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DISJOINT 5-CYCLES IN A GRAPH

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Abstract

We prove that if G is a graph of order $5k$ and the minimum degree of G is at least $3k$ then G contains k disjoint cycles of length 5.

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1. Introduction and Notation

A set of graphs is said to be disjoint if no two of them have any common vertex. Corrádi and Hajnal [3] investigated the maximum number of disjoint cycles in a graph. They proved that if G is a graph of order at least $3k$ with minimum degree at least $2k$, then G contains k disjoint cycles. In particular, when the order of G is exactly $3k$, then G contains k disjoint triangles. Erdős and Faudree [5] conjectured that if G is a graph of order $4k$ with minimum degree at least $2k$, then G contains k disjoint cycles of length 4. This conjecture has been confirmed by Wang [8]. El-Zahar [4] conjectured that if G is a graph of order $n = n_1 + n_2 + \cdots + n_k$ with $n_i \geq 3$ $(1 \leq i \leq k)$ and the minimum degree of G is at least $\lceil n_1/2 \rceil + \lceil n_2/2 \rceil + \cdots + \lceil n_k/2 \rceil$, then G contains k disjoint cycles of lengths $n_1, n_2, \ldots, n_k$, respectively. He proved this conjecture for $k = 2$. When $n_1 = n_2 = \cdots = n_k = 3$, this conjecture holds by Corrádi and Hajnal's result. When $n_1 = n_2 = \cdots = n_k = 4$, El-Zahar's conjecture reduces to the above conjecture of Erdős and Faudree. Abbasi [1] announced a solution to El-Zahar's conjecture for very large n.

In this paper, we develop a constructive method to show that El-Zahar's conjecture is true for all $n = 5k$ with $n_i = 5$ $(1 \leq i \leq k)$.

Theorem 1. *If G is a graph of order $5k$ and the minimum degree of G is at least $3k$, then G contains k disjoint cycles of length 5.*

We shall use the terminology and notation from [2] except as indicated. Let G be a graph. Let $u \in V(G)$. The neighborhood of u in G is denoted by $N(u)$. Let H be a subgraph of G or a subset of $V(G)$ or a sequence of distinct vertices of G. We define $N(u, H)$ to be the set of neighbors of u contained in H, and let $e(u, H) = |N(u, H)|$. Clearly, $N(u, G) = N(u)$ and $e(u, G)$ is the degree of u in G. If X is a subgraph of G or a subset of $V(G)$ or a sequence of distinct vertices of G, we define $N(X, H) = \cup_u N(u, H)$ and $e(X, H) = \sum_u e(u, H)$ where u runs over all the vertices in X. Let x and y be two distinct vertices. We define $I(xy, H)$ to be $N(x, H) \cap N(y, H)$ and let $i(xy, H) = |I(xy, H)|$. Let each of $X_1, X_2, \ldots, X_r$ be a subgraph of G or a subset of $V(G)$. We use $[X_1, X_2, \ldots, X_r]$ to denote the subgraph of G induced by the set of all the vertices that belong to at least one of $X_1, X_2, \ldots, X_r$. We use C_i to denote a cycle of length i for all integers $i \geq 3$, and use P_j to denote a path of order j for all integers $j \geq 1$. For a cycle C of G, a chord of C is an edge of $G - E(C)$ which joins two vertices of C, and we use $\tau(C)$ to denote the number of chords of C in G. Furthermore, if $x \in V(C)$, we use $\tau(x, C)$ to denote the number of chords of C that are incident with x. For each integer $k \geq 3$, a k-cycle is a cycle of length k. If S is a set of subgraphs of G, we write $G \supseteq S$.

For an integer $k \geq 1$ and a graph G', we use kG' to denote a set of k disjoint graphs isomorphic to G'. If $G_1, \ldots, G_r$ are r graphs and $k_1, \ldots, k_r$ are r positive integers, we use $k_1G_1 \uplus \cdots \uplus k_rG_r$ to denote a set of $k_1 + \cdots + k_r$ disjoint graphs which consist of k_1 copies of $G_1, \ldots, k_{r-1}$ copies of G_{r-1} and k_r copies of G_r. For two graphs H_1 and H_2, the union of H_1 and H_2 is still denoted by $H_1 \cup H_2$ as usual, that is, $H_1 \cup H_2 = (V(H_1) \cup V(H_2), E(H_1) \cup E(H_2))$. Let each of Y and Z be a subgraph of G, or a subset of $V(G)$, or a sequence of distinct vertices of G. If Y and Z do not have any common vertices, we define $E(Y, Z)$ to be the set of all the edges of G between Y and Z. Clearly, $e(Y, Z) = |E(Y, Z)|$. If $C = x_1x_2 \ldots x_rx_1$ is a cycle, then the operations on the subscripts of the x_i's will be taken by modulo r in $\{1, 2, \ldots, r\}$.

We use B to denote a graph of order 5 and size 6 such that B has two edge-disjoint triangles. We use F to denote a graph of order 5 and size 5 such that F has a vertex of degree 1 and a 4-cycle. Let F_1 be the graph of order 5 obtained from F by adding a new edge to F such that the new edge joins the two vertices of F whose degrees in F are 2. Let F_2 be the graph of order 5 and size 7 obtained from $K_{2,3}$ by adding a new edge to $K_{2,3}$ such that F_2 has two adjacent vertices of degree 4. We use K_4^+ to denote the graph of order 5 and size 7 such that K_4^+ has a vertex of degree 1. Finally, we use K_5^- to denote a graph of order 5 with 9 edges.

Let $\{H, L_1, \ldots, L_t\}$ be a set of $t+1$ disjoint subgraphs of G such that $L_i \cong C_5$

for $i = 1, \ldots, t$. We say that $\{H, L_1, \ldots, L_t\}$ is optimal if for any $t+1$ disjoint subgraphs $H', L'_1, \ldots, L'_t$ in $[H, L_1, \ldots, L_t]$ with $H' \cong H$ and $L'_i \cong C_5 (1 \le i \le t)$, we have that $\sum_{i=1}^{t} \tau(L'_i) \le \sum_{i=1}^{t} \tau(L_i)$. Let L be a 5-cycle of G and H a subgraph of order 5 in G. We write $H \ge L$ if H has a 5-cycle L' such that $\tau(L') \ge \tau(L)$. Moreover, if $\tau(L') > \tau(L)$, we write $H > L$.

Let L be a 5-cycle of G. Let $u \in V(L)$ and $x_0 \in V(G) - V(L)$. We write $x_0 \to (L, u)$ if $[L - u + x_0] \supseteq C_5$. Moreover, if $[L - u + x_0] \ge L$ then we write $x_0 \Rightarrow (L, u)$ and if $[L - u + x_0] > L$ then we write $x_0 \stackrel{a}{\to} (L, u)$. In addition, if it does not hold that $x_0 \stackrel{a}{\to} (L, u)$ then we write $x_0 \stackrel{na}{\to} (L, u)$. Clearly, $x_0 \Rightarrow (L, u)$ when $x_0 \stackrel{a}{\to} (L, u)$. If $x_0 \to (L, u)$ for all $u \in V(L)$ then we write $x_0 \to L$. Similarly, we define $x_0 \Rightarrow L$ and $x_0 \stackrel{a}{\to} L$. If $[L - u + x_0] \supseteq B$, we write $x_0 \stackrel{z}{\to} (L, u)$.

Let P be a path of order at least 2 or a sequence of at least two distinct vertices in $G - V(L + x_0)$. Let X be a subset of $V(G) - V(L + x_0)$ with $|X| \ge 2$. We write $x_0 \to (L, u; P)$ if $x_0 \to (L, u)$ and u is adjacent to the two end vertices of P. In this case, if P is a path of order 4, then $[x_0, L, P] \supseteq 2C_5$. We write $x_0 \to (L, u; X)$ if $x_0 \to (L, u; xy)$ for some $\{x, y\} \subseteq X$ with $x \ne y$. We write $x_0 \to (L; P)$ if $x_0 \to (L, u; P)$ for some $u \in V(L)$ and $x_0 \to (L; X)$ if $x_0 \to (L, u; X)$ for some $u \in V(L)$. Similarly, we define the notation $x_0 \stackrel{z}{\to} (L; P)$ and $x_0 \stackrel{z}{\to} (L; X)$. If it does not hold that $x_0 \stackrel{z}{\to} (L; P)$, we write $x_0 \stackrel{nz}{\to} (L; P)$. If it does not hold that $x_0 \stackrel{z}{\to} (L; X)$, we write $x_0 \stackrel{nz}{\to} (L; X)$.

2. Sketch of the Proof of Theorem 1 and Preliminary Lemmas

2.1. Sketch of the proof of Theorem 1

Let G be a graph of order $5k$ with minimum degree at least $3k$. Suppose, by way of contradiction, that $G \not\supseteq kC_5$. We may assume that G is maximal, i.e., $G + xy \supseteq kC_5$ for each pair of non-adjacent vertices x and y of G. Thus $G \supseteq P_5 \uplus (k-1)C_5$. Our first goal is to show that $G \supseteq K_4^+ \uplus (k-1)C_5$. This will be accomplished through a series of lemmas in Section 2.2. Say $G \supseteq \{D, L_1, \ldots, L_{k-1}\}$ with $D \cong K_4^+$ and $L_i \cong C_5 (1 \le i \le k)$. Let $x_0 \in V(D)$ with $e(x_0, D) = 1$ and let $Q = D - x_0$. We shall estimate the upper bound on $2e(x_0, G) + e(Q, G) \ge 18k$. This needs an estimation on each $2e(x_0, L_i) + e(Q, L_i)$. The idea is to show that if $e(x_0, L_i)$ is increasing then $e(Q, L_i)$ is decreasing for otherwise $[D, L_i] \supseteq 2C_5$, a contradiction. This is accomplished in Lemma 3.3. It turns out that $2e(x_0, G) + e(Q, G) < 18k$, a contradiction.

2.2. Preliminary lemmas

Our proof of Theorem 1 will use the following lemmas. Let $G = (V, E)$ be a given graph in the following.

Lemma 2.1. *The following statements hold:*

(a) *If P' and P'' are two disjoint paths of G such that $|V(P')| = 2$, $2 \leq |V(P'')| \leq 3$ and $e(P', P'') \geq 3$, then $[P', P''] \supseteq C_4$.*

(b) *If x and y are two distinct vertices and P is a path of order 3 in G such that $\{x, y\} \cap V(P) = \emptyset$ and $e(xy, P) \geq 5$, then $[x, y, P]$ contains a 5-cycle C such that $\tau(C) \geq 2$.*

(c) *If D is a graph of order 5 with $e(D) \geq 7$, then $D \supseteq C_5$, unless $D \cong K_4^+$ or $D \cong F_2$.*

(d) *If R is a subset of $V(G)$ and L is a 5-cycle of $G - R$ such that $|R| = 4$ and $e(R, L) \geq 13$, then $u \to (L; R - \{u\})$ for some $u \in R$, or there exist two labellings $R = \{y_1, y_2, y_3, y_4\}$ and $L = b_1b_2b_3b_4b_5b_1$ such that $N(y_1, L) = N(y_2, L) = \{b_1, b_2, b_3, b_4\}$, $N(y_3, L) = \{b_1, b_5, b_4\}$ and $N(y_4, L) = \{b_1, b_4\}$.*

Proof. It is easy to check (a), (b) and (c). To prove (d), we suppose, for a contradiction, that $u \not\to (L; R - \{u\})$ for all $u \in R$. Let $R = \{y_1, y_2, y_3, y_4\}$ be such that $e(y_1, L) \geq e(y_i, L)$ for all $y_i \in R$. As $e(R, L) \geq 13$, $e(y_1, L) \geq 4$ and there exists $b \in V(L)$ such that $e(b, R - \{y_1\}) \geq 2$. If $e(y_1, L) = 5$ then $y_1 \to (L, b; R - \{y_1\})$, a contradiction. Hence we may assume that $L = b_1b_2b_3b_4b_5b_1$ and $e(y_1, b_1b_2b_3b_4) = 4$. Thus $e(b_i, R - \{y_1\}) \leq 1$ for $i \in \{2, 3, 5\}$. Then $6 \geq e(b_1b_4, R - \{y_1\}) \geq 13 - 4 - 3 = 6$. It follows that $e(b_1b_4, R - \{y_1\}) = 6$ and $e(b_i, R - \{y_1\}) = 1$ for $i \in \{2, 3, 5\}$. W.l.o.g., say $b_2y_2 \in E$. Then $e(b_3, y_3y_4) = 0$ as $y_2 \not\to (L, b_3; R - \{y_2\})$. Hence $b_3y_2 \in E$. W.l.o.g., say $b_5y_3 \in E$. Thus (d) holds. ■

Lemma 2.2. *Let D and L be disjoint subgraphs of G such that $D \cong B$ and $L \cong C_5$. Say $D = x_0x_1x_2x_0x_3x_4x_0$. Suppose that $e(D - x_0, L) \geq 13$. Then $[D, L] \supseteq 2C_5$.*

Proof. Let $H = [D, L]$. On the contrary, suppose $H \not\supseteq 2C_5$. Then it is easy to see that

$$x_i \not\to (L; x_jx_s) \text{ and } x_i \not\to (L; x_jx_t) \text{ for}$$
$$\{\{i, j\}, \{s, t\}\} = \{\{1, 2\}, \{3, 4\}\}. \tag{1}$$

Let $R = \{x_1, x_2, x_3, x_4\}$. W.l.o.g., say $e(x_1, L) \geq e(x_i, L)$ for all $x_i \in R$. Then $e(x_1, L) \geq 4$. First, assume that $e(x_1, L) = 5$. By (1), $I(x_2x_3, L) = I(x_2x_4, L) = \emptyset$. Thus $e(x_2x_3, L) \leq 5$ and $e(x_2x_4, L) \leq 5$. Since $e(R, L) \geq 13$, it follows that $e(x_4, L) \geq 3$ and $e(x_3, L) \geq 3$. As $x_3 \not\to (L; x_1x_4)$, we see that $e(x_3, L) = 3$. Similarly, $e(x_4, L) = 3$. Then $e(x_2, L) = 2$. As $x_2 \not\to (L; x_1x_3)$, we see that the two vertices of $N(x_2, L)$ must be consecutive on L. Say $N(x_2, L) = \{a_1, a_2\}$. Then $[x_0, x_1, x_2, a_1, a_2] \supseteq C_5$ and $[x_3, x_4, a_3, a_4, a_5] \supseteq C_5$, a contradiction. Therefore $e(x_1, L) = 4$. Say $N(x_1, L) = \{a_1, a_2, a_3, a_4\}$. By (1), $I(x_2x_j, \{a_2, a_3, a_5\}) = \emptyset$ for $j \in \{3, 4\}$. Thus $e(x_2x_j, L) \leq 7$ for $j \in \{3, 4\}$ and so $e(x_j, L) \geq 2$ for $j \in \{3, 4\}$.

