$. In this paper we study the fractional version of distance k-domination and related parameters.

Keywords: domination, distance k-domination, distance k-dominating function, k-packing, fractional distance k-domination .

2010 Mathematics Subject Classification: 05C69, 05C72.

1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected and connected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For basic terminology in graphs we refer to Chartrand and Lesniak [3]. For basic terminology in domination related concepts we refer to Haynes *et al.* [9].

Let $G = (V, E)$ be a graph. A subset D of V is called a *dominating set* of G if every vertex in $V - D$ is adjacent to at least one vertex in D. A dominating set D is called a *minimal dominating set* if no proper subset of D is a dominating set of G. The minimum (maximum) cardinality of a minimal dominating set of G is called the *domination number* (*upper domination number*) of G and is denoted by $\gamma(G)$ $(\Gamma(G))$. Let A and B be two subsets of V. We say that B *dominates* A if every vertex in $A - B$ is adjacent to at least one vertex in B. If B dominates A, then we write $B \to A$. Meir and Moon [12] introduced the concept of a k-packing and distance k-domination in a graph as a natural generalisation of the concept of domination. Let $G = (V, E)$ be a graph and $v \in V$. For any positive integer k, let $N_k(v) = \{u \in V : d(u, v) \leq k\}$ and $N_k[v] = N_k(v) \cup \{v\}$. A set $S \subseteq V$ is a *distance k-dominating set* of G if $N_k[v] \cap S \neq \emptyset$ for every vertex $v \in V - S$. The minimum (maximum) cardinality among all minimal distance k-dominating sets of G is called the *distance k-domination number* (*upper distance k-domination number*) of G and is denoted by $\gamma_k(G)$ $(\Gamma_k(G))$. A set $S \subseteq V$ is said to be an *efficient distance k-dominating set* of G if $|N_k[v] \cap S| = 1$ for all $v \in V - S$. Clearly, $\gamma(G) = \gamma_1(G)$. A distance k-dominating set of cardinality $\gamma_k(G)$ $(\Gamma_k(G))$ is called a γ_k (Γ_k)-set. Hereafter, we shall use the term k-domination for distance k-domination.

Note that, $\gamma_k(G) = \gamma(G^k)$, where G^k is the k^{th} power of G, which is obtained from G by joining all pairs of distinct vertices u, v with $d(u, v) \leq k$. A subset $S \subseteq V(G)$ of a graph $G = (V, E)$ is said to be a *k-packing* ([12]) of G, if $d(u, v) > k$ for all pairs of distinct vertices u and v in S. The *k-packing number* $\rho_k(G)$ is defined to be the maximum cardinality of a k-packing set in G. The *corona* of a graph G, denoted by $G \circ K_1$, is the graph formed from a copy of G by attaching to each vertex v a new vertex v' and an edge $\{v, v'\}$. The *Cartesian product* of graphs G and H, denoted by $G \Box H$, is the graph with vertex set $V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \Box H$ if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G)$. For a survey of results on distance domination we refer to Chapter 12 of Haynes *et al.* [10].

Hedetniemi *et al.* [11] introduced the concept of fractional domination in graphs. Grinstead and Slater [6] and Domke *et al.* [5] have presented several results on fractional domination and related parameters in graphs. Arumugam *et al.* [1] have investigated the fractional version of global domination in graphs.

Let $G = (V, E)$ be a graph. Let $g : V \to \mathbb{R}$ be any function. For any subset S of V, let $g(S) = \sum_{v \in S} g(v)$. The *weight* of g is defined by $|g| = g(V) = \sum_{v \in V} g(v)$. For a subset S of V, the function $\chi_S : V \to \{0, 1\}$ defined by

$$\chi_S(v) = \begin{cases} 1 & \text{if } v \in S, \\ 0 & \text{if } v \notin S, \end{cases}$$

is called the *characteristic function* of S.

A function $g : V \to [0, 1]$ is called a *dominating function* (DF) of the graph $G = (V, E)$ if $g(N[v]) = \sum_{u \in N[v]} g(u) \geq 1$ for all $v \in V$. For functions f, g from $V \to [0, 1]$ we write $f \leq g$ if $f(v) \leq g(v)$ for all $v \in V$. Further, we write $f < g$ if $f \leq g$ and $f(v) < g(v)$ for some $v \in V$. A DF g of G is *minimal* (MDF) if f is not a DF for all functions $f : V \to [0, 1]$ with $f < g$.

