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A CHARACTERIZATION OF COMPLETE TRIPARTITE DEGREE-MAGIC GRAPHS[1]

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Abstract

A graph is called degree-magic if it admits a labelling of the edges by integers $1, 2, \ldots, |E(G)|$ such that the sum of the labels of the edges incident with any vertex v is equal to $\frac{1+|E(G)|}{2}\deg(v)$. Degree-magic graphs extend supermagic regular graphs. In this paper we characterize complete tripartite degree-magic graphs.

Keywords: supermagic graphs, degree-magic graphs, complete tripartite graphs.

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1. Introduction

We consider finite undirected graphs without loops, multiple edges and isolated vertices. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and the edge set of G, respectively. Cardinalities of these sets are called the *order* and *size* of G.

Let a graph G and a mapping f from $E(G)$ into positive integers be given. The *index mapping* of f is the mapping f^* from $V(G)$ into positive integers defined by

$$f^*(v) = \sum_{e \in E(G)} \eta(v,e) f(e) \quad \text{for every } v \in V(G),$$

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where $\eta(v, e)$ is equal to 1 when e is an edge incident with a vertex v, and 0 otherwise. An injective mapping f from $E(G)$ into positive integers is called a *magic labelling* of G for an *index* λ if its index mapping f^* satisfies

$$f^*(v) = \lambda \quad \text{for all } v \in V(G).$$

A magic labelling f of a graph G is called a *supermagic labelling* if the set $\{f(e) : e \in E(G)\}$ consists of consecutive positive integers. We say that a graph G is *supermagic* (*magic*) whenever there exists a supermagic (magic) labelling of G.

A bijection f from $E(G)$ into $\{1, 2, \dots, |E(G)|\}$ is called a *degree-magic labelling* (or only d-*magic labelling*) of a graph G if its index mapping f^* satisfies

$$f^*(v) = \frac{1 + |E(G)|}{2} \deg(v) \quad \text{for all } v \in V(G).$$

A d-magic labelling f of a graph G is called *balanced* if for all $v \in V(G)$ it holds

$$|\{e \in E(G) : \eta(v, e) = 1, f(e) \leq \lfloor |E(G)|/2 \rfloor\}|$$
$$= |\{e \in E(G) : \eta(v, e) = 1, f(e) > \lfloor |E(G)|/2 \rfloor\}|.$$

We say that a graph G is *degree-magic* (*balanced degree-magic*) (or only d-*magic*) when there exists a d-magic (balanced d-magic) labelling of G.

The concept of magic graphs was introduced by Sedláček [7]. Supermagic graphs were introduced by M.B. Stewart [8]. There is by now a considerable number of papers published on magic and supermagic graphs; we refer the reader to [4] for comprehensive references. The concept of degree-magic graphs was introduced in [1] as some extension of supermagic regular graphs. Basic properties of degree-magic graphs were also established in [1]. Let us recall those, which we shall use hereinafter.

Theorem 1. *Let G be a regular graph. Then G is supermagic if and only if it is degree-magic.*

Theorem 2. *Let G be a* d-*magic graph of even size. Then every vertex of G has an even degree and every component of G has an even size.*

Theorem 3. *Let H_1 and H_2 be edge-disjoint subgraphs of a graph G which form its decomposition. If H_1 is* d-*magic and H_2 is balanced* d-*magic then G is a* d-*magic graph. Moreover, if H_1 and H_2 are both balanced* d-*magic then G is a balanced* d-*magic graph.*

A *complete k-partite graph* is a graph whose vertices can be partitioned into $k \geq 2$ disjoint classes $V_1, \dots, V_k$ such that two vertices are adjacent whenever they belong to distinct classes. If $|V_i| = n_i$, $i = 1, \dots, k$, then the complete k-partite graph is denoted by $K_{n_1,\dots,n_k}$.

Stewart [9] characterized supermagic complete graphs. Supermagic regular complete multipartite graphs were characterized in [6]. Thus, according to Theorem

1, degree-magic regular complete multipartite graphs are characterized as well. All balanced d-magic complete multipartite graphs are characterized in [2]. In particular for the complete bipartite graphs we have

Theorem 4 [1]**.** *The complete bipartite graph $K_{m,n}$ is balanced* d*-magic if and only if the following statements hold:*

(i) $m \equiv n \equiv 0 \pmod 2$,

(ii) *if* $m \equiv n \equiv 2 \pmod 4$, *then* $\min\{m, n\} \geq 6$.