First, assume $e(x_2x_j, L) = 7$ for some $j \in \{3,4\}$. Say $e(x_2x_3, L) = 7$. Then $I(x_2x_3, L) = \{a_1, a_4\}$ and $e(a_i, x_2x_3) = 1$ for $i \in \{2,3,5\}$. If $e(x_4, a_2a_3) \geq 1$, say w.l.o.g. $x_4a_2 \in E$, then $[a_1, a_2, x_4, x_0, x_3] \supseteq C_5$ and so $x_2a_5 \notin E$ as $H \not\supseteq 2C_5$. Consequently, $x_3a_5 \in E$ and so $H \supseteq 2C_5 = \{x_3a_5a_1a_2x_4x_3, x_1x_0x_2a_4a_3x_1\}$, a contradiction. Hence $e(x_4, a_2a_3) = 0$ and so $e(x_4, a_1a_4) \geq 1$. W.l.o.g., say $x_4a_1 \in E$. Then $[x_3, x_4, a_1, a_5, a_4] \supseteq C_5$ and so $e(x_2, a_2a_3) = 0$ as $H \not\supseteq 2C_5$. Thus $e(x_3, a_2a_3) = 2$. As $e(x_3, L) \leq e(x_1, L) = 4$, $x_3a_5 \notin E$. Thus $x_2a_5 \in E$, and consequently, $H \supseteq 2C_5 = \{x_3x_4a_1a_2a_3x_3, x_1x_0x_2a_5a_4x_1\}$, a contradiction. Therefore $e(x_2x_j, L) \leq 6$ for $j \in \{3,4\}$ and so $e(x_j, L) \geq 3$ for $j \in \{3,4\}$. Similarly, if $e(x_3, L) = 4$ then $e(x_1x_4, L) \leq 6$, a contradiction. Hence $e(x_3, L) = 3$. Similarly, $e(x_4, L) = 3$. Then $e(x_2, L) = 3$ as $e(R, L) \geq 13$. Assume $x_2a_5 \in E$. Then $e(a_5, x_3x_4) = 0$ by (1). As $e(x_3x_4, L) = 6$, either $e(x_3x_4, a_1a_2) \geq 3$ or $e(x_3x_4, a_3a_4) \geq 3$. Say w.l.o.g. the former holds. Then $[x_3, x_0, x_4, a_1, a_2] \supseteq C_5$ and $[x_1, x_2, a_5, a_4, a_3] \supseteq C_5$, a contradiction. Hence $x_2a_5 \notin E$. As $e(x_2, L) = 3$, either $e(x_2, a_1a_3) = 2$ or $e(x_2, a_2a_4) = 2$. W.l.o.g., say the former holds. As $x_2 \not\to (L; x_1x_j)$ for $j \in \{3,4\}$, $e(a_2, x_3x_4) = 0$. As $e(x_3x_4, L) = 6$, either $e(x_3x_4, a_3a_5) \geq 3$ or $e(x_3x_4, a_1a_4) \geq 3$. Thus either $[x_3, x_4, a_3, a_4, a_5] \supseteq C_5$ or $[x_3, x_4, a_4, a_5, a_1] \supseteq C_5$. In each situation, we see that $H \supseteq 2C_5$, a contradiction. ■

Lemma 2.3. *Let P and L be disjoint subgraphs of G such that $P \cong P_5$ and $L \cong C_5$. Suppose that $\{P, L\}$ is optimal, $e(P, L) \geq 16$ and $[P, L] \not\supseteq 2C_5$. Then $[P, L] \supseteq F \uplus C_5$.*

Proof. Say $P = x_1x_2x_3x_4x_5$ with $e(x_1, L) \geq e(x_5, L)$ and $L = a_1a_2a_3a_4a_5a_1$. Then $e(x_1, L) \geq 1$. Let $H = [P, L]$. On the contrary, suppose $H \not\supseteq F \uplus C_5$. Assume first that $e(x_1, L) = 1$. Say $x_1a_1 \in E$. As $e(P, L) \geq 16$ and $e(x_5, L) \leq 1$, $e(x_2x_3x_4, L) \geq 14$. Thus $e(x_2, a_3a_4) \geq 1$. W.l.o.g., say $x_2a_3 \in E$. Then $[x_1, x_2, a_3, a_2, a_1] \supseteq C_5$. As $e(x_3x_4, L) \geq 14 - e(x_2, L) \geq 9$, $e(x_3x_4, a_4a_5) \geq 3$. By Lemma 2.1(a), $[x_5, x_4, x_3, a_4, a_5] \supseteq F$ and so $H \supseteq F \uplus C_5$, a contradiction. Hence $e(x_1, L) \geq 2$.

As $e(P, L) \geq 16$, $I(x_2x_4, L) \neq \emptyset$ or $I(x_3x_5, L) \neq \emptyset$. Therefore $x_1 \not\to L$ for otherwise $H \supseteq F \uplus C_5$. Hence $e(x_1, L) \leq 4$. We divide the proof into the following cases.

Case 1. $e(x_1, L) = 4$. Say $N(x_1, L) = \{a_1, a_2, a_3, a_4\}$. Then $[L - a_i + x_1] \supseteq F$ for all $a_i \in V(L)$. Thus $I(x_2x_5, L) = \emptyset$ as $H \not\supseteq F \uplus C_5$. As $x_1 \not\to L$, $\tau(a_5, L) = 0$. Then $x_1 \xrightarrow{a} (L, a_5)$. By the optimality of $\{P, L\}$, $[P - x_1 + a_5] \not\supseteq P_5$ and so $e(a_5, x_2x_5) = 0$ and $e(a_5, x_3x_4) \leq 1$. Thus $e(x_2x_5, L) \leq 4$ and so $e(x_3x_4, L) \geq 8$. Suppose $e(x_2, L) \geq 1$. Then $e(x_2, a_2a_4) \geq 1$ or $e(x_2, a_1a_3) \geq 1$. W.l.o.g., say the former holds. Then $[x_1, x_2, a_2, a_3, a_4] \supseteq C_5$. As $H \not\supseteq F \uplus C_5$ and by Lemma 2.1(a), we see that $e(x_3x_4, a_1a_5) \leq 2$. It follows that $e(x_3x_4, a_2a_3a_4) = 6$ and $e(x_2x_5, L - a_5) = 4$. Thus $e(a_2, x_2x_5) > 0$. Then $[P - x_1 + a_2] \supseteq F$. As $x_1 \to$

(L, a_2), $H \supseteq F \uplus C_5$, a contradiction. Hence $e(x_2, L) = 0$. Similarly, if $e(x_5, L) = 4$ then $e(x_4, L) = 0$ and so $e(P, L) < 16$, a contradiction. Hence $e(x_5, L) \leq 3$ and so $e(x_3x_4, L) \geq 9$. As $e(a_5, x_3x_4) \leq 1$, it follows that $e(x_3x_4, L - a_5) = 8$, $e(a_5, x_3x_4) = 1$ and $e(x_5, L) = 3$. Then $e(a_i, x_3x_5) = 2$ for some $i \in \{2, 3\}$ and so $H \supseteq F \uplus C_5$ as $x_1 \to (L, a_i)$, a contradiction.

Case 2. $e(x_1, L) = 3$. Then $e(x_5, L) \leq 3$. First, suppose that the three vertices in $N(x_1, L)$ are not consecutive on L. Say $N(x_1, L) = \{a_1, a_2, a_4\}$. Clearly, $I(x_2x_5, L) \subseteq \{a_4\}$ since $H \not\supseteq 2C_5$ and $H \not\supseteq F \uplus C_5$. Hence $e(x_2x_5, L) \leq 6$. If $x_2a_4 \in E$ then $[x_1, x_2, a_1, a_5, a_4] \supseteq C_5$. As $H \not\supseteq F \uplus C_5$, $e(x_3x_4, a_2a_3) \leq 2$. Similarly, $[x_1, x_2, a_2, a_3, a_4] \supseteq C_5$ and so $e(x_3x_4, a_1a_5) \leq 2$. Consequently, $e(P, L) \leq 15$, a contradiction. Hence $x_2a_4 \notin E$. Thus $e(x_2x_5, L) \leq 5$ and so $e(x_3x_4, L) \geq 8$. If $e(x_2, L) > 0$, then $[x_1, x_2, P'] \supseteq C_5$ where $P' = L - \{a_i, a_{i+1}\}$ for some $\{a_i, a_{i+1}\} \subseteq V(L)$. As $H \not\supseteq F \uplus C_5$, $e(x_3x_4, a_ia_{i+1}) \leq 2$. Consequently, $e(x_3x_4, P') = 6$, $e(x_3x_4, a_ia_{i+1}) = 2$ and $e(x_2x_5, L) = 5$. Hence $e(a_t, x_2x_5) = 1$ for all $a_t \in V(L)$. Thus $[P - x_1 + a_j] \supseteq F$ and $x_1 \to (L, a_j)$ where $a_j \in V(P') \cap \{a_3, a_5\}$, a contradiction.

Therefore $e(x_2, L) = 0$ and so $e(x_3x_4, L) = 10$ and $e(x_5, L) = 3$. Consequently, $H \supseteq 2C_5$ or $H \supseteq F \uplus C_5$, a contradiction. Therefore the three vertices in $N(x_1, L)$ are consecutive on L. Say $N(x_1, L) = \{a_1, a_2, a_3\}$. Then $I(x_2x_5, L) \subseteq \{a_1, a_3\}$ since $H \not\supseteq 2C_5$ and $H \not\supseteq F \uplus C_5$. Thus $e(x_2x_5, L) \leq 7$ and so $e(x_3x_4, L) \geq 6$. Assume $e(x_2, a_4a_5) \geq 1$. Say w.l.o.g. $x_2a_4 \in E$. Then $[x_1, x_2, a_2, a_3, a_4] \supseteq C_5$ and $[x_1, x_2, a_1, a_5, a_4] \supseteq C_5$. As $H \not\supseteq F \uplus C_5$ and by Lemma 2.1(a), $e(x_3x_4, a_1a_5) \leq 2$ and $e(x_3x_4, a_2a_3) \leq 2$. It follows that $e(x_2x_5, L) = 7$, $e(x_3x_4, L) = 6$, $e(a_4, x_3x_4) = 2$, and $e(x_2x_5, a_1a_3) = 4$. Then $[x_1, x_5, a_1, a_2, a_3] \supseteq C_5$ and $[a_5, a_4, x_2, x_3, x_4] \supseteq F$, a contradiction. Hence $e(x_2, a_4a_5) = 0$ and so $e(x_2, L) \leq 3$. Thus $e(x_3x_4, L) \geq 7$. Assume $e(x_2, a_1a_3) \geq 1$. Then $[x_1, x_2, a_1, a_2, a_3] \supseteq C_5$. Then $e(x_3x_4, a_4a_5) \leq 2$ as $H \not\supseteq F \uplus C_5$. Thus $e(x_3x_4, a_1a_2a_3) \geq 5$. As $H \not\supseteq F \uplus C_5$ and $x_1 \to (L, a_2)$, we have $e(a_2, x_2x_4) \leq 1$. As $e(P, L) \geq 16$, it follows that $e(a_2, x_2x_4) = 1$, $e(x_3, a_1a_2a_3) = 3$, $e(x_3x_4, a_4a_5) = 2$ and $e(x_5, L) = 3$. As $H \not\supseteq F \uplus C_5$ and $x_1 \to (L, a_2)$, we see that $x_5a_2 \notin E$. Then $e(x_5, a_4a_5) \geq 1$ and so $[x_3, x_4, x_5, a_4, a_5] \supseteq F$, a contradiction. Hence $e(x_2, a_1a_3) = 0$ and so $e(x_2, L) \leq 1$. If $e(x_5, L) = 3$ then we also have $e(x_4, L) \leq 1$ by the symmetry and so $e(P, L) \leq 13$, a contradiction. Hence $e(x_5, L) \leq 2$. It follows that so $e(x_3x_4, L) = 10$, $e(x_2, L) = 1$ and $e(x_5, L) = 2$. Thus $e(a_2, x_2x_4) = 2$ and so $H \supseteq F \uplus C_5$, a contradiction.