The *fractional domination number* $\gamma_f(G)$ and the *upper fractional domination number* $\Gamma_f(G)$ are defined as follows:

$$\gamma_f(G) = \min\{|g| : g \text{ is a minimal dominating function of } G\},$$
$$\Gamma_f(G) = \max\{|g| : g \text{ is a minimal dominating function of } G\}.$$

For a dominating function f of G, the *boundary set* $\mathcal{B}_f$ and the *positive set* $\mathcal{P}_f$ are defined by $\mathcal{B}_f = \{u \in V(G) : f(N[u]) = 1\}$ and $\mathcal{P}_f = \{u \in V(G) : f(u) > 0\}$. A function $g : V \to [0,1]$ is called a *packing function* (PF) of the graph $G = (V, E)$ if $g(N[v]) = \sum_{u \in N[v]} g(u) \le 1$ for all $v \in V$. The *lower fractional packing number* $p_f(G)$ and the *fractional packing number* $P_f(G)$ are defined as follows:

$$p_f(G) = \min\{|g| : g \text{ is a maximal packing function of } G\},$$
$$P_f(G) = \max\{|g| : g \text{ is a maximal packing function of } G\}.$$

It was observed in Chapter 3 of [10] that for every graph G, $1 \le \gamma_f(G) = P_f(G) \le \gamma(G) \le \Gamma(G) \le \Gamma_f(G)$. We need the following theorems:

Theorem 1.1 [5]**.** *For a graph G, $p_f(G) \le \rho_2(G) \le P_f(G)$.*

Theorem 1.2 [2]**.** *A DF f of G is an MDF if and only if $\mathcal{B}_f \to \mathcal{P}_f$.*

Theorem 1.3 [2]**.** *If f and g are MDFs of G and $0 < \lambda < 1$ then $h_\lambda = \lambda f + (1-\lambda)g$ is an MDF of G if and only if $\mathcal{B}_f \cap \mathcal{B}_g \to \mathcal{P}_f \cup \mathcal{P}_g$.*

Theorem 1.4 [5]**.** *If G is an r-regular graph of order n, then $\gamma_f(G) = \frac{n}{r+1}$.*

Theorem 1.5 [4]**.** *Let G be a block graph. Then for any integer $k \ge 1$, we have $\rho_{2k}(G) = \gamma_k(G)$.*

For other families of graphs satisfying $\rho_2(G) = \gamma(G)$, we refer to Rubalcaba *et al.* [13].

Definition 1.6 [15]**.** A *linear Benzenoid chain $B(h)$* of length h is the graph obtained from $P_2 \square P_{h+1}$ by subdividing exactly once each edge of the two copies of P_{h+1}. Hence $B(h)$ is a subgraph of $P_2 \square P_{2h+1}$. The graph $B(4)$ is given in Figure 1.

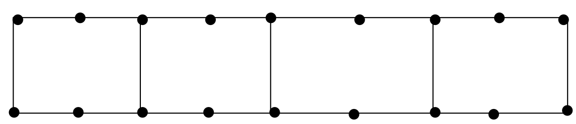

Figure 1. $B(4)$.

Theorem 1.7 [15]**.** *For the linear benzenoid chain $B(h)$, we have*

$$\gamma_k(B(h)) = \begin{cases} \lceil \frac{h+1}{k} \rceil & \text{if } k \ne 2, \\ \lceil \frac{h+2}{k} \rceil & \text{if } k = 2. \end{cases}$$

We refer to Scheinerman and Ullman [14] for fractionalization techniques of various graph parameters. Hattingh *et al.* [8] introduced the distance k-dominating function and proved that the problem of computing the upper distance fractional domination number is NP-complete. In this paper we present further results on fractional distance k-domination.

2. Distance k-dominating Function

Hattingh *et al.* [8] introduced the following concept of fractional distance k-domination.

Definition 2.1. A function $g : V \to [0,1]$ is called a *distance k-dominating function* or simply a *k-dominating function* (kDF) of a graph $G = (V,E)$, if for every $v \in V$, $g(N_k[v]) = \sum_{u \in N_k[v]} g(u) \geq 1$. A k-dominating function (kDF) g of a graph G is called a *minimal k-dominating function* ($MkDF$) if f is not a k-dominating function of G for all functions $f : V \to [0,1]$ with $f < g$. The *fractional k-domination number* $\gamma_{kf}(G)$ and the *upper fractional k-domination number* $\Gamma_{kf}(G)$ are defined as follows:

$$\gamma_{kf}(G) = \min\{|g| : g \text{ is an } MkDF \text{ of } G\},$$
$$\Gamma_{kf}(G) = \max\{|g| : g \text{ is an } MkDF \text{ of } G\}.$$

We observe that if $k \geq rad(G)$, then $\Delta(G^k) = n-1$ and $\gamma_{kf}(G) = 1$. Hence throughout this paper, we assume that $k < rad(G)$.