The complete bipartite graph $K_{m,n}$ is d-magic if and only if there exists a magic (m, n)-rectangle (see [1] for details). Thus, the known result on magic rectangles (e.g., Theorem 1 in [5] or Theorem 2 in [3]) can be rewritten as follows.

Theorem 5. *The complete bipartite graph $K_{m,n}$, for $m \geq n$, is* d*-magic if and only if the following statements hold:*

(i) $m \equiv n \pmod 2$,

(ii) *if* $n = 2$ *then* $m > 2$,

(iii) *if* $n = 1$ *then* $m = 1$.

The problem of characterizing d-magic complete multipartite graphs seems to be difficult. It is solved in this paper for complete tripartite graphs.

2. Complete Tripartite Graphs

First we present some sufficient conditions for complete tripartite graphs to possess the d-magic property.

Lemma 1. *Let m, n and o be even positive integers. Then the complete tripartite graph $K_{m,n,o}$ is balanced* d*-magic.*

Proof. Suppose that $m \geq n \geq o$ and consider the following cases.

Case A. Let $o > 2$, or $n > o = 2$ and $m + n \equiv 0 \pmod 4$. Evidently, the graph $K_{m,n,o}$ is decomposable into edge-disjoint subgraphs isomorphic to $K_{m,n}$ and $K_{m+n,o}$. According to Theorem 4, both of these subgraphs are balanced d-magic. Thus, by Theorem 3, $K_{m,n,o}$ is balanced d-magic, too.

Case B. Let $n > o = 2$ and $m + n \not\equiv 0 \pmod 4$. In this case we have either $m \equiv 0 \pmod 4$, or $n \equiv 0 \pmod 4$. Without loss of generality, assume that $m \equiv 0 \pmod 4$. The graph $K_{m,n,o}$ is decomposable into subgraphs isomorphic to $K_{m,o}$ and $K_{n,m+o}$. By Theorem 4, both of these subgraphs are balanced d-magic. Therefore, $K_{m,n,o}$ is balanced d-magic because of Theorem 3.

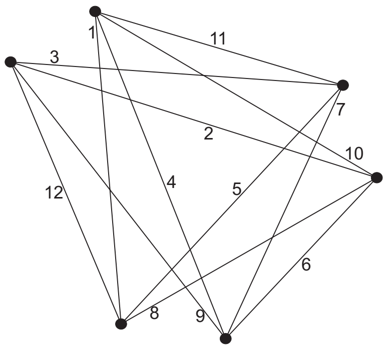

Figure 1. Balanced d-magic labelling of $K_{2,2,2}$.

Case C. Let $n = o = 2$. A balanced d-magic labelling of $K_{2,2,2}$ is given in Figure 1. Thus, $K_{2,2,2}$ is balanced d-magic. If $m > 2$, then the graph $K_{m,n,o}$ is decomposable into edge-disjoint subgraphs isomorphic to $K_{2,n,o}$ and $K_{m-2,n+o}$. As $K_{2,2,2}$ and $K_{m-2,4}$ are balanced d-magic, $K_{m,n,o}$ is balanced d-magic by Theorem 3. ■

Lemma 2. *Let $m \geq n \geq o$ be odd positive integers such that $m \equiv 3 \pmod 4$ whenever $n = 1$. Then the complete tripartite graph $K_{m,n,o}$ is* d-*magic.*

Proof. Let us assume to the contrary that $K_{m,n,o}$ (where $m \geq n \geq o$ are odd positive integers such that $m \equiv 3 \pmod 4$ whenever $n = 1$) is a complete tripartite graph with a minimum number of vertices which is not d-magic. Consider the following cases.