Case 3. $e(x_1, L) = 2$. Then $e(x_5, L) \leq 2$ and $e(x_3x_4, L) \geq 7$. First, suppose that the two vertices in $N(x_1, L)$ are not consecutive on L. Say $N(x_1, L) = \{a_1, a_3\}$. Assume $e(x_2, a_1a_3) \geq 1$. Then $[x_1, x_2, a_1, a_2, a_3] \supseteq C_5$. As $H \not\supseteq F \uplus C_5$ and by Lemma 2.1(a), $e(x_3x_4, a_4a_5) \leq 2$. Hence $e(x_3x_4, a_1a_2a_3) \geq 5$. As $x_1 \to (L, a_2)$ and $H \not\supseteq F \uplus C_5$, $e(a_2, x_2x_4) \leq 1$. As $e(P, L) \geq 16$, it follows that $e(a_2, x_2x_4) = 1$, $e(x_5, L) = 2$, $e(x_2, L - a_2) = 4$, $e(x_3, a_1a_2a_3) = 3$ and

$e(x_3x_4, a_4a_5) = 2$. As $[x_3, x_4, x_5, a_4, a_5] \not\supseteq F$, $e(x_5, a_4a_5) = 0$ by Lemma 2.1(a). As $x_1 \to (L, a_2)$ and $H \not\supseteq F \uplus C_5$, $a_2x_5 \notin E$. Thus $e(x_5, a_1a_3) = 2$. It follows that $[x_1, x_2, a_1, a_5, a_4] \supseteq C_5$ and $[x_3, x_4, x_5, a_3, a_2] \supseteq C_5$, a contradiction. Hence $e(x_2, a_1a_3) = 0$. Thus $e(x_3x_4, L) \geq 9$. As $e(x_3x_4, L) \leq 10$, $e(x_2, L) \geq 2$ and so $e(x_2, a_4a_5) \geq 1$. Say w.l.o.g. $x_2a_4 \in E$. Then $[x_1, x_2, a_4, a_5, a_1] \supseteq C_5$. As $H \not\supseteq F \uplus C_5$ and by Lemma 2.1(a), $e(x_3x_4, a_2a_3) \leq 2$ and so $e(x_3x_4, L) \leq 8$, a contradiction. Therefore the two vertices in $N(x_1, L)$ are consecutive on L. Say $N(x_1, L) = \{a_1, a_2\}$. Assume $x_2a_4 \in E$. Then $[x_1, x_2, a_4, a_5, a_1] \supseteq C_5$ and $[x_1, x_2, a_4, a_3, a_2] \supseteq C_5$. Thus $e(x_3x_4, a_2a_3) \leq 2$ and $e(x_3x_4, a_1a_5) \leq 2$ since $H \not\supseteq F \uplus C_5$. Hence $e(x_3x_4, L) \leq 6$, a contradiction. Hence $x_2a_4 \notin E$. Thus $e(x_3x_4, L) \geq 8$. Assume $e(x_2, a_3a_5) \geq 1$. Say $x_2a_3 \in E$. Then $[x_1, x_2, a_3, a_2, a_1] \supseteq C_5$ and so $e(x_3x_4, a_4a_5) \leq 2$. It follows that $e(x_3x_4, a_1a_2a_3) = 6$, $e(x_3x_4, a_4a_5) = 2$, $e(x_2, L - a_4) = 4$ and $e(x_5, L) = 2$. As $x_2a_5 \in E$ and by the symmetry, we also have $e(x_3x_4, a_5a_1a_2) = 6$. Then $H \supseteq F \uplus C_5$, a contradiction. Therefore $e(x_2, a_3a_5) = 0$. It follows that $e(x_2, a_1a_2) = 2$, $e(x_3x_4, L) = 10$ and $e(x_5, L) = 2$. Then $H \supseteq F \uplus C_5$, a contradiction ■

Lemma 2.4. *Let D and L be disjoint subgraphs of G with $D \cong F_2$ and $L \cong C_5$. Let R be the set of the three vertices of D with degree 2 in D. If $e(R, L) \geq 10$, then $[D, L] \supseteq F_1 \uplus C_5$.*

Proof. As $e(R, L) \geq 10$, $e(u, L) \geq 4$ for some $u \in R$. Thus $u \to (L, v)$ for some $v \in V(L)$ with $e(v, R - \{u\}) \geq 1$. Clearly, $[D - u + v] \supseteq F_1$. ■

Lemma 2.5. *Let D and L be disjoint subgraphs of G with $D \cong F$ and $L \cong C_5$. Suppose that $\{D, L\}$ is optimal and $e(D, L) \geq 16$. Then $[D, L]$ contains one of $F_1 \uplus C_5$, $F_2 \uplus C_5$, $B \uplus C_5$ and $2C_5$, or there exist two labellings $D = x_0x_1x_2x_3x_4x_1$ and $L = a_1a_2a_3a_4a_5a_1$ such that $e(x_0, L) = 0$, $e(x_1x_3, L) = 10$, $N(x_2, L) = N(x_4, L) = \{a_1, a_2, a_4\}$, $\tau(L) = 4$ and $a_3a_5 \notin E$.*

Proof. Say $H = [D, L]$. Suppose that H does not contain any of $F_1 \uplus C_5$, $F_2 \uplus C_5$, $B \uplus C_5$ and $2C_5$. We shall prove that there exist two labellings of D and L satisfying the property in the lemma. Say $D = x_0x_1x_2x_3x_4x_1$ and $L = a_1a_2a_3a_4a_5a_1$. Then $x_2x_4 \notin E$. Let $Q = x_1x_2x_3x_4x_1$. If $e(x_0, L) \geq 4$, then for each $a_i \in V(L)$, $[L - a_i + x_0] \supseteq C_5$ or $[L - a_i + x_0] \supseteq F_1$. Thus $[Q + a_i] \not\supseteq C_5$ and so $e(a_i, Q) \leq 2$ for each $a_i \in V(L)$. Consequently, $e(D, L) \leq 15$, a contradiction. Therefore $e(x_0, L) \leq 3$. We divide the proof into the following cases.

Case 1. $e(x_0, L) = 0$. First, suppose that $e(x_2, L) \geq 4$ or $e(x_4, L) \geq 4$. Say, $\{a_1, a_2, a_3, a_4\} \subseteq N(x_2, L)$. Assume $e(x_1, a_2a_3) \geq 1$. Say w.l.o.g. $x_1a_2 \in E$.

Then $[x_0, x_1, x_2, a_2, a_1] \supseteq F_1$ and $[x_0, x_1, x_2, a_2, a_3] \supseteq F_1$. As $H \not\supseteq F_1 \uplus C_5$, we see that $e(x_3x_4, a_3a_5) \leq 2$ and $e(x_3x_4, a_1a_4) \leq 2$. As $e(Q, L) \geq 16$, it follows that $e(x_1x_2, L) = 10$ and $e(a_2, x_3x_4) = 2$. Thus $[x_0, x_1, a_2, x_3, x_4] \supseteq F_1$ and $x_2 \to$

(L,a_2), a contradiction. Hence $e(x_1,a_2a_3)=0$. As $e(x_1,L)\ge 1$, this argument implies that $e(x_2,L)\ne 5$. Similarly, $e(x_4,L)\ne 5$. As $e(Q,L)\ge 16$, it follows that $e(x_1,a_1a_5a_4)=3$, $e(x_3,L)=5$ and $e(x_4,L)=4$. Then $[x_0,x_1,x_2,a_1,a_2]\supseteq F_1$ and $[x_3,x_4,a_3,a_4,a_5]\supseteq C_5$, a contradiction. Hence $e(x_2,L)\le 3$ and $e(x_4,L)\le 3$. Consequently, $e(x_1x_3,L)=10$, $e(x_2,L)=e(x_4,L)=3$. Then x_2 is adjacent two consecutive vertices of L. Say w.l.o.g. $e(x_2,a_1a_2)=2$. Then $[x_0,x_1,x_2,a_1,a_2]\supseteq F_1$. Thus $e(x_4,a_3a_5)=0$ as $H\not\supseteq F_1\uplus C_5$. Hence $e(x_4,a_1a_2a_4)=3$. Similarly, $e(x_2,a_1a_2a_4)=3$. Clearly, $[D-x_3+a_i]\supseteq F$ for $i\in\{1,2\}$. As $\{D,L\}$ is optimal, $x_3\overset{na}{\to}(L,a_i)$ for $i\in\{1,2\}$. This implies that $\tau(a_1,L)=\tau(a_2,L)=2$. As $[x_0,x_1,x_2,a_1,a_2]\supseteq F_1$, $[x_3,x_4,a_3,a_4,a_5]\not\supseteq C_5$. This implies that $a_3a_5\notin E$. Therefore these two labellings satisfy the property described in the lemma.

Case 2. $e(x_0,L)=1$. Then $e(Q,L)\ge 15$. Say $x_0a_1\in E$. First, suppose $e(x_1,a_3a_4)\ge 1$. Say w.l.o.g. $x_1a_3\in E$. Then $[x_1,x_0,a_1,a_2,a_3]\supseteq C_5$. By Lemma 2.1(c), we have $e(a_4a_5,x_2x_3x_4)\le 3$ since $H\not\supseteq 2C_5$, $H\not\supseteq F_1\uplus C_5$ and $H\not\supseteq F_2\uplus C_5$. Thus $e(a_4a_5,Q)\le 5$. Similarly, if $x_1a_4\in E$ then $e(a_2a_3,Q)\le 5$ and so $e(Q,L)\le 14$, a contradiction. Hence $x_1a_4\notin E$. Thus $e(a_4a_5,Q)\le 4$ and so $e(a_1a_2a_3,Q)\ge 11$. This implies that if $e(a_2,x_1x_3)=2$ then there is a choice $\{i,j\}=\{2,4\}$ such that $e(x_i,a_1a_3)=2$ and $e(a_2,x_1x_jx_3)=3$. Thus $[x_0,x_1,x_j,x_3,a_2]\supseteq F_1$ and $x_i\to(L,a_2)$, a contradiction. Hence $e(a_2,x_1x_3)=1$, $e(a_1a_3,Q)=8$, $e(a_2,x_2x_4)=2$ and $e(a_4a_5,Q)=4$ with $a_5x_1\in E$. Consequently, $[a_4,a_5,a_1,x_0,x_1]\supseteq F_1$ and $[a_2,a_3,x_2,x_3,x_4]\supseteq C_5$, a contradiction. Therefore $e(x_1,a_3a_4)=0$.

Next, suppose $e(x_1,a_1a_5)=2$ or $e(x_1,a_1a_2)=2$. Say w.l.o.g. $e(x_1,a_1a_5)=2$. Then $[a_4,a_5,a_1,x_0,x_1]\supseteq F_1$. Thus $e(a_2a_3,x_2x_4)\le 2$. Hence $e(a_2a_3,Q)\le 5$ and so $e(a_1a_5a_4,x_2x_3x_4)\ge 8$. This implies that if $x_3a_5\in E$ then there is a choice $\{i,j\}=\{2,4\}$ such that $e(a_5,x_1x_ix_3)=3$, $e(x_j,a_1a_4)=2$ and consequently, $H\supseteq F_1\uplus C_5$, a contradiction. Hence $a_5x_3\notin E$ and it follows that $e(a_1,x_2x_3x_4)=3$, $e(a_5,x_2x_4)=2$, $e(a_4,x_2x_3x_4)=3$, $e(a_2a_3,Q)=5$ with $a_2x_1\in E$. Then $[a_3,a_2,a_1,x_0,x_1]\supseteq F_1$ and $[a_4,a_5,x_2,x_3,x_4]\supseteq C_5$, a contradiction. Therefore $e(x_1,a_1a_5)\le 1$ and $e(x_1,a_1a_2)\le 1$. Thus $e(x_1,L)\le 2$. Assume that $a_1x_3\in E$. Then $x_2\not\to(L,a_1)$ as $H\not\supseteq 2C_5$. Hence $e(x_2,a_2a_5)\le 1$, and similarly, $e(x_4,a_2a_5)\le 1$. As $e(Q,L)\ge 15$, it follows that $e(x_1,a_2a_5)=2$, $e(x_3,L)=5$, $e(x_2x_4,a_1a_3a_4)=6$ and $e(x_2,a_2a_5)=e(x_4,a_2a_5)=1$. Say w.l.o.g. $a_5x_4\in E$. Then $[D-x_2+a_5]\supseteq F_1$ and $x_2\to(L,a_5)$, a contradiction. Therefore $a_1x_3\notin E$. If $x_1a_1\in E$ then $e(x_1,a_2a_5)=0$ and so $e(a_1,Q-x_3)+e(L-a_1,Q-x_1)\ge 15$. Then $[D-x_2+a_1]\supseteq F_1$ and $x_2\to(L,a_1)$, a contradiction. Hence $N(x_1,L)\subseteq\{a_2,a_5\}$. As $e(Q,L)\ge 15$, $e(a_2a_5,x_2x_4)\ge 3$ and $e(a_2a_4,x_3x_i)\ge 3$ for $i\in\{2,4\}$. Say w.l.o.g. $x_2a_5\in E$. Then $[x_0,x_1,x_2,a_5,a_1]\supseteq C_5$ and $[x_3,x_4,a_2,a_3,a_4]\supseteq C_5$, a contradiction.

Case 3. $N(x_0,L)=\{a_i,a_{i+2}\}$ for some $i\in\{1,2,3,4,5\}$. Say $N(x_0,L)=\{a_1,a_3\}$. Then $e(Q,L)\ge 14$. As $H\not\supseteq 2C_5$, $e(a_2,Q)\le 2$. We claim that