Lemma 2.2 [8]**.** *Let f be a k-dominating function of a graph $G = (V,E)$. Then f is minimal k-dominating if and only if whenever $f(v) > 0$ there exists some $u \in N_k[v]$ such that $f(N_k[u]) = 1$.*

Remark 2.3. The characteristic function of a γ_k-set and that of a Γ_k-set of a graph G are $MkDF$s of G. Hence it follows that $1 \leq \gamma_{kf}(G) \leq \gamma_k(G) \leq \Gamma_k(G) \leq \Gamma_{kf}(G)$.

Definition 2.4. A function $g : V \to [0,1]$ is called a *distance k-packing function* or simply a *k-packing function* of a graph $G = (V,E)$, if for every $v \in V$, $g(N_k[v]) \leq 1$. A k-packing function g of a graph G is *maximal* if f is not a k-packing function of G for all functions $f : V \to [0,1]$ with $f > g$. The *fractional k-packing number* $p_{kf}(G)$ and the *upper fractional k-packing number* $P_{kf}(G)$ are defined as follows:

$$p_{kf}(G) = \min\{|g| : g \text{ is a maximal k-packing function of } G\},$$
$$P_{kf}(G) = \max\{|g| : g \text{ is a maximal k-packing function of } G\}.$$

Observation 2.5. The fractional k-domination number $\gamma_{kf}(G)$ is the optimal solution of the following linear programming problem (LPP).

Minimize $z = \sum_{i=1}^{n} f(v_i)$, subject to

$$\sum_{u \in N_k[v]} f(u) \geq 1 \text{ and } 0 \leq f(v) \leq 1 \text{ for all } v \in V.$$

The dual of the above LPP is

Maximize $z = \sum_{i=1}^{n} f(v_i)$, subject to

$$\sum_{u \in N_k[v]} f(u) \leq 1 \text{ and } 0 \leq f(v) \leq 1 \text{ for all } v \in V.$$

The optimal solution of the dual LPP is the upper fractional k-packing number $P_{kf}(G)$. It follows from the strong duality theorem that $P_{kf}(G) = \gamma_{kf}(G)$. Hence if there exists a minimal k-dominating function g and a maximal k-packing function h with $|g| = |h|$, then $P_{kf}(G) = |h| = |g| = \gamma_{kf}(G)$.

Lemma 2.6. *For any graph G of order n we have $\gamma_{kf}(G) \leq \frac{n}{k+1}$ and the bound is sharp.*

Proof. Since $|N_k[u]| \geq k+1$ for all $u \in V$, it follows that the constant function f defined on V by $f(v) = \frac{1}{k+1}$ for all $v \in V$, is a k-dominating function with $|f| = \frac{n}{k+1}$. Hence $\gamma_{kf}(G) \leq \frac{n}{k+1}$. To prove the sharpness of this bound, consider the graph G consisting of a cycle of length $2k$ with a path of length k attached to each vertex of the cycle. Clearly $n = 2k(k+1)$. Further the set S of all pendant vertices of G forms an efficient k-dominating set of G and hence $\sum_{u \in N_k[v]} f(u) = 1$ for all $v \in V$ where f is the characteristic function of S. Hence $\gamma_k(G) = \gamma_{kf}(G) = 2k = \frac{n}{k+1}$. ■

Observation 2.7. We observe that $\gamma_{kf}(G) = \gamma_f(G^k)$. Hence the following is an immediate consequence of Theorem 1.2.

Let G be a graph and let $A, B \subseteq V$. We say that A, *k-dominates* B if $N_k[v] \cap A \neq \emptyset$ for all $v \in B$ and we write $A \to_k B$. Now for any kDF f of G let $\mathcal{P}_f = \{u \in V(G) : f(u) > 0\}$ and $\mathcal{B}_f = \{u \in V(G) : f(N_k[u]) = 1\}$. Then f is an $MkDF$ of G if and only if $\mathcal{B}_f \to_k \mathcal{P}_f$.

Observation 2.8. If f and g are $kDFs$ of a graph $G = (V, E)$ and $\lambda \in (0,1)$, then the convex combination of f and g defined by $h_\lambda(v) = \lambda f(v) + (1-\lambda) g(v)$ for all $v \in V$ is a kDF of G. However, the convex combination of two $MkDFs$ of a graph G need not be minimal, as shown in the following example.