Case A. $n = 1$. Then $o = 1$ and $m \equiv 3 \pmod 4$ in this case. If $m > 3$ then $K_{m,n,o}$ is decomposable into edge-disjoint subgraphs isomorphic to $K_{m-4,n,o}$ and $K_{4,n+o}$. By the minimality of $K_{m,n,o}$, the graph $K_{m-4,n,o}$ is d-magic and according to Theorem 4, $K_{4,2}$ is balanced d-magic. Thus, by Theorem 3, $K_{m,n,o}$ is d-magic, contrary to the choice of $K_{m,n,o}$. Therefore, $m = 3$. However, $K_{3,1,1}$ admits a d-magic labelling (see Figure 2) and so it is d-magic, a contradiction.

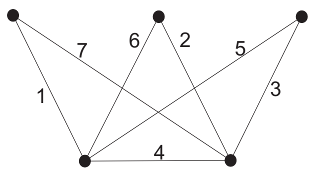

Figure 2. Degree-magic labelling of $K_{3,1,1}$

Case B. $o = 1$ and $n = 3$. As $m \geq n$, the graph $K_{m,n,o}$ is decomposable into subgraphs isomorphic to $K_{m-2,n,o}$ and $K_{2,n+o}$. By the minimality of $K_{m,n,o}$, the graph $K_{m-2,n,o}$ is d-magic and according to Theorem 4, $K_{2,4}$ is balanced d-magic. Thus, by Theorem 3, $K_{m,n,o}$ is d-magic, a contradiction.

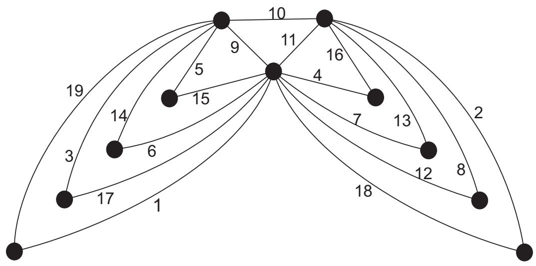

Figure 3. Degree-magic labelling of G_1

Case C. $o = 1$ and $n > 3$. If $m > 5$ then $K_{m,n,o}$ is decomposable into edge-disjoint subgraphs isomorphic to $K_{m-4,n,o}$ and $K_{4,n+o}$. By the minimality of $K_{m,n,o}$, the graph $K_{m-4,n,o}$ is d-magic and by Theorem 4, $K_{4,n+o}$ is balanced d-magic. According to Theorem 3, $K_{m,n,o}$ is d-magic, a contradiction. Therefore, $m = n = 5$. The graph $K_{5,5,1}$ is decomposable into edge-disjoint subgraphs isomorphic to $K_{4,4}$ and G_1 which is depicted in Figure 3. The graph $K_{4,4}$ is balanced d-magic by Theorem 4 and G_1 is d-magic (see Figure 3). Thus, using Theorem 3, $K_{5,5,1}$ is d-magic, a contradiction.

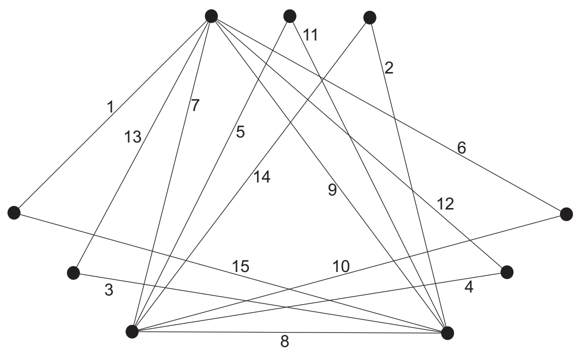

Figure 4. Degree-magic labelling of G_2

Case D. $o > 1$. If $m > 3$ then $K_{m,n,o}$ is decomposable into subgraphs isomorphic to $K_{m-4,n,o}$ and $K_{4,n+o}$. By the minimality of $K_{m,n,o}$, the graph $K_{m-4,n,o}$ is d-magic and by Theorem 4, $K_{4,n+o}$ is balanced d-magic. According to Theorem 3, $K_{m,n,o}$ is d-magic, a contradiction. Therefore, $m = n = o = 3$. The graph $K_{3,3,3}$ is decomposable into subgraphs isomorphic to $K_{2,2,2}$ and G_2 which is depicted in Figure 4. The graph $K_{2,2,2}$ is balanced d-magic by Lemma 1 and G_2 is d-magic (see Figure 4). Thus by Theorem 3, $K_{3,3,3}$ is d-magic, a contradiction.