$e(x_1, a_1a_3) = 0$. On the contrary, say $e(x_1, a_1a_3) \geq 1$. Then $[x_0, x_1, a_1, a_2, a_3] \supseteq C_5$. Since $H \not\supseteq 2C_5$, $H \not\supseteq F_1 \uplus C_5$ and $H \not\supseteq F_2 \uplus C_5$, we see that $e(a_4a_5, x_2x_3x_4) \leq 3$ by Lemma 2.1(c). Thus $e(a_4a_5, Q) \leq 5$ and $e(a_1a_3, Q) \geq 14 - e(a_2, Q) - e(a_4a_5, Q) \geq 7$. As $e(a_1a_3, Q) \leq 8$, it follows that either $e(a_1, Q) = 4$ and $x_1a_5 \in E$ or $e(a_3, Q) = 4$ and $x_1a_4 \in E$. Say w.l.o.g. the former holds. Then $[D - x_3 + a_1] \supseteq F_2$, $[x_0, x_1, a_1, a_5, a_4] \supseteq F_1$ and $[x_0, x_1, a_1, a_5, x_i] \supseteq F_2$ for $i \in \{2, 4\}$. Furthermore, if $x_1a_2 \in E$ then $[x_0, x_1, a_1, a_5, a_2] \supseteq F_2$ and $[x_0, x_1, a_1, a_2, x_i] \supseteq F_2$ for $i \in \{2, 4\}$. Assume for the moment that $e(a_3, x_2x_4) = 2$. Then we see that $e(a_2, x_2x_4) = 0$ as $H \not\supseteq F_1 \uplus C_5$. If $x_1a_2 \in E$, then $e(a_4, x_2x_4) = 0$ as $H \not\supseteq F_2 \uplus C_5$ and for the same reason, $[a_3, a_4, a_5, x_3, x_i] \not\supseteq C_5$ for $i \in \{2, 4\}$. This implies that $x_3a_5 \notin E$ and so $e(a_5, x_2x_4) \geq 1$ since $8 \geq e(a_1a_3, Q) \geq 14 - e(a_2, Q) - e(a_4a_5, Q) \geq 7$. Thus $x_3a_3 \notin E$ since $[a_3, a_4, a_5, x_3, x_i] \not\supseteq C_5$ for $i \in \{2, 4\}$. It follows that $\{a_3x_1, x_3a_4\} \subseteq E$. Consequently, $[a_1, a_5, a_4, x_2, x_3] \supseteq C_5$ and $[x_0, x_1, x_4, a_2, a_3] \supseteq F_2$, a contradiction. Hence $x_1a_2 \notin E$. As $e(Q, L) \geq 14$, it follows that $a_2x_3 \in E$, $e(a_1a_3, Q) = 8$, $e(x_1, a_4a_5) = 2$ and $e(a_4a_5, x_2x_3x_4) = 3$. Say w.l.o.g. $a_4x_2 \in E$. Then $[a_2, a_3, a_4, x_2, x_3] \supseteq C_5$ and so $H \supseteq F_2 \uplus C_5$, a contradiction. Hence $e(a_3, x_2x_4) \leq 1$. It follows that $e(a_3, x_2x_4) = 1$, $e(a_3, x_1x_3) = 2$, $e(a_2, Q) = 2$ and $e(a_4a_5, Q) = 5$ with $e(x_1, a_4a_5) = 2$. Thus $[x_0, x_1, a_5, a_4, a_3] \supseteq C_5$ and so $e(a_2, x_1x_3) = 2$ as $H \not\supseteq 2C_5$. Say w.l.o.g. $a_3x_2 \in E$. As $H \not\supseteq F_2 \uplus C_5$, we see that $[x_2, x_3, a_5, a_4, a_3] \not\supseteq C_5$ and $[a_3, a_4, x_2, x_3, x_4] \not\supseteq C_5$. This implies that $e(a_5, x_2x_3) = 0$ and $a_4x_4 \notin E$. As $e(a_4a_5, x_2x_3x_4) = 3$, it follows that $[a_4, a_5, x_2, x_3, x_4] \supseteq C_5$ and so $H \supseteq 2C_5$, a contradiction. Therefore $e(x_1, a_1a_3) = 0$. Assume $e(x_1, a_4a_5) = 0$. As $e(Q, L) \geq 14$, it follows that $e(x_2x_3x_4, L - a_2) = 12$ and $e(a_2, Q) = 2$. Thus $[x_2, x_3, x_4, a_4, a_5] \supseteq K_5^-$. As $[x_1, x_0, a_1, a_2, a_3] \supseteq F$, we have $\tau(L) \geq 4$ by the optimality of $\{D, L\}$. Consequently, $x_0 \to (L, a_r)$ for some $r \in \{4, 5\}$ and so $H \supseteq 2C_5$ as $[Q + a_r] \supseteq C_5$, a contradiction. Hence $e(x_1, a_4a_5) \geq 1$. Say w.l.o.g. $x_1a_5 \in E$. Then $[x_0, x_1, a_5, a_4, a_3] \supseteq C_5$. Since $H \not\supseteq 2C_5$, $H \not\supseteq F_1 \uplus C_5$ and $H \not\supseteq F_2 \uplus C_5$, we see that $e(a_1a_2, x_2x_3x_4) \leq 3$ by Lemma 2.1(c). Thus $e(a_1a_2, Q) \leq 4$ and so $e(a_3a_4a_5, Q) \geq 10$. Hence $e(a_4a_5, Q) \geq 7$. As above, we shall have that $[x_2, x_3, x_4, a_4, a_5] \not\supseteq K_5^-$. This implies that $e(a_4a_5, x_2x_3x_4) \neq 6$. Thus $e(a_4a_5, x_2x_3x_4) = 5$, $e(x_1, a_4a_5) = 2$, $e(a_3, x_2x_3x_4) = 3$ and $e(a_1a_2, Q) = 4$. Similarly, we shall have $e(a_1, x_2x_3x_4) = 3$ as $[x_0, x_1, a_4, a_5, a_1] \supseteq C_5$. As $e(a_4a_5, x_2x_3x_4) = 5$, we may assume w.l.o.g. that $e(a_4, x_2x_3x_4) = 3$. Thus $[a_3, a_4, x_2, x_3, x_4] \supseteq K_5^-$ and $[a_2, a_1, a_5, x_1, x_0] \supseteq F$. By the optimality of $\{D, L\}$, we shall have $\tau(L) \geq 4$. Thus $x_0 \to (L, a_r)$ for some $r \in \{4, 5\}$ and so $H \supseteq 2C_5$, a contradiction.

Case 4. $N(x_0, L) = \{a_i, a_{i+1}\}$ for some $i \in \{1, 2, 3, 4, 5\}$. Say, $N(x_0, L) = \{a_1, a_2\}$. First, suppose that $x_1a_4 \in E$. Then $[x_0, x_1, a_4, a_5, a_1] \supseteq C_5$ and $[x_0, x_1, a_4, a_3, a_2] \supseteq C_5$. Since $H \not\supseteq 2C_5$, $H \not\supseteq F_1 \uplus C_5$ and $H \not\supseteq F_2 \uplus C_5$, we see that $e(a_2a_3, Q - x_1) \leq 3$ and $e(a_1a_5, Q - x_1) \leq 3$ by Lemma 2.1(c). As $e(Q, L) \geq 14$, it follows that $e(x_1, L) = 5$, $e(a_4, Q) = 4$, $e(a_2a_3, Q - x_1) = 3$ and

$e(a_1a_5, Q-x_1) = 3$. Then $[x_0, x_1, a_5, a_1, a_2] \supseteq C_5$ and so $e(a_3a_4, Q-x_1) \leq 3$. Thus $e(a_3, Q-x_1) = 0$ as $e(a_4, Q-x_1) = 3$. Similarly, $e(a_5, Q-x_1) = 0$. Thus $e(a_1a_2, Q-x_1) = 6$. Then $[a_1, x_2, x_3, a_4, a_5] \supseteq C_5$ and $[a_3, a_2, x_0, x_1, x_4] \supseteq F_2$, a contradiction. Hence $x_1a_4 \notin E$.

Next, suppose $e(x_3, a_1a_2) = 2$. Then $e(x_i, a_1a_3) \leq 1$ and $e(x_i, a_2a_5) \leq 1$ for each $i \in \{2,4\}$ as $H \not\supseteq 2C_5$. Thus $e(x_2x_4, L-a_4) \leq 4$ and so $e(x_1, L-a_4) + e(x_3, L) + e(a_4, x_2x_4) \geq 10$. Then $e(x_1, a_1a_2) \geq 1$. Thus $[x_i, x_1, x_0, a_1, a_2] \supseteq F_1$ for $i \in \{2,4\}$. Clearly, $e(x_3, a_3a_5) \geq 1$. Assume $e(x_3, a_3a_5) = 2$. Then $e(x_2x_4, a_3a_5) = 0$ as $H \not\supseteq F_1 \uplus C_5$. If $e(a_4, x_2x_4) = 1$, then $e(x_1, L-a_4) = 4$, $e(x_3, L) = 5$ and $e(x_2x_4, a_1a_2) = 4$. Thus $[x_0, x_1, x_4, a_2, a_3] \supseteq F_2$ and $[x_3, a_4, a_5, a_1, x_2] \supseteq C_5$, a contradiction. Hence $e(a_4, x_2x_4) = 2$. If $x_3a_4 \in E$ then $[x_2, x_3, x_4, a_4, a_i] \supseteq F_2$ for $i \in \{3,5\}$. As $e(x_1, a_3a_5) \geq 1$, we see that $H \supseteq F_2 \uplus C_5$, a contradiction. Thus $x_3a_4 \notin E$, $e(x_1, L-a_4) = 4$, $e(x_3, L-a_4) = 4$, $e(a_4, x_2x_4) = 2$ and $e(x_2x_4, a_1a_2) = 4$. Thus $[x_0, x_1, x_4, a_2, a_3] \supseteq F_2$ and $[x_3, a_1, a_5, a_4, x_2] \supseteq C_5$, a contradiction. We conclude that $e(x_3, a_3a_5) = 1$. Thus $e(x_1, L-a_4) = 4$, $e(x_3, L) = 4$ and $e(a_4, x_2x_4) = 2$. Say w.l.o.g. $x_3a_5 \in E$. Then $[x_2, x_4, a_5, a_4, x_3] \supseteq F_2$ and $[x_0, x_1, a_1, a_2, a_3] \supseteq C_5$, a contradiction. Therefore $e(x_3, a_1a_2) \leq 1$. Next, suppose that $e(x_2, a_1a_2) \geq 1$ and $e(x_4, a_1a_2) \geq 1$. Then $[x_i, x_1, x_0, a_1, a_2] \supseteq C_5$ for $i \in \{2,4\}$. Since $H \not\supseteq 2C_5$, $H \not\supseteq F_1 \uplus C_5$ and $H \not\supseteq F_2 \uplus C_5$, we see that $e(x_3x_i, a_3a_4a_5) \leq 3$ for $i \in \{2,4\}$ by Lemma 2.1(c). Furthermore, if for some $i \in \{2,4\}$, say $i = 2$, we have $e(x_2, a_3a_4a_5) = 3$, then $[x_2, a_3, a_4, a_5, a_j] \supseteq F_1$ for $j \in \{1,2\}$ and so $e(x_3, a_1a_2) = 0$ since $H \not\supseteq C_5 \uplus F_1$. Consequently, $e(x_1, L-a_4) = 4$, $e(x_2x_4, L) = 10$ and so $H \supseteq 2C_5$, a contradiction. Therefore if $e(x_3, a_3a_4a_5) = 0$ then $e(x_i, a_3a_4a_5) \leq 2$ for $i \in \{2,4\}$. Together with $x_1a_4 \notin E$ and $e(x_3, a_1a_2) \leq 1$, we see that if $e(x_3, a_3a_4a_5) = 0$ or $e(x_3, a_3a_4a_5) > 1$ then $e(Q, L) \leq 13$, a contradiction. Hence $e(x_3, a_3a_4a_5) = 1$. It follows that $e(x_1, L-a_4) = 4$, $e(x_3, a_1a_2) = 1$, $e(x_2x_4, a_1a_2) = 4$, $e(x_2, a_3a_4a_5) = 2$ and $e(x_4, a_3a_4a_5) = 2$. If $e(x_3, a_3a_5) = 1$, then either $[x_2, x_3, a_3, a_4, a_5] \supseteq C_5$ or $[x_2, x_3, a_3, a_4, a_5] \supseteq F_1$, and consequently, $H \supseteq C_5 \uplus F_1$, a contradiction. Hence $x_3a_4 \in E$. Then we see that $[x_2, x_3, a_4, a_5, a_1] \supseteq C_5$ and $[x_0, x_1, x_4, a_2, a_3] \supseteq F_2$, a contradiction. Therefore either $e(x_2, a_1a_2) = 0$ or $e(x_4, a_1a_2) = 0$. Say w.l.o.g. $e(x_4, a_1a_2) = 0$.

Finally, if $e(x_2, a_1a_2) \geq 1$ then, as above, we would have $e(x_3x_4, a_3a_4a_5) \leq 3$ and so $e(Q, L) \leq 13$, a contradiction. Hence $e(x_2, a_1a_2) = 0$. As $e(Q, L) \geq 14$, it follows that $e(x_1, L-a_4) = 4$, $e(x_3, L-a_i) = 4$ for some $i \in \{1,2\}$ and $e(x_2x_4, a_3a_4a_5) = 6$. As $[x_2, x_3, x_4, a_4, a_5] \supseteq C_5$, we see $H \supseteq 2C_5$, a contradiction.

Case 5. $N(x_0, L) = \{a_i, a_{i+1}, a_{i+2}\}$ for some $i \in \{1,2,3,4,5\}$.
Say $N(x_0, L) = \{a_1, a_2, a_3\}$. Then for each $i \in \{2,4,5\}$, $[L - a_i + x_0] \supseteq C_5$ or $[L - a_i + x_0] \supseteq F_1$ and so $e(a_i, Q) \leq 2$. Thus $e(a_1a_3, Q) \geq 7$. Hence $[Q + a_i] \supseteq C_5$ for each $i \in \{1,3\}$. Therefore $[L - a_i + x_0] \not\supseteq C_5$ and $[L - a_i + x_0] \not\supseteq B$ for each $i \in \{1,3\}$. This implies that $\tau(L) \leq 1$. As $e(a_1a_3, Q) \leq 8$, $e(a_4a_5, Q) \geq 3$. Say

w.l.o.g. $e(a_5, Q) = 2$. As $[Q + a_5] \not\supseteq C_5$, $N(a_5, Q) = \{x_2, x_4\}$ or $N(a_5, Q) = \{x_1, x_3\}$. First, assume $N(a_5, Q) = \{x_2, x_4\}$. Then $[a_4, a_5, x_2, x_3, x_4] \supseteq F$. As $e(a_1a_3, Q) \geq 7$, $e(x_1, a_1a_3) \geq 1$ and so $[x_0, x_1, a_1, a_2, a_3] \supseteq C' \cong C_5$ with $\tau(C') \geq 2$, contradicting the optimality of $\{D, L\}$. Hence $N(a_5, Q) = \{x_1, x_3\}$. Then $[a_4, a_5, x_1, x_i, x_3] \supseteq F$ for each $i \in \{2, 4\}$. By the optimality of $\{D, L\}$ and Lemma 2.1(b), we get $e(x_i, a_1a_3) \leq 1$ for each $i \in \{2, 4\}$ and so $e(a_1a_3, Q) \leq 6$, a contradiction.