Consider the cycle $G = C_7 = (u_1 u_2 \ldots u_7 u_1)$ with $k = 2$. The function $f : V(G) \to [0,1]$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \{u_1, u_5\}, \\ 0 & \text{otherwise}, \end{cases}$$

is a minimal 2-dominating function of G with $\mathcal{P}_f = \{u_1, u_5\}$, $\mathcal{B}_f = \{u_1, u_2, u_4, u_5\}$. Also, the function $g : V(G) \to [0,1]$ defined by

$$g(x) = \begin{cases} 1 & \text{if } x \in \{u_3, u_6\}, \\ 0 & \text{otherwise}, \end{cases}$$

is a minimal 2-dominating function of G with $\mathcal{P}_g = \{u_3, u_6\}$, $\mathcal{B}_g = \{u_2, u_3, u_6, u_7\}$. Let $h = \frac{1}{2}f + \frac{1}{2}g$. Then $h(u_1) = h(u_3) = h(u_5) = h(u_6) = \frac{1}{2}$, $h(u_2) = h(u_4) = h(u_7) = 0$, $h(N_2[u_i]) = \frac{3}{2}$ for $i \neq 2$ and $h(N_2[u_2]) = 1$. Hence $\mathcal{P}_h = \{u_1, u_3, u_5, u_6\}$ and $\mathcal{B}_h = \{u_2\}$. Since $u_5, u_6 \notin N_2[u_2]$ we have $\mathcal{B}_h$ does not 2-dominate $\mathcal{P}_h$ and hence the kDF h is not minimal.

Observation 2.9. If f and g are $MkDFs$ of G and $0 < \lambda < 1$, then $h_\lambda = \lambda f + (1-\lambda)g$ is an $MkDF$ of G if and only if $\mathcal{B}_f \cap \mathcal{B}_g \to_k \mathcal{P}_f \cup \mathcal{P}_g$.

Observation 2.10. For the cycle C_n, the graph $G = C_n^k$ is $2k$-regular and hence it follows from Theorem 1.4 that $\gamma_{kf}(C_n) = \frac{n}{2k+1}$.

We now proceed to determine the fractional k-domination number of several families of graphs.

Proposition 2.11. *For the hypercube Q_n, $\gamma_{kf}(Q_n) = \frac{2^n}{\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{k}}$.*

Proof. For any two vertices $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in Q_n, $d(x, y) \leq k$ if and only if x and y differ in at most k coordinates and hence Q_n^k is r-regular where $r = \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k}$. Hence by Theorem 1.4, we have $\gamma_{kf}(Q_n) = \frac{2^n}{r+1} = \frac{2^n}{\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{k}}$. ■

Proposition 2.12. *For the graph $G = P_2 \square C_n$, we have*

$$\gamma_{kf}(G) = \begin{cases} \frac{8}{7} & \text{if } n = 4 \text{ and } k = 2, \\ \frac{n}{2k} & \text{if } n \geq 5. \end{cases}$$

Proof. If $n = 4$ and $k = 2$, then G^2 is a 6-regular graph and hence $\gamma_{2f}(G) = \frac{8}{7}$. If $n \geq 5$, G^k is a $(4k-1)$-regular graph and hence $\gamma_{kf}(G) = \frac{2n}{4k-1+1} = \frac{n}{2k}$. ■

Theorem 2.13. *Let $G = C_n \circ K_1$. Then $\gamma_{kf}(G) = \frac{n}{2k-1}$.*

Proof. Let $C_n = (v_1 v_2 \ldots v_n v_1)$. Let u_i be the pendant vertex adjacent to v_i. Clearly, $|N_k[u_i] \cap V(C_n)| = 2k-1$ and $N_k[u_i] \subset N_k[v_i]$, $1 \leq i \leq n$. Hence the function $g : V(G) \to [0, 1]$ defined by

$$g(x) = \begin{cases} 0 & \text{if } x = u_i, \\ \frac{1}{2k-1} & \text{if } x = v_i \end{cases}$$

is a minimal k-dominating function of G with $|g| = \frac{n}{2k-1}$. Also we have $|N_k[v_i] \cap \{u_j : 1 \leq j \leq n\}| = 2k-1$, $1 \leq i \leq n$. Hence the function $h : V(G) \to [0, 1]$ defined by

$$h(x) = \begin{cases} \frac{1}{2k-1} & \text{if } x = u_i, \\ 0 & \text{if } x = v_i \end{cases}$$

is a maximal k-packing function of G with $|h| = \frac{n}{2k-1}$. Hence by Observation 2.5, we have $\gamma_{kf}(G) = \frac{n}{2k-1}$. ■

Theorem 2.14. *For the grid $G = P_2 \Box P_n$, we have*

$$\gamma_{kf}(G) = \begin{cases} \frac{n(n+2k)}{2k(n+k)} & \text{if } n \equiv 0 \pmod{2k}, \\ \lceil \frac{n}{2k} \rceil & \text{otherwise.} \end{cases}$$

Proof. Let $P_2 = (u_0, u_1)$ and $P_n = (v_0, v_1, \ldots, v_{n-1})$, so that $V(G) = \{(u_i, v_j) : i = 0, 1,\ 0 \leq j \leq n-1\}$.