■

Lemma 3. *Let $n \geq o$ be odd positive integers and let m be an even positive integer such that $m \equiv 0 \pmod 4$ whenever $n = 1$. Then the complete tripartite graph $K_{m,n,o}$ is* d*-magic.*

Proof. Let us assume to the contrary that $K_{m,n,o}$ (where $n \geq o$ are odd positive integers and m is an even positive integer such that $m \equiv 0 \pmod 4$ whenever $n = 1$) is a complete tripartite graph with a minimum number of vertices which is not d-magic. Consider the following cases.

Case A. $m > 4$. The graph $K_{m,n,o}$ is decomposable into edge-disjoint subgraphs isomorphic to $K_{m-4,n,o}$ and $K_{4,n+o}$. By the minimality of $K_{m,n,o}$, the graph $K_{m-4,n,o}$ is d-magic and by Theorem 4, $K_{4,n+o}$ is balanced d-magic. According to Theorem 3, $K_{m,n,o}$ is d-magic, contrary to the choice of $K_{m,n,o}$.

Case B. $m = 4$. The graph $K_{m,n,o}$ is decomposable into subgraphs isomorphic to $K_{m,n+o}$ and $K_{n,o}$. Thus, if $n = 1$ or $o > 1$, then by Theorems 4, 5 and 3, $K_{m,n,o}$ is d-magic, a contradiction. Therefore, $o = 1$ and $n > 1$. $K_{m,n,o}$ can be decomposed into subgraphs isomorphic to $K_{m-2,n,o}$ and $K_{2,n+o}$. If $n \equiv 3 \pmod 4$, then, according to the minimality of $K_{m,n,o}$ and Theorems 4, 3, the graph $K_{m,n,o}$ is d-magic, a contradiction. So, $1 < n \equiv 1 \pmod 4$, i.e., there is a positive integer k such that $n = 4k + 1$. Denote the vertices of $K_{4,n,1}$ by $u_1, \ldots, u_4, v_1, \ldots, v_n, w$ in such a way that $\{u_1, \ldots, u_4\}$, $\{v_1, \ldots, v_n\}$ and $\{w\}$ are its maximal independent sets. Consider the mapping $f : E(K_{4,n,1}) \to \{1, 2, \ldots, 5n + 4\}$ given by

$$f(u_1v_j) = \begin{cases} 1 + 2k - \frac{j+1}{2} & \text{if } \ j < n, \ \ j \equiv 1 \pmod 2, \\ 10 + 20k - \frac{j}{2} & \text{if } \ j \equiv 0 \pmod 2, \\ 1 + 3k & \text{if } \ j = n, \end{cases}$$

$$f(u_2v_j) = \begin{cases} 8 + 16k - \frac{j+1}{2} & \text{if } \ j < n, \ \ j \equiv 1 \pmod 2, \\ 2 + 4k + \frac{j}{2} & \text{if } \ j \equiv 0 \pmod 2, \\ 7 + 13k & \text{if } \ j = n, \end{cases}$$

$$f(u_3v_j) = \begin{cases} 8 + 16k & \text{if } \ j = 1, \\ 10 + 18k - \frac{j-1}{2} & \text{if } \ 1 < j \leq 1 + 2k, \ \ j \equiv 1 \pmod 2, \\ 9 + 18k - \frac{j-1}{2} & \text{if } \ j > 1 + 2k, \ \ j \equiv 1 \pmod 2, \\ 2 + 4k - \frac{j}{2} & \text{if } \ j \leq 2k, \ \ j \equiv 0 \pmod 2, \\ 1 + 4k - \frac{j}{2} & \text{if } \ j > 2k, \ \ j \equiv 0 \pmod 2, \end{cases}$$

$$f(u_4v_j) = \begin{cases} 4 + 8k & \text{if } \ j = 1, \\ 2 + 6k + \frac{j-1}{2} & \text{if } \ 1 < j \leq 1 + 2k, \ \ j \equiv 1 \pmod 2, \\ 3 + 6k + \frac{j-1}{2} & \text{if } \ j > 1 + 2k, \ \ j \equiv 1 \pmod 2, \\ 8 + 14k - \frac{j}{2} & \text{if } \ j \leq 2k, \ \ j \equiv 0 \pmod 2, \\ 7 + 14k - \frac{j}{2} & \text{if } \ j > 2k, \ \ j \equiv 0 \pmod 2, \end{cases}$$