Case 6. $N(x_0, L) = \{a_i, a_{i+1}, a_{i+3}\}$ for some $i \in \{1, 2, 3, 4, 5\}$.
Say $N(x_0, L) = \{a_1, a_2, a_4\}$. Clearly, $x_0 \to (L, a_3)$ and $x_0 \to (L, a_5)$. Thus $e(a_3, Q) \leq 2$ and $e(a_5, Q) \leq 2$ for otherwise $H \supseteq 2C_5$. As $H \not\supseteq 2C_5$, we see that $x_0 \not\to L$ and so $a_3a_5 \notin E$. As $e(Q, L) \geq 13$, $e(a_3a_5, Q) \geq 1$. Say w.l.o.g. $e(a_5, Q) \geq 1$. Then $[Q + a_5] \supseteq F$. By the optimality of $\{D, L\}$, $\tau(L) \geq \tau(x_0a_1a_2a_3a_4x_0)$. This implies that $a_2a_5 \in E$. Similarly, if $e(a_3, Q) \geq 1$ then $a_1a_3 \in E$. Assume $a_1a_3 \notin E$. Then $e(a_3, Q) = 0$ and so $e(a_1a_2a_4, Q) \geq 11$. Then $e(a_r, Q) = 4$ for some $r \in \{1, 2\}$ and $[L - a_r + x_0] \supseteq F$. As $\tau(a_rx_1x_2x_3x_4a_r) \geq 3$, it follows that $\tau(L) = 3$ and so $\{a_1a_4, a_2a_4\} \subseteq E$. Thus $[L - a_1 + x_0] \supseteq F_2$ and $[Q + a_1] \supseteq C_5$, a contradiction. Therefore $a_1a_3 \in E$. Thus $[L - a_4 + x_0] \supseteq F_2$. Hence $[Q + a_4] \not\supseteq C_5$ and so $e(a_4, Q) \leq 2$. Consequently, $e(a_1a_2, Q) \geq 7$ and so $[Q + a_i] \supseteq C_5$ for each $i \in \{1, 2\}$. Hence $a_1a_4 \notin E$ and $a_2a_4 \notin E$ for otherwise $H \supseteq F_2 \uplus C_5$. Hence $\tau(L) = 2$. By the optimality of $\{D, L\}$, $[Q + a_i] \not\supseteq C$ with $C \cong C_5$ and $\tau(C) \geq 3$ for each $i \in \{1, 2\}$. This implies that $e(a_i, Q) \leq 3$ for each $i \in \{1, 2\}$ and therefore $e(a_1a_2, Q) \leq 6$, a contradiction. ■

Lemma 2.6. *Let D, L_1 and L_2 be disjoint subgraphs of G with $D \cong F$ and $L_1 \cong L_2 \cong C_5$. Suppose that $L_1 = a_1a_2a_3a_4a_5a_1$, $V(D) = \{x_0, x_1, x_2, x_3, x_4\}$ and $E(D) = \{x_0x_1, x_1x_2, x_2x_3, x_3x_4, x_4x_1\}$ such that $e(x_0, L_1) = 0$, and $e(x_1x_3, L_1) = 10$, $N(x_2, L_1) = N(x_4, L_1) = \{a_1, a_2, a_4\}$, $\tau(L_1) = 4$ and $a_3a_5 \notin E$. Suppose that $e(x_0x_2a_3a_5, L_2) \geq 13$. Then $[D, L_1, L_2]$ contains either of $F_1 \uplus 2C_5$ or $3C_5$.*

Proof. For the proof, we may assume that none of x_0x_3, x_1x_3 and x_2x_4 is an edge as they will not be used in the proof. Set $G_1 = [D, L_1]$, $G_2 = [G_1, L_2]$ and $R = \{x_0, x_2, a_3, a_5\}$. It is easy to see that for any permutation f of $\{x_2, a_3, a_5\}$, we can extend f to be an automorphism of G_1 such that every vertex of $G_1 - \{x_2, a_3, a_5\}$ is fixed under f. Therefore x_2, a_3 and a_5 are in the symmetric position in the following argument. On the contrary, suppose that $G_2 \not\supseteq F_1 \uplus 2C_5$ and $G_2 \not\supseteq 3C_5$. It is easy to check that if $u \to (L_2; R - \{u\})$ for some $u \in R$ then $G_2 \supseteq F_1 \uplus 2C_5$ or $G_2 \supseteq 3C_5$. Therefore $u \not\to (L_2; R - \{u\})$ for each $u \in R$. By Lemma 2.1(d), there exist two labellings $R = \{y_1, y_2, y_3, y_4\}$ and $L_2 = b_1b_2b_3b_4b_5b_1$ such that $e(y_1y_2, b_1b_2b_3b_4) = 8$, $e(y_3, b_1b_5b_4) = 3$ and $e(y_4, b_1b_4) = 2$. If $x_0 \in \{y_1, y_2\}$, we may assume that $\{y_1, y_2\} = \{x_0, x_2\}$. Then $[x_0, x_1, x_2, b_2, b_3] \supseteq C_5$, $[a_3, a_5, b_1, b_5, b_4] \supseteq C_5$ and $[x_3, x_4, a_1, a_2, a_4] \supseteq C_5$, a contradiction. Hence $x_0 \notin$

$\{y_1, y_2\}$. Say w.l.o.g. that $\{y_1, y_2\} = \{a_3, a_5\}$. Thus $[a_3, a_4, a_5, b_2, b_3] \supseteq C_5$, $[x_0, x_2, b_1, b_5, b_4] \supseteq C_5$ and $[x_1, x_4, x_3, a_1, a_2] \supseteq C_5$, a contradiction. ∎

Lemma 2.7. *Let D and L be disjoint subgraphs of G with $D \cong K_4^+$ and $L \cong B$. Let R be the set of the four vertices of L with degree 2 in L. Suppose that $e(D, R) \geq 13$. Then either $[D, L] \supseteq K_4^+ \uplus C_5$ or $[D, L] \supseteq 2C_5$ or $[D, L] \supseteq B \uplus C_5$.*

Proof. Say $H = [D, L]$. On the contrary, suppose that H contains none of $K_4^+ \uplus C_5$, $2C_5$ and $B \uplus C_5$. Say $V(D) = \{x_0, x_1, x_2, x_3, x_4\}$ with $e(x_0, D) = 1$ and $x_0x_1 \in E$. Let $Q = [x_1, x_2, x_3, x_4]$. Say $L = a_0a_1a_2a_0a_3a_4a_0$. Then $Q \cong K_4$ and $R = \{a_1, a_2, a_3, a_4\}$. If $e(x_0, R) \geq 3$, say w.l.o.g. $e(x_0, a_1a_2a_3) = 3$, then $[L - a_i + x_0] \supseteq C_5$ and so $Q + a_i \not\supseteq C_5$ for each $i \in \{1, 2, 4\}$. Consequently, $e(a_i, Q) \leq 1$ for all $i \in \{1, 2, 4\}$ and so $e(D, R) \leq 11$, a contradiction. Hence $e(x_0, R) \leq 2$. Suppose that $e(x_0, R) = 2$. Then $e(R, Q) \geq 11$. First, assume $e(x_0, a_1a_2) = 1$ and $e(x_0, a_3a_4) = 1$. Say w.l.o.g. $e(x_0, a_1a_3) = 2$. Then $e(a_2, Q) \leq 1$ and $e(a_4, Q) \leq 1$ as $H \not\supseteq 2C_5$. Consequently, $e(R, Q) \leq 10$, a contradiction. Therefore we may assume w.l.o.g. that $e(x_0, a_1a_2) = 2$. We claim $e(x_1, a_1a_2) = 0$. To see this, suppose $e(x_1, a_1a_2) \geq 1$. Then $[x_0, x_1, a_1, a_2, a_0] \supseteq C_5$. Thus $e(a_3a_4, x_2x_3x_4) \leq 2$ for otherwise $[a_3, a_4, x_2, x_3, x_4] \supseteq C_5$ or $[a_3, a_4, x_2, x_3, x_4] \supseteq K_4^+$. Thus $e(a_3a_4, Q) \leq 4$ and so $e(a_1a_2, Q) \geq 7$. Say w.l.o.g. $e(a_1, Q) = 4$. Then $[D - x_i + a_1] \supseteq K_4^+$ for each $i \in \{2, 3, 4\}$ and so $[L - a_1 + x_i] \not\supseteq C_5$ for each $i \in \{2, 3, 4\}$. Thus $I(a_2a_3, Q - x_1) = \emptyset$ and so $e(a_2a_3, Q) \leq 5$. Hence $e(a_4, Q) \geq 2$. Similarly, $e(a_3, Q) \geq 2$. It follows that $[a_3, a_4, x_2, x_3, x_4] \supseteq C_5$ or $[a_3, a_4, x_2, x_3, x_4] \supseteq B$, a contradiction. This shows that $e(x_1, a_1a_2) = 0$. Suppose $e(a_1, Q - x_1) = 3$ or $e(a_2, Q - x_1) = 3$. Then $[x_0, x_1, x_i, a_1, a_2] \supseteq C_5$ for each $i \in \{2, 3, 4\}$. Thus $[x_i, x_j, a_0, a_3, a_4] \not\supseteq C_5$ and $[x_i, x_j, a_0, a_3, a_4] \not\supseteq B$ for each $2 \leq i < j \leq 4$. This implies that $e(a_3a_4, Q - x_1) \leq 2$. Hence $e(a_1a_2, Q) \geq 7$ and so $e(x_1, a_1a_2) \geq 1$, a contradiction. Hence $e(a_i, Q - x_1) \leq 2$ for each $i \in \{1, 2\}$ and so $e(a_3a_4, Q) \geq 7$. Say w.l.o.g. $e(a_4, Q) = 4$. Then $[D - x_i + a_4] \supseteq K_4^+$ for each $i \in \{2, 3, 4\}$ and therefore $I(a_1a_3, Q - x_1) = \emptyset$ as $H \not\supseteq K_4^+ \uplus C_5$. Thus $e(a_1a_3, Q) \leq 4$ and so $e(a_2, Q) \geq 3$, a contradiction. Next, suppose $e(x_0, R) = 1$. Then $e(Q, R) \geq 12$. Say $x_0a_1 \in E$. Suppose $e(x_1, a_1a_2) \geq 1$. Then $[x_0, x_1, a_1, a_2, a_0] \supseteq C_5$ or $[x_0, x_1, a_1, a_2, a_0] \supseteq B$. Thus $[x_2, x_3, x_4, a_3, a_4] \not\supseteq C_5$. This implies that $e(a_3a_4, Q - x_1) \leq 3$. Thus $e(a_3a_4, Q) \leq 5$ and so $e(a_1a_2, Q) \geq 7$. Thus $[D - x_i + a_1] \supseteq C_5$ for all $i \in \{2, 3, 4\}$. As $H \not\supseteq 2C_5$, $I(a_2a_3, Q - x_1) = \emptyset$ and $I(a_2a_4, Q - x_1) = \emptyset$. Hence $e(a_2a_3, Q) \leq 5$ and so $e(a_4, Q) \geq 3$. Then $I(a_2a_4, Q - x_1) \neq \emptyset$, a contradiction. Hence $e(x_1, a_1a_2) = 0$. Thus $e(a_1a_2, Q) \leq 6$ and $e(a_3a_4, Q) \geq 6$. Then $[x_i, x_j, a_3, a_4, a_0] \supseteq C_5$ for some $2 \leq i < j \leq 4$. Say $\{i, j, k\} = \{2, 3, 4\}$. Then $a_2x_k \notin E$ as $H \not\supseteq 2C_5$. Therefore $e(a_1a_2, Q) \leq 5$ and so $e(a_3a_4, Q) \geq 7$. Thus $[x_r, x_t, a_3, a_4, a_0] \supseteq C_5$ for all $2 \leq r < t \leq 4$. Therefore $e(a_2, Q - x_1) = 0$ as $H \not\supseteq 2C_5$. Consequently, $e(Q, R) \leq 11$, a contradiction.

Finally, suppose $e(x_0, R) = 0$. As $e(R, Q) \geq 13$, $e(a_i, Q) = 4$ for some $a_i \in R$.

Say $e(a_1, Q) = 4$. Then $I(a_2a_3, Q - x_1) = \emptyset$ as $H \not\supseteq K_4^+ \uplus C_5$. Thus $e(a_4, Q) = 4$ as $e(R, Q) \geq 13$. Similarly, $e(a_3, Q) = 4$. Then we readily see that $H \supseteq K_4^+ \uplus C_5$, a contradiction. ∎

Lemma 2.8. *Let B_1 and B_2 be disjoint subgraphs of G such that $B_1 \cong B$ and $B_2 \cong B$. Let R be the set of the four vertices of B_1 with degree 2 in B_1. Suppose that $e(R, B_2) \geq 13$. Then $[B_1, B_2] \supseteq 2C_5$ or $[B_1, B_2] \supseteq B \uplus C_5$.*

Proof. On the contrary, suppose that $[B_1, B_2] \not\supseteq 2C_5$ and $[B_1, B_2] \not\supseteq B \uplus C_5$. Say $B_1 = a_0a_1a_2a_0a_3a_4a_0$ and $B_2 = b_0b_1b_2b_0b_3b_4b_0$. Then $R = \{a_1, a_2, a_3, a_4\}$ and $e(R, B_2 - b_0) \geq 9$. This implies that $e(a_ia_{i+1}, b_jb_{j+1}) \geq 3$ for some $i \in \{1, 3\}$ and $j \in \{1, 3\}$. Say w.l.o.g. $e(a_1a_2, b_1b_2) \geq 3$. Then $[a_1, a_2, b_0, b_1, b_2] \supseteq C_5$ and $[b_1, b_2, a_0, a_1, a_2] \supseteq C_5$.

Therefore $[a_0, a_3, a_4, b_3, b_4] \not\supseteq C_5$, $[a_0, a_3, a_4, b_3, b_4] \not\supseteq B$, $[b_0, b_3, b_4, a_3, a_4] \not\supseteq C_5$ and $[b_0, b_3, b_4, a_3, a_4] \not\supseteq B$. This implies that $e(a_3a_4, b_3b_4) \leq 1$ and $e(b_0, a_3a_4) \leq 1$. If $e(a_1a_2, b_3b_4) \geq 3$, then we also have that $e(a_3a_4, b_1b_2) \leq 1$ and it follows that $e(a_1a_2, B_2) = 10$ and $e(a_3a_4, b_3b_4) = 1$ as $e(R, B_2) \geq 13$. Consequently, $[B_2 - b_r + a_1] \supseteq C_5$ and $[B_1 - a_1 + b_r] \supseteq C_5$ where $r \in \{3, 4\}$ with $e(b_r, a_3a_4) = 1$, a contradiction. Hence $e(a_1a_2, b_3b_4) \leq 2$. Suppose $e(a_3a_4, b_1b_2) \geq 3$. Similarly, we shall have $e(a_1a_2, b_3b_4) \leq 1$, $e(b_0, a_1a_2) \leq 1$ and so $e(R, B_2) \leq 12$, a contradiction. Therefore, $e(a_3a_4, b_1b_2) \leq 2$. Thus $e(a_3a_4, B_2) \leq 4$ and so $e(a_1a_2, B_2) \geq 9$. Consequently, $e(a_1a_2, b_3b_4) \geq 3$, a contradiction. ∎

Lemma 2.9. *Let D and L be disjoint subgraphs of G with $D \cong F_1$ and $L \cong C_5$. Suppose that $\{D, L\}$ is optimal and $e(D, L) \geq 16$. Then $[D, L]$ contains one of $K_4^+ \uplus C_5$, $K_4^+ \uplus B$, $2C_5$ and $B \uplus C_5$, or there exist two labellings $L = a_1a_2a_3a_4a_5a_1$ and $V(D) = \{x_0, x_1, x_2, x_3, x_4\}$ with $E(D) = \{x_0x_1, x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_2x_4\}$ such that $e(x_0, L) = 0$, $e(a_1a_2a_4, D - x_0) = 12$, $N(a_3, D) = N(a_5, D) = \{x_2, x_4\}$, $\tau(L) = 4$ and $a_3a_5 \notin E$.*

Proof. Say $H = [D, L]$. Say that H does not contain any of $K_4^+ \uplus C_5$, $K_4^+ \uplus B$, $2C_5$ and $B \uplus C_5$.
Let $V(D) = \{x_0, x_1, x_2, x_3, x_4\}$, $E(D) = \{x_0x_1, x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_2x_4\}$ and $L = a_1a_2a_3a_4a_5a_1$, Set $Q = [x_1, x_2, x_3, x_4]$. Since $H \not\supseteq 2C_5$ and $H \not\supseteq B \uplus C_5$, we see that for each $a_i \in V(L)$, if $x_0 \to (L, a_i)$ or $x_0 \xrightarrow{z} (L, a_i)$ then $e(a_i, Q) \leq 2$. Thus $x_0 \not\to L$ for otherwise $e(D, L) \leq 15$. Hence $e(x_0, L) \leq 4$.