Case 1. $n \equiv 0 \pmod{2k}$. Let $n = 2kp$, $p > 1$. Define $f : V(G) \to [0,1]$ by

$$f((u_i, v_j)) = \begin{cases} (\frac{1}{2p+1})(p - \lfloor \frac{j}{2k} \rfloor) & \text{if } j \equiv (k-1) \pmod{2k}, \\ (\frac{1}{2p+1})(\lfloor \frac{j}{2k} \rfloor + 1) & \text{if } j \equiv k \pmod{2k}, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is a k-dominating function of G. Also, since $f((u_0, v_j)) = f((u_1, v_j))$ for all j, we have $|f| = 2(\sum_{j=0}^{n-1} f((u_0, v_j))) = \frac{2}{2p+1}[(p + (p-1) + \cdots + 3 + 2 + 1) + (1 + 2 + 3 + \cdots + p)] = \frac{2p(p+1)}{2p+1} = \frac{n(n+2k)}{2k(n+2)}$. Now consider the function $h : V(G) \to [0,1]$ defined by

$$h((u_i, v_j)) = \begin{cases} (\frac{1}{2p+1})(p - \lfloor \frac{j}{2k} \rfloor) & \text{if } j \equiv 0 \pmod{2k}, \\ (\frac{1}{2p+1})(\lfloor \frac{j}{2k} \rfloor + 1) & \text{if } j \equiv (2k-1) \pmod{2k}, \\ 0 & \text{otherwise.} \end{cases}$$

Then h is a k-packing function of G with $|h| = \frac{2p(p+1)}{2p+1} = \frac{n(n+2k)}{2k(n+2k)}$. Hence $\gamma_{kf}(G) = \frac{n(n+2k)}{2k(n+k)}$.

Case 2. $n \not\equiv 0 \pmod{2k}$. Let $n = 2kq + r$, $1 \leq r \leq 2k-1$. Let $S = S_1 \cup S_2$ and

$$S_1 = \begin{cases} \{(u_0, v_j) : j \equiv 0 \pmod{4k}\} & \text{if } 1 \leq r \leq k, \\ \{(u_0, v_j) : j \equiv (k-1) \pmod{4k}\} & \text{if } k+1 \leq r \leq 2k-1. \end{cases}$$

$$S_2 = \begin{cases} \{(u_1, v_j) : j \equiv 2k \pmod{4k}\} & \text{if } 1 \leq r \leq k, \\ \{(u_1, v_j) : j \equiv (3k-1) \pmod{4k}\} & \text{if } k+1 \leq r \leq 2k-1. \end{cases}$$

Let f be the characteristic function of S. Since $d(x, y) \geq 2k+1$ for all $x, y \in S$, it follows that $f(N_k[u]) = 1$ for all $u \in V(G)$. Thus f is both a minimal k-dominating function and a maximal k-packing function of G and hence $\gamma_{kf}(G) = |f| = |S| = \lceil \frac{n}{2k} \rceil$. ■

A special case of the above theorem gives the following result of Hare [7].

Corollary 2.15. *For the grid graph $G = P_2 \Box P_n$, we have*

$$\gamma_f(G) = \begin{cases} \frac{n(n+2)}{2(n+1)} & \text{if } n \text{ is even,} \\ \lceil \frac{n}{2} \rceil & \text{if } n \text{ is odd.} \end{cases}$$

3. Graphs with $\gamma_{kf}(G) = \gamma_k(G)$

In this section we obtain several families of graphs for which the fractional k-domination number and the k-domination number are equal.