$$f(wv_j) = \begin{cases} 5+8k+j & \text{if } \ j<n, \ \ j \equiv 1 \pmod 2, \\ 3+8k+j & \text{if } \ j \le 2k, \ \ j \equiv 0 \pmod 2, \\ 5+8k+j & \text{if } \ j > 2k, \ \ j \equiv 0 \pmod 2, \\ 5+10k & \text{if } \ j = n, \end{cases}$$

$$f(wu_i) = \begin{cases} 9+17k & \text{if } \ i=1, \\ 3+7k & \text{if } \ i=2, \\ 2+4k & \text{if } \ i=3, \\ 6+12k & \text{if } \ i=4. \end{cases}$$

It is not difficult to check that f is a bijection, $f^*(u_i) = (5+10k)(1+n)$ for all $i = 1, \dots, 4$, $f^*(v_j) = 5(5+10k)$ for all $j = 1, \dots, n$ and $f^*(w) = (5+10k)(4+n)$. Thus, $K_{4,n,1}$ is d-magic, a contradiction.

Case C. $m = 2$ and $o > 1$. In this case $K_{n,o}$ is d-magic by Theorem 5. If $n + o \equiv 0 \pmod 4$, then $K_{2,n+o}$ is balanced d-magic by Theorem 4. The graph $K_{2,n,o}$ is decomposable into edge-disjoint subgraphs isomorphic to $K_{2,n+o}$ and $K_{n,o}$ and so, using Theorem 3, it is d-magic, a contradiction. Therefore, $n+o \equiv 2 \pmod 4$. As $K_{n,o}$ is d-magic, there is its d-magic labelling $g : E(K_{n,o}) \to \{1, 2, \dots, \varepsilon\}$, where $\varepsilon = no$ is its number of edges. Suppose that e', e^* are edges of $K_{n,o}$ such that $g(e') = 1$ and $g(e^*) = \varepsilon$. Consider the following subcases.

Subcase C1. If e' and e^* are adjacent edges (note that $n = o = 3$ belongs to this subcase), then denote the vertices of $K_{2,n,o}$ by $u_1, u_2, v_1, v_2, \dots, v_{n+o}$ in such a way that $\{u_1, u_2\}$ is its maximal independent set, the subgraph $K_{n,o}$ is induced by $\{v_1, \dots, v_{n+o}\}$ and $e' = v_1v_3$, $e^* = v_2v_3$. The graph $K_{2,n,o}$ is decomposable into edge-disjoint subgraphs G_3 (induced by $\{u_iv_j : i \in \{1,2\}, j \in \{7, \dots, n+o\}\}$, if $n+o > 6$) and G_4 (induced by remaining edges). Evidently, if $n+o > 6$ then G_3 is isomorphic to $K_{2,n+o-6}$, and by Theorem 4, it is balanced d-magic. Consider the mapping $h_1 : E(G_4) \to \{1, 2, \dots, \varepsilon + 12\}$ given by

$$h_1(e) = \begin{cases} 6 + g(e) & \text{if } \ e \in E(K_{n,o}) - \{e', e^*\}, \\ 6 & \text{if } \ e = e', \\ 7 + \varepsilon & \text{if } \ e = e^*, \end{cases}$$

and the values of edges u_iv_j are described in the following matrix

$h_1(u_iv_j)$	v_1	v_2	v_3	v_4	v_5	v_6
u_1	$\varepsilon+9$	$\varepsilon+8$	7	1	$\varepsilon+11$	3
u_2	5	4	$\varepsilon+6$	$\varepsilon+12$	2	$\varepsilon+10$

It is easy to see that h_1 is a bijection. Since $\deg_{G_4}(v_j) = \deg_{K_{n,o}}(v_j)$, for each $j \in \{7, \dots, n+o\}$, we have

$$\begin{aligned} h_1^*(v_j) &= g^*(v_j) + 6\deg_{G_4}(v_j) = \tfrac{1+\varepsilon}{2}\deg_{G_4}(v_j) + 6\deg_{G_4}(v_j) \\ &= \tfrac{13+\varepsilon}{2}\deg_{G_4}(v_j). \end{aligned}$$