Assume $e(x_0, L) = 4$. Say $e(x_0, a_1a_2a_3a_4) = 4$. As $x_0 \not\to L$, $\tau(a_5, L) = 0$. Clearly, $e(a_i, Q) \leq 2$ for each $i \in \{2, 3, 5\}$ since $H \not\supseteq 2C_5$. Thus $e(a_1a_4, Q) \geq 6$. Say $e(a_1, Q) \geq 3$. Then $[Q + a_1] \supseteq C$ with $C \cong C_5$ and $\tau(C) \geq 3$. Then $a_2a_4 \notin E$ for otherwise $[L - a_1 + x_0] \supseteq K_4^+$. Thus $\tau(L) \leq 2$. As $[L - a_1 + x_0] \supseteq F_1$, we see that $2 \geq \tau(L) \geq \tau(C) \geq 3$ by the optimality of $\{D, L\}$, a contradiction. Therefore $e(x_0, L) \leq 3$ and so $e(Q, L) \geq 13$. Set $T = x_2x_3x_4x_2$. We divide the proof into the following six cases.

Case 1. $N(x_0, L) = \{a_i, a_{i+1}, a_{i+2}\}$ for some $i \in \{1, 2, 3, 4, 5\}$.
Say $N(x_0, L) = \{a_1, a_2, a_3\}$. Then $Q + a_2 \not\supseteq C_5$ and so $e(a_2, Q) \le 2$. As $x_0 \not\to L$, we see that $\tau(a_2, L) \le 1$. If $\{a_1a_4, a_3a_5\} \subseteq E$ then $x_0 \to (L, a_i)$ or $x_0 \overset{z}{\to} (L, a_i)$ and so $e(a_i, Q) \le 2$ for each $a_i \in V(L)$. Consequently, $e(Q, L) \le 10$, a contradiction. Hence $a_1a_4 \notin E$ or $a_3a_5 \notin E$. Thus $\tau(L) \le 3$. Suppose $\tau(a_2, L) = 1$. Say w.l.o.g. $a_2a_4 \in E$. Then $x_0 \to (L, a_i)$ for $i \in \{3, 5\}$. Thus $e(a_i, Q) \le 2$ for $i \in \{3, 5\}$. As $e(Q, L) \ge 13$, $e(a_1a_4, Q) \ge 7$. Thus $[Q + a_r]$ contains a 5-cycle with at least 4 chords, where $e(a_r, Q) = 4$ with $r \in \{1, 4\}$. As $[L - a_r + x_0] \supseteq F_1$ and by the optimality of $\{D, L\}$, we have $\tau(L) \ge 4$, a contradiction. Hence $\tau(a_2, L) = 0$. Suppose $a_1a_3 \in E$. Then $[L - a_i + x_0] \supseteq K_4^+$ for each $i \in \{4, 5\}$. As $H \not\supseteq K_4^+ \uplus C_5$, $e(a_i, Q) \le 2$ for $i \in \{4, 5\}$. As $e(Q, L) \ge 13$, $e(a_1a_3, Q) \ge 7$ and $e(a_4a_5, Q) \ge 3$. Say w.l.o.g. $e(a_5, Q) = 2$. As $[Q + a_5] \not\supseteq C_5$, $e(a_5, x_2x_4) = 2$. As $e(x_1, a_1a_3) \ge 1$, $[x_1, x_0, a_1, a_2, a_3] \supseteq C_5$. Thus $e(a_4, T) = 0$ as $H \not\supseteq 2C_5$. It follows that $e(a_1a_3, Q) = 8$ and $a_4x_1 \in E$. Consequently, $H \supseteq 2C_5$, a contradiction. Hence $a_1a_3 \notin E$ and so $\tau(L) \le 1$. Since $[L - a_i + x_0] \supseteq F_1$ for each $i \in \{4, 5\}$, we see that $[Q + a_i]$ does not contain a 5-cycle with at least 2 chords for each $i \in \{4, 5\}$ by the optimality of $\{D, L\}$. This implies that for each $i \in \{4, 5\}$, $e(a_i, Q) \le 2$ and if $e(a_i, Q) = 2$ then $e(a_i, x_2x_4) = 2$. Similar to the above, we see that $H \supseteq 2C_5$, a contradiction.

Case 2. $N(x_0, L) = \{a_i, a_{i+1}, a_{i+3}\}$ for some $i \in \{1, 2, 3, 4, 5\}$.
Say $N(x_0, L) = \{a_1, a_2, a_4\}$. Then for each $i \in \{3, 5\}$, $x_0 \to (L, a_i)$ and so $e(a_i, Q) \le 2$. Thus $e(a_1a_2a_4, Q) \ge 13 - e(a_3a_5, Q) \ge 9$. Suppose that $e(a_3, Q) = 2$ or $e(a_5, Q) = 2$. Say w.l.o.g. $e(a_5, Q) = 2$. Then $e(a_5, x_2x_4) = 2$ as $[Q + a_5] \not\supseteq C_5$. If $a_3x_3 \in E$ then $[a_3, a_4, a_5, x_3, x_i] \supseteq C_5$ for $i \in \{2, 4\}$ and so $e(x_i, a_1a_2) = 0$ for $i \in \{2, 4\}$ since $H \not\supseteq 2C_5$. Consequently, $e(a_1a_2a_4, Q) \le 8$, a contradiction. Hence $a_3x_3 \notin E$. If $a_3x_1 \in E$ then $[x_1, x_0, a_1, a_2, a_3] \supseteq C_5$ and so $e(a_4, T) = 0$ as $H \not\supseteq 2C_5$. Thus $e(a_1a_2a_4, Q) = 9$ and so $e(a_3, Q) = 2$. Consequently, $[Q + a_3] \supseteq C_5$, a contradiction. Hence $N(a_3, Q) \subseteq \{x_2, x_4\}$. If $e(x_1, a_2a_4) \ge 1$ then $[x_1, x_0, a_2, a_3, a_4] \supseteq C_5$ and so $e(a_1, T) = 0$ as $H \not\supseteq 2C_5$. It follows that $e(a_3, x_2x_4) = 2$ and $e(a_2a_4, Q) = 8$. Consequently, $H \supseteq 2C_5$, a contradiction. Hence $e(x_1, a_2a_4) = 0$. Thus $e(a_2a_4, T) \ge 5$ as $e(a_1a_2a_4, Q) \ge 9$. Hence $[x_3, x_4, a_2, a_3, a_4] \supseteq C_5$ and $[x_0, x_1, x_2, a_5, a_1] \supseteq C_5$, a contradiction.

Therefore $e(a_3, Q) \le 1$ and $e(a_5, Q) \le 1$. Then $e(a_1a_2a_4, Q) \ge 11$. Thus $e(a_1a_2, Q) \ge 7$. Say w.l.o.g. $e(a_1, Q) = 4$. Then $[a_5, a_1, x_2, x_3, x_4] \supseteq K_4^+$. As $e(x_1, a_2a_4) \ge 1$, $[x_1, x_0, a_2, a_3, a_4] \supseteq C_5$ and so $H \supseteq K_4^+ \uplus C_5$, a contradiction.

Case 3. $N(x_0, L) = \{a_i, a_{i+1}\}$ for some $i \in \{1, 2, 3, 4, 5\}$. In this case, $e(Q, L) \ge 14$. Say $e(x_0, a_1a_2) = 2$. Suppose $x_1a_4 \in E$. Then $[x_1, x_0, a_1, a_5, a_4] \supseteq C_5$. As H does not contain one of $2C_5$ and $K_4^+ \uplus C_5$, we see that $e(a_2a_3, T) \le 2$. Similarly, $e(a_1a_5, T) \le 2$ as $[x_1, x_0, a_2, a_3, a_4] \supseteq C_5$. Thus $e(Q, L) \le 12$, a contradiction. Hence $x_1a_4 \notin E$. Next, suppose that $e(x_1, a_3a_5) \ge 1$. Say w.l.o.g. $x_1a_3 \in E$. Then $[x_1, x_0, a_1, a_2, a_3] \supseteq C_5$. As H does not contain one of $2C_5$,

$B \uplus C_5$ and $K_4^+ \uplus C_5$, we have that $e(a_4a_5, T) \leq 2$ and either $e(a_4, T) = 0$ or $e(a_5, T) = 0$. If we also have $x_1a_5 \in E$ then $e(a_3a_4, T) \leq 2$ and either $e(a_4, T) = 0$ or $e(a_3, T) = 0$. Consequently, it follows, as $e(Q, L) \geq 14$, that $e(a_5, T) = 2$, $e(a_3, T) = 2$, $e(a_4, T) = 0$ and $e(a_1a_2, Q) = 8$. Then $x_i \to (L, a_1)$ for some $x_i \in V(T)$ with $e(x_i, a_2a_5) = 2$ and so $H \supseteq 2C_5$, a contradiction. Hence $x_1a_5 \notin E$. Thus $e(a_1a_2a_3, Q) \geq 12$. Then $x_3 \to (L, a_2)$ and so $H \supseteq 2C_5$, a contradiction. We conclude that $e(x_1, a_3a_4a_5) = 0$.

As $e(Q, L) \geq 14$, $e(x_2x_4, a_1a_2) \geq 1$. Say w.l.o.g. $e(x_2, a_1a_2) \geq 1$. Then $[x_2, x_1, x_0, a_1, a_2] \supseteq C_5$. As $H \not\supseteq 2C_5$ and by Lemma 2.1(c), $e(x_3x_4, a_3a_4a_5) \leq 4$. Thus $e(a_3a_4a_5, Q) \leq 7$. Hence $e(a_1a_2, Q) \geq 7$. Say w.l.o.g. $e(a_1, Q) = 4$. Then $x_i \not\to (L, a_1)$ for each $x_i \in V(T)$ since $H \not\supseteq 2C_5$. This implies that $I(a_2a_5, T) = \emptyset$ and so $e(a_2a_5, Q) \leq 4$. Consequently, $e(a_3a_4, T) = 6$ as $e(Q, L) \geq 14$. Thus $[a_5, a_4, a_3, x_3, x_4] \supseteq K_4^+$ and $[x_2, x_1, x_0, a_2, a_1] \supseteq C_5$, a contradiction.

Case 4. $N(x_0, L) = \{a_i, a_{i+2}\}$ for some $i \in \{1, 2, 3, 4, 5\}$. Say, $N(x_0, L) = \{a_1, a_3\}$. The $e(a_2, Q) \leq 2$ as $H \not\supseteq 2C_5$. First, suppose $e(x_1, a_1a_3) \geq 1$. Then $[x_1, x_0, a_1, a_2, a_3] \supseteq C_5$ and therefore $e(a_4a_5, T) \leq 2$. Thus $e(a_1a_3, Q) \geq 14 - 2 - 2 - e(x_1, a_4a_5) \geq 8$. It follows that $e(a_1a_3, Q) = 8$, $e(a_2, Q) = 2$, $e(a_4a_5, T) = 2$ and $e(x_1, a_4a_5) = 2$. Consequently, $H \supseteq 2C_5$, a contradiction. Hence $e(x_1, a_1a_3) = 0$. Next, suppose $e(x_1, a_4a_5) \geq 1$. Say w.l.o.g. $x_1a_4 \in E$. Then $[x_1, x_0, a_1, a_5, a_4] \supseteq C_5$ and so $e(a_2a_3, T) \leq 2$. Thus $e(a_1a_5a_4, Q) \geq 14 - 3 = 11$. It follows that $e(a_4a_5, Q) = 8$, $e(a_1, T) = 3$, $x_1a_2 \in E$ and $e(a_2a_3, T) = 2$. Then $[D - x_1 + a_1] \supseteq K_4^+$ and $[L - a_1 + x_1] \supseteq C_5$, a contradiction. Hence $e(x_1, a_4a_5) = 0$. As $e(Q, L) \geq 14$, it follows that $e(T, L - a_2) = 12$ and $e(a_2, Q) = 2$. Then we readily see that $H \supseteq 2C_5$, a contradiction.