Lemma 3.1. *If a graph G has an efficient k-dominating set, then $\gamma_{kf}(G) = \gamma_k(G)$.*

Proof. Let D be an efficient k-dominating set of G. Then $|N_k[u] \cap D| = 1$ for all $u \in V(G)$. Hence the characteristic function of D is both a minimal k-dominating function and a maximal k-packing function of G and so $\gamma_{kf}(G) = \gamma_k(G)$. ■

Lemma 3.2. *For any graph G, $\gamma_{kf}(G) = 1$ if and only if $\gamma_k(G) = 1$.*

Proof. Suppose $\gamma_k(G) = 1$. Since $\gamma_{kf}(G) \le \gamma_k(G)$, it follows that $\gamma_{kf}(G) = 1$. Conversely, let $\gamma_{kf}(G) = 1$. Then $\gamma_f(G^k) = 1$ and hence $\gamma(G^k) = 1$. Since $\gamma(G^k) = \gamma_k(G)$ the result follows. ■

Lemma 3.3. *For any graph G, $p_{kf}(G) \le \rho_{2k}(G) \le P_{kf}(G)$.*

Proof. Let $u \in V(G)$. Since $N_k[u] = N_{G^k}[u]$, we have $p_{kf}(G) = p_f(G^k)$, $P_{kf}(G) = P_f(G^k)$ and $\rho_{2k}(G) = \rho_2(G^k)$.

Hence the result follows from Theorem 1.1. ■

Corollary 3.4. *For any graph G, $1 \le p_{kf}(G) \le \rho_{2k}(G) \le P_{kf}(G) = \gamma_{kf}(G) \le \gamma_k(G) \le \Gamma_k(G) \le \Gamma_{kf}(G)$.*

Corollary 3.5. *If G is any graph with $\rho_{2k}(G) = \gamma_k(G)$, then $\gamma_{kf}(G) = \gamma_k(G)$.*

Corollary 3.6. *If G is a block graph, then $\gamma_{kf}(G) = \gamma_k(G)$.*

Proof. It follows from Theorem 1.5 that $\rho_{2k}(G) = \gamma_k(G)$ and hence the result follows. ■

Corollary 3.7. *For any tree T, we have $\gamma_{kf}(T) = \gamma_k(T)$.*

Theorem 3.8. *For the graph $G = P_{k+1} \square P_n$ where $n \equiv 1 \pmod{(k+1)}$, $k \ge 1$, we have $\gamma_{kf}(G) = \gamma_k(G) = \lceil \frac{n}{k+1} \rceil$.*

Proof. Let $n = (k+1)q + 1$, $q \ge 1$. Clearly $|V(G)| = n(k+1) = (k+1)^2 q + (k+1)$. Let $P_{k+1} = (u_0, u_1, u_2, \ldots, u_k)$ and $P_n = (v_0, v_1, \ldots, v_{n-1})$ so that $V(G) = \{(u_i, v_j) : 0 \le i \le k,\ 0 \le j \le n-1\}$.

Now let $S_1 = \{(u_0, v_i) : i \equiv 0 \pmod{2(k+1)}\}$, $S_2 = \{(u_k, v_i) : i \equiv (k+1) \pmod{2(k+1)}\}$ and $S = S_1 \cup S_2$. Clearly, $d(x, y) = (2k+1)r$, $r \ge 1$, for all $x, y \in S$ and $|S| = \lceil \frac{n}{k+1} \rceil = q + 1$. Also, (u_0, v_0) and exactly one of the vertices (u_0, v_{n-1}) or (u_k, v_{n-1}) are in S and each of these two vertices k-dominates $\frac{(k+1)(k+2)}{2}$ vertices of G. Also, if $u \in N_k[x] \cap N_k[y]$, where $x, y \in S$, then $d(u,x) \leq k$, $d(u,y) \leq k$ and so $d(x,y) \leq d(x,u) + d(u,y) \leq 2k$, which is a contradiction. Thus $N_k[x] \cap N_k[y] = \emptyset$ for all $x, y \in S$. Each of the remaining vertices of S k-dominates $(k+1)^2$ vertices of G. Further, $|V(G)| - (k+1)(k+2)$ is a multiple of $(k+1)^2$ and hence it follows that S is an efficient k-dominating set of G. Hence, by Lemma 3.1, we have $\gamma_{kf}(G) = \gamma_k(G) = |S| = \lceil \frac{n}{k+1} \rceil$. ■

Theorem 3.9. *For the graph $G = P_3 \Box P_n$, we have $\gamma_{2f}(G) = \gamma_2(G) = \lceil \frac{n}{3} \rceil$.*