For $3 \le j \le 6$, $\deg_{G_4}(v_j) = 2 + \deg_{K_{n,o}}(v_j)$ and so

$$\begin{aligned} h_1^*(v_j) &= g^*(v_j) + 6\deg_{K_{n,o}}(v_j) + \varepsilon + 13 \quad = \tfrac{13+\varepsilon}{2}\deg_{K_{n,o}}(v_j) + \varepsilon + 13 \\ &= \tfrac{13+\varepsilon}{2}\deg_{G_4}(v_j). \end{aligned}$$

Similarly

$$\begin{aligned} h_1^*(v_1) &= g^*(v_1) - 1 + 6\deg_{K_{n,o}}(v_1) + \varepsilon + 14 = \tfrac{13+\varepsilon}{2}\deg_{G_4}(v_1), \\ h_1^*(v_2) &= g^*(v_2) + 1 + 6\deg_{K_{n,o}}(v_2) + \varepsilon + 12 = \tfrac{13+\varepsilon}{2}\deg_{G_4}(v_2) \end{aligned}$$

and for $i \in \{1, 2\}$

$$h_1^*(u_i) = 3\varepsilon + 39 = \tfrac{13+\varepsilon}{2}\deg_{G_4}(u_i).$$

Therefore, G_4 is a d-magic graph and by Theorem 3, the graph $K_{2,n,o}$ is also d-magic, a contradiction.

Subcase C2. If e' and e^* are not adjacent edges ($n + o \geq 10$ in this subcase), then denote the vertices of $K_{2,n,o}$ by $u_1, u_2, v_1, v_2, \ldots, v_{n+o}$ in such a way that $\{u_1, u_2\}$ is its maximal independent set, the subgraph $K_{n,o}$ is induced by $\{v_1, \ldots, v_{n+o}\}$ and $e' = v_1v_2$, $e^* = v_3v_4$. The graph $K_{2,n,o}$ is decomposable into edge-disjoint subgraphs G_5 (induced by $\{u_iv_j : i \in \{1,2\}, j \in \{11, \ldots, n+o\}\}$, if $n + o > 10$) and G_6 (induced by remaining edges). Evidently, if $n + o > 10$ then G_5 is isomorphic to $K_{2,n+o-10}$, and by Theorem 4, it is balanced d-magic. Consider the mapping $h_2 : E(G_6) \to \{1, 2, \ldots, \varepsilon + 20\}$ given by

$$h_2(e) = \begin{cases} 10 + g(e) & \text{if } \ e \in E(K_{n,o}) - \{e', e^*\}, \\ 10 & \text{if } \ e = e', \\ 11 + \varepsilon & \text{if } \ e = e^*, \end{cases}$$

and the values of edges u_iv_j are described in the following matrix

$h_1(u_iv_j)$	u_1	u_2
v_1	$\varepsilon + 19$	3
v_2	5	$\varepsilon + 17$
v_3	$\varepsilon + 18$	2
v_4	4	$\varepsilon + 16$
v_5	1	$\varepsilon + 20$
v_6	$\varepsilon + 15$	6
v_7	7	$\varepsilon + 14$
v_8	$\varepsilon + 13$	8
v_9	$\varepsilon + 12$	9
v_{10}	11	$\varepsilon + 10$

Analogously as in the *Case* C1 it is easy to verify that h_2 is a d-magic labelling. Thus, G_6 is a d-magic graph and consequently, the graph $K_{2,n,o}$ is d-magic, a contradiction.

Case D. $m = 2$ and $o = 1$. In this case there is a positive integer k such that $n = 2k + 1$. Denote the vertices of $K_{2,n,1}$ by u_0, u_1, u_2, v_{-k}, $\ldots$, v_k in such a way that $\{u_1, u_2\}$, $\{v_{-k}, \ldots, v_k\}$ and $\{u_0\}$ are its maximal independent sets.