Case 5. $e(x_0, L) = 1$. Then $e(Q, L) \geq 15$. Say $x_0a_1 \in E$. First, suppose $e(x_1, a_3a_4) \geq 1$. Say w.l.o.g. $x_1a_3 \in E$. Then $[x_1, x_0, a_1, a_2, a_3] \supseteq C_5$. Thus $e(a_4a_5, T) \leq 2$ and so $e(a_4a_5, Q) \leq 4$. If we also have $x_1a_4 \in E$ then $e(a_2a_3, T) \leq 2$ as $[x_1, x_0, a_1, a_5, a_4] \supseteq C_5$. But then we obtain $e(Q, L) \leq 12$, a contradiction. Hence $x_1a_4 \notin E$. As $e(Q, L) \geq 15$, it follows that $e(a_1a_2a_3, Q) = 12$, $e(a_4a_5, T) = 2$ and $x_1a_5 \in E$. Then $[a_4, a_5, x_1, x_0, a_1] \supseteq F_1$ and $[T, a_2, a_3] \supseteq K_5$. By the optimality of $\{D, L\}$, $[L] \cong K_5$ and so $H \supseteq 2C_5$, a contradiction. Hence $e(x_1, a_3a_4) = 0$. Then $e(a_2a_5, Q) \geq 15 - e(a_1a_3a_4, Q) \geq 15 - 10 = 5$. Thus $e(x_2x_4, a_2a_5) \geq 1$. Say w.l.o.g. $x_2a_5 \in E$. Then $[x_0, x_1, x_2, a_5, a_1] \supseteq C_5$. As $H \not\supseteq 2C_5$, $e(a_2a_4, x_3x_4) \leq 2$. Clearly, $e(a_2a_3a_4, x_1x_2) \leq 4$. Then $e(a_1a_5, Q) \geq 15 - 6 - e(a_3, x_3x_4) \geq 7$ and so $e(a_1, T) \geq 2$. Suppose that $a_1x_3 \in E$. Then $x_i \not\to (L, a_1)$ for all $x_i \in V(T)$ for otherwise $H \supseteq 2C_5$. This implies that $I(a_2a_5, T) = \emptyset$. As $x_2a_5 \in$, $x_2a_2 \notin E$ and so $e(a_2a_3a_4, x_1x_2) \leq 3$. As $e(Q, L) \geq 15$, it follows that $e(a_1a_5, Q) = 8$, $e(a_2a_3a_4, x_3x_4) = 4$ and so $e(x_3x_4, a_3a_4) = 4$. Thus $[a_2, a_3, a_4, x_3, x_4] \supseteq K_4^+$ and so $H \supseteq K_4^+ \uplus C_5$, a contradiction. Hence $a_1x_3 \notin E$. Thus $e(a_1a_5, Q) = 7$. It follows that $e(a_1, Q - x_3) = 3$, $e(a_5, Q) = 4$, $e(a_2a_4, x_3x_4) = 2$, $e(a_3, x_3x_4) = 2$, $e(x_2, a_3a_4) = 2$ and $e(a_2, x_1x_2) = 2$. Then

$[x_2, x_1, x_0, a_1, a_2] \supseteq C_5$ and $[a_5, a_4, a_3, x_3, x_4] \supseteq C_5$, a contradiction.

Case 6. $e(x_0, L) = 0$. As $H \not\supseteq K_4^+ \uplus C_5$, we see that for each $a_i \in V(L)$, if $e(a_i, Q - x_3) = 3$ then $x_3 \not\to (L, a_i)$. Since $e(a_i, Q) = 4$ for some $a_i \in V(L)$ as $e(Q, L) \geq 16$, it follows that $x_3 \not\to L$ and so $e(x_3, L) \leq 4$. First, suppose $e(x_3, L) = 4$. Say $e(x_3, L - a_5) = 4$. Then $e(a_i, Q - x_3) \leq 2$ for each $i \in \{2, 3, 5\}$. As $e(Q, L) \geq 16$, it follows that $e(a_i, Q - x_3) = 2$ for $i \in \{2, 3, 5\}$ and $e(a_1a_4, Q - x_3) = 6$. If $x_1a_5 \in E$, then $e(a_5, x_1x_2) = 2$ or $e(a_5, x_1x_4) = 2$. Say w.l.o.g. $e(a_5, x_1x_2) = 2$. Then $[x_0, x_1, x_2, a_1, a_5] \supseteq K_4^+$ and $[x_3, x_4, a_2, a_3, a_4] \supseteq C_5$, a contradiction. Hence $e(a_5, x_2x_4) = 2$. Then $[D - x_3 + a_5] \supseteq F_1$. By the optimality of $\{D, L\}$, $\tau(L) \geq \tau(x_3a_1a_2a_3a_4x_3)$. This implies that $\tau(a_5, L) = 2$ and so $x_3 \to (L, a_1)$, a contradiction.

Next, suppose that $e(x_3, L) = 3$ and $N(x_3, L) = \{a_i, a_{i+1}, a_{i+3}\}$ for some $i \in \{1, 2, 3, 4, 5\}$. Say $N(x_3, L) = \{a_1, a_2, a_4\}$. Then $e(a_3, Q - x_3) \leq 2$ and $e(a_5, Q - x_3) \leq 2$. As $e(Q, L) \geq 16$, it follows that $e(a_1a_2a_4, Q - x_3) = 9$, $e(a_3, Q - x_3) = 2$ and $e(a_5, Q - x_3) = 2$. If $e(x_1, a_3a_5) \geq 1$, then we may assume w.l.o.g. that $e(a_3, x_1x_2) = 2$. Consequently, $[x_0, x_1, x_2, a_2, a_3] \supseteq K_4^+$ and $[x_3, x_4, a_1, a_5, a_4] \supseteq C_5$, a contradiction. Hence $e(a_3a_5, x_2x_4) = 4$. Clearly, $[x_0, x_1, x_2, a_2, a_3] \supseteq F_1$ and $\tau(x_4x_3a_1a_5a_4x_4) \geq 3$. Thus $\tau(L) \geq 3$ by the optimality of $\{D, L\}$. As $x_3 \not\to (L, a_1)$, $a_3a_5 \notin E$. Thus $a_1a_4 \in E$ or $a_2a_4 \in E$. Say w.l.o.g. $a_1a_4 \in E$. Then $\tau(x_4x_3a_1a_5a_4x_4) = 4$. Thus $\tau(L) = 4$ and so the lemma holds.

Next, suppose that $N(x_3, L) = \{a_i, a_{i+1}, a_{i+2}\}$ for some $i \in \{1, 2, 3, 4, 5\}$. Say $N(x_3, L) = \{a_1, a_2, a_3\}$. Then $e(a_2, Q - x_3) \leq 2$. As $e(D, L) \geq 16$, either $e(a_1a_5, Q - x_3) = 6$ or $e(a_3a_4, Q - x_3) = 6$. Say w.l.o.g. $e(a_1a_5, Q - x_3) = 6$. Then $[x_0, x_1, x_i, a_1, a_5] \supseteq K_4^+$ and so $[x_3, x_j, a_2, a_3, a_4] \not\supseteq C_5$ for each $\{i, j\} = \{2, 4\}$. This implies that $e(a_4, x_2x_4) = 0$ and so $e(D, L) \leq 15$, a contradiction.

Next, suppose that $e(x_3, L) = 2$ and $N(x_3, L) = \{a_i, a_{i+2}\}$ for some $i \in \{1, 2, 3, 4, 5\}$. Say $N(x_3, L) = \{a_1, a_3\}$. Then $e(a_2, Q - x_3) \leq 2$. As $e(Q, L) \geq 16$, it follows that $e(L - a_2, Q - x_3) = 12$ and $e(a_2, Q - x_3) = 2$. Then $[x_0, x_1, x_2, a_4, a_5] \supseteq K_4^+$ and $[x_3, x_4, a_1, a_2, a_3] \supseteq C_5$, a contradiction.

Next, suppose that $e(x_3, L) = 2$ and $N(x_3, L) = \{a_i, a_{i+1}\}$ for some $i \in \{1, 2, 3, 4, 5\}$. Say $N(x_3, L) = \{a_1, a_2\}$. As $e(Q, L) \geq 16$, either $e(a_1a_5, Q - x_3) = 6$ or $e(a_2a_3, Q - x_3) = 6$. Say w.l.o.g. $e(a_1a_5, Q - x_3) = 6$. Then $[x_0, x_1, x_i, a_1, a_5] \supseteq K_4^+$ and so $[x_j, x_3, a_2, a_3, a_4] \not\supseteq C_5$ for each $\{i, j\} = \{2, 4\}$. This implies that $e(a_4, x_2x_4) = 0$. Consequently, $e(Q, L) \leq 15$, a contradiction.

Finally, we have $e(x_3, L) = 1$. Then $e(L, Q - x_3) = 15$, clearly, $H \supseteq K_4^+ \uplus C_5$, a contradiction. ■

Lemma 2.10. *Let D, L_1 and L_2 be disjoint subgraphs of G with $D \cong F_1$ and $L_1 \cong L_2 \cong C_5$. Suppose that $L_1 = a_1a_2a_3a_4a_5a_1$, $V(D) = \{x_0, x_1, x_2, x_3, x_4\}$ and $E(D) = \{x_0x_1, x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_2x_4\}$ such that*

$e(x_0,L_1)=0$, $e(a_1a_2a_4,D-x_0)=12$, $N(a_3,D)=N(a_5,D)=\{x_2,x_4\}$, $\tau(L_1)=4$ and $a_3a_5\notin E$. Suppose that $\{D,L_1,L_2\}$ is optimal and $e(x_0x_3a_3a_5,L_2)\geq 13$. Then $[D,L_1,L_2]$ contains either $K_4^+\uplus 2C_5$ or $3C_5$.

Proof. Let $G_1=[D,L_1]$, $G_2=[D,L_1,L_2]$ and $R=\{x_0,x_3,a_3,a_5\}$. On the contrary, suppose that G_2 does not contain any of $K_4^+\uplus 2C_5$ and $3C_5$. It is easy to see that for any permutation f of $\{x_3,a_3,a_5\}$, we can extend f to be an automorphism of G_1 such that any vertex in $G_1-\{x_3,a_3,a_5\}$ is fixed under f. Thus x_3, a_3 and a_5 are in the symmetric position in the following argument. It is easy to check that if $u\to(L_2;R-\{u\})$ for some $u\in R$, then $G_2\supseteq K_4^+\uplus 2C_5$ or $G_2\supseteq 3C_5$. Thus $u\not\to(L_2;R-\{u\})$ for each $u\in R$. By Lemma 2.1(d), there exist two labellings $R=\{y_1,y_2,y_3,y_4\}$ and $L_2=b_1b_2b_3b_4b_5b_1$ such that $e(y_1y_2,b_1b_2b_3b_4)=8$, $e(y_3,b_1b_5b_4)=3$ and $e(y_4,b_1b_4)=2$. If $x_0\in\{y_1,y_2\}$, we may assume w.l.o.g. that $\{x_0,x_3\}=\{y_1,y_2\}$. Then $[G_1-x_0+b_5]\supseteq F_1\uplus K_5^-$. By the optimality of $\{D,L_1,L_2\}$, $x_0\overset{na}{\to}(L_2,b_5)$. This implies that $\tau(b_5,L_2)=2$. Thus $x_0\to(L_2,b_1;R-\{x_0\})$, a contradiction. Hence $x_0\notin\{y_1,y_2\}$. W.l.o.g., say $\{a_3,a_5\}=\{y_1,y_2\}$. Then $[a_3,a_4,a_5,b_2,b_3]\supseteq C_5$, $[x_0,x_3,b_1,b_5,b_4]\supseteq C_5$ and $[x_2,x_1,x_4,a_1,a_2]\supseteq C_5$, a contradiction. ■

3. Proof of Theorem 1

Let G be a graph of order $5k$ with minimum degree at least $3k$. Suppose, for a contradiction, that $G\not\supseteq kC_5$. We may assume that G is maximal, i.e., $G+xy\supseteq kC_5$ for each pair of non-adjacent vertices x and y of G. Thus $G\supseteq P_5\uplus(k-1)C_5$. Our proof will follow from the following three lemmas.

Lemma 3.1. *For each $s\in\{1,2,\ldots,k\}$, $G\not\supseteq sB\uplus(k-s)C_5$.*

Proof. On the contrary, suppose that $G\supseteq sB\uplus(k-s)C_5$ for some $s\in\{1,2,\ldots,k\}$. Let s be the minimum number in $\{1,2,\ldots,k\}$ such that $G\supseteq sB\uplus(k-s)C_5$. Say $G\supseteq sB\uplus(k-s)C_5=\{B_1,\ldots,B_s,L_1,\ldots,L_{k-s}\}$ with $B_i\cong B$ for $i\in\{1,2,\ldots,s\}$. Let R be the set of the four vertices of B_1 whose degrees in B_1 are 2. By Lemma 2.2, Lemma 2.8 and the minimality of s, we see that $e(R,B_i)\leq 12$ and $e(R,L_j)\leq 12$ for all $i\in\{2,3,\ldots,s\}$ and $j\in\{1,2,\ldots,k-s\}$. Therefore $e(R,G)\leq 12(k-1)+8=12k-4$. As the minimum degree of G is $3k$, we obtain $12k-4\geq e(R,G)\geq 12k$, a contradiction. ■

Lemma 3.2. *There exists a sequence $(D,L_1,L_2,\ldots,L_{k-1})$ of disjoint subgraphs of G such that $D\cong K_4^+$ and $L_i\cong C_5$ for all $i\in\{1,2,\ldots,k-1\}$.*

Proof. First, we claim that $G\supseteq F\uplus(k-1)C_5$. We choose a sequence $(P,L_1,L_2,\ldots,L_{k-1})$ of disjoint subgraphs of G such that $P\cong P_5$ and $L_i\cong C_5$ for all $i \in \{1,2,\ldots,k-1\}$ with $\sum_{i=1}^{k-1}\tau(L_i)$ as large as possible. As $G \not\supseteq kC_5$ and by Lemma 2.1(c), $e(P,P) \le 14$ and so $e(P, G-V(P)) \ge 15k-14 = 15(k-1)+1$. Thus $e(P,L_i) \ge 16$ for some $i \in \{1,2,\ldots,k-1\}$. By Lemma 2.3, $[P,L_i] \supseteq F \uplus C_5$ and so $G \supseteq F \uplus (k-1)C_5$.

Next, we claim that $G \supseteq F_1 \uplus (k-1)C_5$. Assume for the moment that $G \supseteq F_2 \uplus (k-1)C_5 = \{D, L_1, L_2, \ldots, L_{k-1}\}$ with $D \cong F_2$. Let R be the three vertices of D with degree 2 in D. Then $e(R, G-V(D)) \ge 9k-6 = 9(k-1)+3$. Thus $e(R,L_i) \ge 10$ for some $i \in \{1,2,\ldots,k-1\}$. By Lemma 2.4, $[D,L_i] \supseteq F_1 \uplus C_5$ and so $G \supseteq F_1 \uplus (k-1)C_5$. Hence we may assume that $G \not\supseteq F_2 \uplus (k-1)C_5$. Then we choose a sequence $(D, L_1, L_2, \ldots, L_{k-1})$ of disjoint subgraphs of G such that $D \cong F$ and $L_i \cong C_5$ for all $i \in \{1,2,\ldots,k-1\}$ with $\sum_{i=1}^{k-1}\tau(L_i)$ as large as possible. Then $e(D,L_i) \ge 16$ for some $i \in \{1,2,\ldots,k-1\}$. By Lemma 2.5 and Lemma 3.1, we may assume that there exist two labellings $D = x_0x_1x_2x_3x_4x_1$ and $L_1 = a_1a_2a_3a_4a_5a_1$ such that $e(x_0,L_1)=0$, $e(x_1x_3,L_1) = 10$, $N(x_2,L_1) = N(x_4,L_1) = \{a_1,a_2,a_4\}$, $\tau(L_1) = 4$ and $a_3a_5 \notin E$. Then $e(x_0x_2a_3a_5, G-V(D\cup L_1)) \ge 12k-17 = 12(k-2)+7$. Thus $e(x_0x_2a_3a_5, L_i) \ge 13$ for some $i \in \{2,3,\ldots,k-1\}$. By Lemma 2.6, we obtain $[D,L_1,L_i] \supseteq F_1 \uplus 2C_5$ and so $G \supseteq F_1 \uplus (k-1)C_5$.