Proof. If $n \equiv 1 \pmod 3$, then the result follows from Theorem 3.8. Suppose $n \equiv 0 \pmod 3$ or $2 \pmod 3$. Let $n = 3q$, $q \geq 1$ or $n = 3q+2$, $q \geq 0$. Let $P_3 = (u_0, u_1, u_2)$ and $P_n = (v_0, v_1, \ldots, v_{n-1})$ so that $V(G) = \{(u_i, v_j) : 0 \leq i \leq 2,\ 0 \leq j \leq n-1\}$. Now $D = \{(u_1, v_j) : j \equiv 1 \pmod 3\}$ is a γ_2-set of G with $|D| = \lceil \frac{n}{3} \rceil$ and hence $\gamma_2(G) = \lceil \frac{n}{3} \rceil$. Further $f = \chi_D$ is a 2-dominating function of G with $|f| = \lceil \frac{n}{3} \rceil$. Also let $S_1 = \{(u_0, v_j) : j \equiv 0 \pmod 6\}$, $S_2 = \{(u_2, v_j) : j \equiv 3 \pmod 6\}$ and $S = S_1 \cup S_2$. Then $g = \chi_S$ is a 2-packing function of G with $|g| = \lceil \frac{n}{3} \rceil$. Hence $\gamma_{2f}(G) = \lceil \frac{n}{3} \rceil$. ■

Observation 3.10. The graph $G = P_3 \Box P_5$ does not have an efficient 2-dominating set. In fact the set $S = \{(u_0, v_0), (u_2, v_3)\}$ efficiently 2-dominates 14 vertices of G and the vertex (u_0, v_4) is not 2-dominated by S. Further if S is any 2-dominating set of G with $|S| = \gamma_2(G) = 2$, then at least one vertex of G is 2-dominated by both vertices of S. This shows that the converse of Lemma 3.1 is not true.

Theorem 3.11. *For the linear benzenoid chain $G = B(h)$, we have*

$$\gamma_{kf}(G) = \gamma_k(G) = \begin{cases} \frac{h}{2} + 1 & \text{if } k = 2 \text{ and } h \equiv 0 \pmod 2, \\ \lceil \frac{h}{k} \rceil & \text{if } k \geq 3 \text{ and } h \equiv \lfloor \frac{k}{2} \rfloor \pmod k. \end{cases}$$

Proof. Since $G = B(h)$ is a subgraph of $P_2 \Box P_{2h+1}$, we take $V(G) = \{(u_i, v_j) : i = 0, 1,\ 0 \leq j \leq 2h\}$, where $P_2 = (u_0, u_1)$ and $P_{2h+1} = (v_0, v_1, \ldots, v_{2h})$. Clearly, $|V(G)| = 4h+2$. Any vertex $u \in V(G)$ k-dominates at most $4k$ vertices of G and hence $\gamma_k(G) \geq \lceil \frac{4h+2}{4k} \rceil$.

Case 1. $k = 2$ and $h \equiv 0 \pmod 2$. In this case we have $\gamma_2(G) \geq \lceil \frac{4h+2}{8} \rceil = \frac{h}{2} + 1$. Now let $S_1 = \{(u_0, v_j) : j \equiv 0 \pmod 8\}$, $S_2 = \{(u_1, v_j) : j \equiv 4 \pmod 8\}$ and $S = S_1 \cup S_2$. Clearly, for any $x, y \in S$, $d(x,y) \geq 5$ and hence $N_2[x] \cap N_2[y] = \emptyset$. Also $|S| = \lceil \frac{2h+1}{4} \rceil = \frac{h}{2} + 1$. Now (u_0, v_0) and exactly one of the vertices (u_0, v_{2h}) or (u_1, v_{2h}) is in S and each of these two vertices 2-dominates exactly 5 vertices of G. Each of the remaining vertices of S 2-dominates 8 vertices of G. Further $|V(G)| - 10 = 4h - 8 = 8(\frac{h}{2} - 1)$, which is a multiple of 8 and hence it follows that S is an efficient 2-dominating set of G. Hence $\gamma_{2f}(G) = \gamma_2(G) = |S| = \frac{h}{2} + 1$.

Case 2. $k \geq 3$ and $h \equiv \lfloor \frac{k}{2} \rfloor \pmod{k}$. Let $h = kq + \lfloor \frac{k}{2} \rfloor$, $q \geq 1$. In this case we have $\gamma_k(G) \geq \lceil \frac{4h+2}{4k} \rceil = \lceil \frac{h}{k} \rceil$. Now let $S_1 = \{(u_0, v_j) : j \equiv (k-1) \pmod{4k}\}$, $S_2 = \{(u_1, v_j) : j \equiv (3k-1) \pmod{4k}\}$ and $S = S_1 \cup S_2$. Clearly, $d(x, y) = (2k + 1)r$, $r \geq 1$ for all $x, y \in S$, hence $N_k[x] \cap N_k[y] = \emptyset$. Also $|S| = \lceil \frac{2h-(k-1)}{2k} \rceil = \lceil \frac{h}{k} \rceil$.