Put $r = \lceil \frac{2k}{3} \rceil$ (note that $3r - 2k \in \{0, 1, 2\}$) and define

$$R = \begin{cases} \{0,1\} & \text{if } \ k = 1, \\ \{0,k\} & \text{if } \ k \text{ is even}, \\ \{0,r\} & \text{if } \ k > 1 \text{ is odd and } 3r - 2k \neq 1, \\ \{0,r,k\} & \text{if } \ k > 1 \text{ is odd and } 3r - 2k = 1. \end{cases}$$

Let P and Q be disjoint subsets of the set $\{0, 1, \ldots, k\} - R$ such that

$$P \cup Q \cup R = \{0, 1, \ldots, k\} \ \text{ and } \ 0 \leq |P| - |Q| \leq 1.$$

Consider the mapping $\xi : E(K_{2,n,1}) \to \{1, 2, \ldots, 6k + 5\}$ given by

$$\xi(u_0u_1) = 6k + 5, \qquad \xi(u_0u_2) = 1,$$

$$\xi(u_jv_i) = \begin{cases} 3k + 3 + i & \text{if } \ j = 0,\ i \in P \cup Q, \\ i + 2 & \text{if } \ j = 1,\ i \in P \text{ or } j = 2,\ i \in Q, \\ 6k + 4 - 2i & \text{if } \ j = 2,\ i \in P \text{ or } j = 1,\ i \in Q, \end{cases}$$

$$\xi(u_jv_{-i}) = \begin{cases} 3k + 3 - i & \text{if } \ j = 0,\ i \in P \cup Q, \\ 2k + 3 - i & \text{if } \ j = 1,\ i \in P \text{ or } j = 2,\ i \in Q, \\ 4k + 3 + 2i & \text{if } \ j = 2,\ i \in P \text{ or } j = 1,\ i \in Q, \end{cases}$$

and the values of edges u_jv_i, $|i| \in R$, are described in the following matrices:

$$\begin{array}{cccc} \xi(u_jv_i) & v_0 & v_1 & v_{-1} \\ u_0 & 10 & 3 & 5 \\ u_1 & 2 & 7 & 4 \\ u_2 & 6 & 8 & 9 \end{array} \quad \text{for } k = 1, \qquad \begin{array}{cccc} \xi(u_jv_i) & v_0 & v_k & v_{-k} \\ u_0 & 6k+4 & k+2 & 2k+3 \\ u_1 & 2 & 4k+3 & 6k+3 \\ u_2 & 3k+3 & 4k+4 & k+3 \end{array} \quad \text{for even } k,$$

$$\begin{array}{cccc} \xi(u_jv_i) & v_0 & v_r & v_{-r} \\ u_0 & 3k+3 & 3k+3+r & 3k+3-r \\ u_1 & 2 & r+2 & 4k+3+2r \\ u_2 & 6k+4 & 6k+4-2r & 2k+3-r \end{array} \quad \text{for } \ 3r - 2k = 0,$$

$$\begin{array}{cccc} \xi(u_jv_i) & v_0 & v_r & v_{-r} \\ u_0 & 3k+3 & 3k+3+r & 3k+3-r \\ u_1 & 2 & 6k+4-2r & 2k+3-r \\ u_2 & 6k+4 & r+2 & 4k+3+2r \end{array} \quad \text{for } 3r - 2k = 2,$$

$$\begin{array}{cccccc} \xi(u_jv_i) & v_0 & v_r & v_{-r} & v_k & v_{-k} \\ u_0 & 6k+4 & r+2 & 3k+3-r & 4k+3 & 2k+3 \\ u_1 & 2 & 3k+3+r & 4k+3+2r & k+2 & 6k+3 \\ u_2 & 3k+3 & 6k+4-2r & 2k+3-r & 4k+4 & k+3 \end{array} \quad \text{for } 3r - 2k = 1.$$