Now, when k is odd, exactly one of the vertices (u_0, v_{2h}) or (u_1, v_{2h}) is in S and it k-dominates $2k+1$ vertices. When k is even, exactly one of the vertices (u_0, v_{2h-1}) or (u_1, v_{2h-1}) are in S and it k-dominates $2k+3$ vertices. The vertex (u_0, v_{k-1}) k-dominates $4k-1$ vertices. In both cases the number of vertices of G which are not k-dominated by these two vertices is a multiple of $4k$ and each of the remaining vertices of S k-dominates $4k$ vertices of G. Hence it follows that S is an efficient k-dominating set of G so that $\gamma_{kf}(G) = \gamma_k(G) = |S| = \lceil \frac{h}{k} \rceil$. ■

Conclusion. In this paper we have determined the fractional k-domination number of several families of graphs. We have also obtained several families of graphs for which $\gamma_{kf}(G) = \gamma_k(G)$. The study of the fractional version of distance k-irredundance and distance k-independence remains open. Slater has mentioned several efficiency parameters such as redundance and influence in Chapter 1 of [10]. One can investigate these parameters for fractional distance domination. The following are some interesting problems for further investigation.

1. Characterize the class of graphs G for which $\gamma_{kf}(G) = \frac{n}{k+1}$.
2. Characterize the class of graphs G with $\gamma_{kf}(G) = \gamma_k(G)$.
3. Determine $\gamma_{kf}(P_r \square P_s)$ for $r, s \geq 4$.

Acknowledgement

We are thankful to the National Board for Higher Mathematics, Mumbai, for its support through the project 48/5/2008/R&D-II/561, awarded to the first author. The second author is thankful to the UGC, New Delhi for the award of FIP teacher fellowship during the XI^{th} plan period. We are also thankful to the referees for their helpful suggestions.

References

[1] S. Arumugam, K. Karuppasamy and I. Sahul Hamid, *Fractional global domination in graphs*, Discuss. Math. Graph Theory **30** (2010) 33–44. doi:10.7151/dmgt.1474

[2] E.J. Cockayne, G. Fricke, S.T. Hedetniemi and C.M. Mynhardt, *Properties of minimal dominating functions of graphs*, Ars Combin. **41** (1995) 107–115.

[3] G. Chartrand and L. Lesniak, Graphs & Digraphs, Fourth Edition, Chapman & Hall/CRC (2005).

[4] G.S. Domke, S.T. Hedetniemi and R.C. Laskar, *Generalized packings and coverings of graphs*, Congr. Numer. **62** (1988) 259–270.

[5] G.S. Domke, S.T. Hedetniemi and R.C. Laskar, *Fractional packings, coverings, and irredundance in graphs*, Congr. Numer. **66** (1988) 227–238.

[6] D.L. Grinstead and P.J. Slater, *Fractional domination and fractional packings in graphs*, Congr. Numer. **71** (1990) 153–172.

[7] E.O. Hare, *k-weight domination and fractional domination of $P_m \times P_n$*, Congr. Numer. **78** (1990) 71–80.

[8] J.H. Hattingh, M.A. Henning and J.L. Walters, *On the computational complexity of upper distance fractional domination*, Australas. J. Combin. **7** (1993) 133–144.

[9] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, New York, 1998).

[10] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Domination in graphs: Advanced Topics (Marcel Dekker, New York, 1998).

[11] S.M. Hedetniemi, S.T. Hedetniemi and T.V. Wimer, Linear time resource allocation algorithms for trees, Technical report URI -014, Department of Mathematics, Clemson University (1987).

[12] A. Meir and J.W. Moon, *Relations between packing and covering numbers of a tree*, Pacific J. Math. **61** (1975) 225–233.

[13] R.R. Rubalcaba, A. Schneider and P.J. Slater, *A survey on graphs which have equal domination and closed neighborhood packing numbers*, AKCE J. Graphs. Combin. **3** (2006) 93–114.

[14] E.R. Scheinerman and D.H. Ullman, Fractional Graph Theory: A Rational Approach to the Theory of Graphs (John Wiley & Sons, New York, 1997).

[15] D. Vukičević and A. Klobučar, *k-dominating sets on linear benzenoids and on the infinite hexagonal grid*, Croatica Chemica Acta **80** (2007) 187–191.

Received 22 December 2010
Revised 12 August 2011
Accepted 16 August 2011