As $\bigcup_{j=0}^{2}\{\xi(u_jv_i)\} = \{i + 2, 3k + 3 + i, 6k + 4 - 2i\}$, for $0 \leq i \leq k$, and $\bigcup_{j=0}^{2}\{\xi(u_jv_{-i})\} = \{2k + 3 - i, 3k + 3 - i, 4k + 3 + 2i\}$, for $1 \leq i \leq k$, it is not difficult to check that ξ is a bijection and $\xi^*(v_t) = 9k+9$ for each $t \in \{-k, \dots, k\}$. Moreover,

$$\xi(u_j v_i) + \xi(u_j v_{-i}) = \begin{cases} 6k+6 & \text{if } j = 0,\ i \in P \cup Q, \\ 2k+5 & \text{if } j = 1,\ i \in P \text{ or } j = 2,\ i \in Q, \\ 10k+7 & \text{if } j = 2,\ i \in P \text{ or } j = 1,\ i \in Q. \end{cases}$$

Therefore, $\xi(u_j v_i) + \xi(u_j v_{-i}) + \xi(u_j v_t) + \xi(u_j v_{-t}) = 12k + 12$, for $i \in P$, $t \in Q$, $j \in \{0, 1, 2\}$. Now, it is easy to verify that $\xi^*(u_0) = (3k + 3)(2k + 3)$ and $\xi^*(u_1) = \xi^*(u_2) = (3k+3)(2k+2)$. Thus, ξ is a d-magic labelling, a contradiction. ∎

Now we are able to prove the main result of the paper.

Proposition. *Let $m \geq n \geq o$ be positive integers. The complete tripartite graph $K_{m,n,o}$ is* d*-magic if and only if both of the following statements hold:*
(i) *if* $n = 1$, *then* $m \equiv 0 \pmod 4$ *or* $m \equiv 3 \pmod 4$,
(ii) *if* $m + n + o \equiv 1 \pmod 2$, *then* $m \equiv n \equiv o \equiv 1 \pmod 2$.

Proof. Denote the vertices of $K_{m,1,1}$ by $u_1, \dots, u_m, v, w$ in such a way that $\{u_1, \dots, u_m\}$, $\{v\}$ and $\{w\}$ are its maximal independent sets. The size of $K_{m,1,1}$ denote by q. Evidently, $q = 2m + 1$. Suppose that f is a d-magic labelling of $K_{m,1,1}$. Then,

$$(1 + q)(1 + m) = f^*(v) + f^*(w) = (1 + 2 + \cdots + q) + f(vw),$$

and consequently, $f(vw) = \frac{1+q}{2} = 1 + m$. Put $A := \{i : f(vu_i) \leq m\}$ and $B := \{i : f(wu_i) \leq m\}$. Clearly, $A \cap B = \emptyset$ and $A \cup B = \{1, 2, \dots, m\}$, because $f(v, u_i) + f(w, u_i) = f^*(u_i) = 1 + q$ for each $i \in \{1, \dots, m\}$. Thus,

$$\sum_{i \in A} f(vu_i) + \sum_{i \in B} f(vu_i) = f^*(v) - f(vw) = \frac{1+q}{2}(1 + m) - \frac{1+q}{2} = (1 + m)m.$$

Consequently,

$$\begin{aligned}(1 + m)m &= \sum_{i \in A} f(vu_i) + \sum_{i \in B} f(vu_i) = \sum_{i \in A} f(vu_i) + \sum_{i \in B}(1 + q - f(wu_i)) \\ &= \sum_{i \in A} f(vu_i) - \sum_{i \in B} f(wu_i) + |B|(1 + q).\end{aligned}$$

Thus, $\sum_{i \in A} f(vu_i) \equiv \sum_{i \in B} f(wu_i) \pmod 2$, because $(1 + m)m$ and $1 + q$ are even integers. This implies that $\sum_{i \in A} f(vu_i) + \sum_{i \in B} f(wu_i)$ is an even integer. However, $\sum_{i \in A} f(vu_i) + \sum_{i \in B} f(wu_i) = 1 + 2 + \cdots + m = \frac{m}{2}(1 + m)$, and it is even only for $m \equiv 0 \pmod 4$ or $m \equiv 3 \pmod 4$.

Suppose that two integers of $\{m, n, o\}$ are even and the third is odd. In this case the graph $K_{m,n,o}$ has an even number of edges and it contains some vertices of odd degree. According to Theorem 2, $K_{m,n,o}$ is not a d-magic graph. This proves that condition (ii) holds.

On the other hand, if conditions (i) and (ii) are satisfied then the complete tripartite graph $K_{m,n,o}$ is d-magic by Lemmas 1, 2 and 3. ∎

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