Suppose that $G \supseteq K_4^+ \uplus B \uplus (k-2)C_5 = \{D, B_1, L_1, L_2, \ldots, L_{k-2}\}$ with $D \cong K_4^+$ and $B_1 \cong B$. Let R be the four vertices of B_1 with degree 2 in B_1. Then either $e(R,D) \ge 13$ or $e(R,L_i) \ge 13$ for some $i \in \{1,2,\ldots,k-2\}$. By Lemma 2.2, Lemma 2.7 and Lemma 3.1, we see that $G \supseteq K_4^+ \uplus (k-1)C_5$. Hence we may suppose that $G \not\supseteq K_4^+ \uplus B \uplus (k-2)C_5$.

We now choose an optimal sequence $(D, L_1, L_2, \ldots, L_{k-1})$ of disjoint subgraphs of G with $D \cong F_1$ and $L_i \cong C_5$ for all $i \in \{1,2,\ldots,k-1\}$. Then $e(D,L_i) \ge 16$ for some $i \in \{1,2,\ldots,k-1\}$. Say w.l.o.g. $e(D,L_1) \ge 16$. By Lemma 2.9 and Lemma 3.1, we may assume that there exist two labellings $L_1 = a_1a_2a_3a_4a_5a_1$ and $V(D) = \{x_0,x_1,x_2,x_3,x_4\}$ with $E(D) = \{x_0x_1, x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_2x_4\}$ such that $e(x_0,L_1) = 0$, $e(a_1a_2a_4, D-x_0) = 12$, $N(a_3,L_1) = N(a_5,L_1) = \{x_2,x_4\}$, $\tau(L_1) = 4$ and $a_3a_5 \notin E$. Let $R = \{x_0,x_3,a_3,a_5\}$ and $G_1 = [D,L_1]$. Then $e(R,G_1) \le 16$ and so $e(R, G-V(G_1)) \ge 12k-16 = 12(k-2)+8$. This implies that $e(R,L_i) \ge 13$ for some $i \in \{2,3,\ldots,k-1\}$. Say w.l.o.g. $e(R,L_2) \ge 13$. By Lemma 2.10, it follows that $[G_1,L_2] \supseteq K_4^+ \uplus 2C_5$ and so $G \supseteq K_4^+ \uplus (k-1)C_5$. ∎

Let $\sigma = (D, L_1, \ldots, L_{k-1})$ be an optimal sequence of disjoint subgraphs in G with $D \cong K_4^+$ and $L_i \cong C_5$ for all $i \in \{1,2,\ldots,k-1\}$. Say $V(D) = \{x_0,x_1,x_2,x_3,x_4\}$ with $N(x_0,D) = \{x_1\}$. Let $Q = D - x_0$ and $T = Q - x_1$. Then $Q \cong K_4$ and $T \cong C_3$.

Lemma 3.3. *For each $t \in \{1,2,\ldots,k-1\}$, the following statements hold:*

(a) *If $e(x_0,L_t) = 5$, then $e(Q,L_t) \le 5$.*

(b) *If* $e(x_0, L_t) = 4$*, then* $e(Q, L_t) \leq 9$.

(c) *If* $e(x_0, L_t) = r$*, then* $e(Q, L_t) \leq 18 - 2r$ *for* $r \in \{1, 3\}$ *and if* $e(x_0, L_t) = 2$*, then* $e(Q, L_t) \leq 15$.

Proof. For convenience, we may assume $L_t = L_1 = a_1a_2a_3a_4a_5a_1$. Let $G_1 = [D, L_1]$. As $G_1 \not\supseteq 2C_5$, we see that if $x_0 \to L_1$, then $e(a_i, Q) \leq 1$ for all $a_i \in V(L_1)$ and so the lemma holds. Hence we may assume that $x_0 \not\to L_1$ and so $e(x_0, L_1) \leq 4$.

To prove (b), say w.l.o.g. $e(x_0, L_1 - a_5) = 4$. On the contrary, suppose $e(Q, L_1) \geq 10$. It is easy to see that $\tau(a_5, L_1) = 0$ for otherwise $x_0 \to L_1$ and so $G_1 \supseteq 2C_5$. As $x_0 \to (L_1, a_i)$ for $i \in \{2, 3, 5\}$, $e(a_i, Q) \leq 1$ for $i \in \{2, 3, 5\}$. If $e(a_5, Q) = 1$ then $[Q + a_5] \cong K_4^+$ and $\tau(x_0a_1a_2a_3a_4x_0) > \tau(L_1)$, contradicting the optimality of σ. Hence $e(a_5, Q) = 0$. It follows that $e(a_2, Q) = e(a_3, Q) = 1$ and $e(a_1a_4, Q) = 8$. Clearly, $\tau(x_0a_3a_4a_5a_1x_0) \geq \tau(L_1)$ with equality only if $a_2a_4 \in E$. As $[Q + a_2] \supseteq K_4^+$ and by the optimality of σ, we obtain $a_2a_4 \in E$. Thus $[a_5, a_4, a_3, a_2, x_0] \supseteq K_4^+$ and $[Q + a_1] \cong K_5$. By the optimality of σ, we obtain $[L_1] \cong K_5$, a contradiction.

To prove (c), we suppose, for a contradiction, that either $e(x_0, L_1) = r$ and $e(Q, L_1) \geq 19 - 2r$ for some $r \in \{1, 3\}$ or $e(x_0, L_1) = 2$ and $e(Q, L_1) \geq 16$. We divide the proof into the following three cases.

Case 1. $e(x_0, L_1) = 3$ and $e(Q, L_1) \geq 13$. First, suppose that $N(x_0, L_1) = \{a_i, a_{i+1}, a_{i+3}\}$ for some $i \in \{1, 2, 3, 4, 5\}$. Say w.l.o.g. $N(x_0, L_1) = \{a_1, a_2, a_4\}$. As $x_0 \not\to L_1$, $a_3a_5 \notin E$. Clearly, $x_0 \to (L_1, a_3)$ and $x_0 \to (L_1, a_5)$. Thus $e(a_3, Q) \leq 1$ and $e(a_5, Q) \leq 1$. It follows that $e(a_1a_2a_4, Q) \geq 11$, $e(x_1, a_1a_4) \geq 1$ and $e(x_1, a_2a_4) \geq 1$. Thus $[x_0, x_1, a_1, a_5, a_4] \supseteq C_5$ and $[x_0, x_1, a_2, a_3, a_4] \supseteq C_5$. As $e(a_i, T) \geq 2$ for $i \in \{1, 2\}$, it is easy to see that $e(a_3a_5, T) = 0$, i.e., $N(a_3a_5, Q) \subseteq \{x_1\}$, for otherwise $G_1 \supseteq 2C_5$.

Let $R = \{x_0, x_3, a_3, a_5\}$. Then $e(R, G_1) \leq 18$ and so $e(R, G - V(G_1)) \geq 12k - 18 = 12(k-2) + 6$. Then $e(R, L_i) \geq 13$ for some $i \in \{2, 3, \ldots, k-1\}$. Say w.l.o.g. $e(R, L_2) \geq 13$. Let $G_2 = [G_1, L_2]$. Then $G_2 \not\supseteq 3C_5$. Since $e(Q, L_1) \geq 13$ and $N(a_3a_5, Q) \subseteq \{x_1\}$, it is easy to check that if $u \to (L_2; R - \{u\})$ for some $u \in R$, then $G_2 \supseteq 3C_5$. Hence $u \not\to (L_2; R - \{u\})$ for all $u \in R$. By Lemma 2.1(d), there exist two labellings $L_2 = b_1b_2b_3b_4b_5b_1$ and $R = \{y_1, y_2, y_3, y_4\}$ such that $e(y_1y_2, L_2 - b_5) = 8$, $e(y_3, b_1b_5b_4) = 3$ and $e(y_4, b_1b_4) = 2$. If $\{y_1, y_2\} = \{x_0, x_3\}$, let $\{s, t\} = \{1, 2\}$ with $a_s \in I(x_0x_3, L_1)$ and then we see that $[x_0, a_s, x_3, b_2, b_3] \supseteq C_5$, $[a_3, a_5, b_1, b_5, b_4] \supseteq C_5$ and $[Q - x_3 + a_4 + a_t] \supseteq C_5$, a contradiction. If $\{y_1, y_2\} = \{x_0, a_i\}$ for some $i \in \{3, 5\}$, we may assume w.l.o.g. that $\{y_1, y_2\} = \{x_0, a_5\}$ and then we see that $[x_0, a_1, a_5, b_2, b_3] \supseteq C_5$, $[a_3, x_3, b_1, b_5, b_4] \supseteq C_5$ and $[a_2, a_4, x_1, x_2, x_4] \supseteq C_5$, a contradiction. If $\{y_1, y_2\} = \{x_3, a_i\}$ for some $i \in \{3, 5\}$, we may assume w.l.o.g. that $\{y_1, y_2\} = \{x_3, a_5\}$ and let $\{s, t\} = \{1, 4\}$ be such that $x_3a_s \in E$. Then we see that $\{x_3, a_s, a_5, b_2, b_3] \supseteq$

C_5, $[x_0, a_3, b_1, b_5, b_4] \supseteq C_5$ and $[x_1, x_2, x_4, a_2, a_t] \supseteq C_5$, a contradiction. Hence $\{y_1, y_2\} = \{a_3, a_5\}$. Then $[a_3, a_4, a_5, b_2, b_3] \supseteq C_5$, $[x_0, x_3, b_1, b_5, b_4] \supseteq C_5$ and $[x_1, x_2, x_4, a_1, a_2] \supseteq C_5$, a contradiction.

Next, suppose that $N(x_0, L_1) = \{a_i, a_{i+1}, a_{i+2}\}$ for some $i \in \{1, 2, 3, 4, 5\}$. Say w.l.o.g. $N(x_0, L_1) = \{a_1, a_2, a_3\}$. Then $e(a_2, Q) \leq 1$ as $G_1 \not\supseteq 2C_5$ and so $e(Q, L_1 - a_2) \geq 12$. First, assume $e(x_1, a_4a_5) \geq 1$. Say w.l.o.g. $x_1a_5 \in E$. Then $[x_0, x_1, a_5, a_1, a_2] \supseteq C_5$. Then $e(a_3a_4, T) \leq 3$ as $G_1 \not\supseteq 2C_5$. If we also have $x_1a_4 \in E$, then similarly, $e(a_1a_5, T) \leq 3$ and so $e(Q, L_1 - a_2) \leq 11$, a contradiction. Hence $x_1a_4 \notin E$. As $e(Q, L_1) \geq 13$, it follows that $e(a_1a_5, Q) = 8$, $e(a_3a_4, T) = 3$, $x_1a_3 \in E$ and $e(a_2, Q) = 1$. Clearly, $[T + a_4 + a_5] \not\supseteq C_5$ as $G_1 \not\supseteq 2C_5$. This implies that $e(a_4, T) = 0$ and so $e(a_3, Q) = 4$. Obviously, $G_1 \supseteq 2C_5$, a contradiction. Hence $e(x_1, a_4a_5) = 0$. Next, assume $e(x_1, a_1a_3) \geq 1$. Then $[x_0, x_1, a_1, a_2, a_3] \supseteq C_5$ and so $e(a_4a_5, T) \leq 3$. It follows that $e(Q, L_1 - a_2) \leq 12$, a contradiction. Hence $e(x_1, L_1 - a_2) = 0$. Thus $e(T, L_1 - a_2) = 12$. Obviously, $G_1 \supseteq 2C_5$, a contradiction.

Case 2. $e(x_0, L_1) = 2$ and $e(Q, L_1) \geq 16$. First, suppose that $N(x_0, L_1) = \{a_i, a_{i+2}\}$ for some $i \in \{1, 2, 3, 4, 5\}$. Say, $N(x_0, L_1) = \{a_1, a_3\}$. Then $e(a_2, Q) \leq 1$ and $e(Q, L_1 - a_2) \geq 15$. Thus $e(x_1, a_1a_3) \geq 1$. Then $[x_0, x_1, a_1, a_2, a_3] \supseteq C_5$ and so $e(a_4a_5, T) \leq 3$. Thus $e(Q, L_1 - a_2) \leq 13$, a contradiction. Therefore we may assume w.l.o.g. that $N(x_0, L_1) = \{a_1, a_2\}$. First, assume $x_1a_4 \in E$. Then $[x_0, x_1, a_4, a_5, a_1] \supseteq C_5$ and $[x_0, x_1, a_4, a_3, a_2] \supseteq C_5$. As $G_1 \not\supseteq 2C_5$, $e(a_2a_3, T) \leq 3$ and $e(a_1a_5, T) \leq 3$. Thus $e(Q, L_1) \leq 14$, a contradiction. Hence $x_1a_4 \notin E$. Next, assume $e(x_1, a_3a_5) \geq 1$. Say w.l.o.g. $x_1a_5 \in E$. Then $[x_0, x_1, a_5, a_1, a_2] \supseteq C_5$ and so $e(a_3a_4, T) \leq 3$. As $e(Q, L_1) \geq 16$, it follows that $e(a_5a_1a_2, Q) = 12$, $e(a_3a_4, T) = 3$ and $x_1a_3 \in E$. Thus $e(x_3, a_2a_5) = 2$ and so $G_1 \supseteq 2C_5$, a contradiction. Hence $e(x_1, a_3a_4a_5) = 0$. Thus $e(T, L_1) \geq 14$. This implies that $e(x_i, a_2a_5) = 2$ and $a_1x_j \in E$ for some $\{i, j\} \subseteq \{2, 3, 4\}$ with $i \neq j$. Consequently, $H \supseteq 2C_5$, a contradiction.

Case 3. $e(x_0, L_1) = 1$ and $e(Q, L_1) \geq 17$. Say w.l.o.g. $x_0a_1 \in E$. Suppose $e(x_1, a_3a_4) \geq 1$. Say $x_1a_3 \in E$. Then $[x_1, x_0, a_1, a_2, a_3] \supseteq C_5$ and so $e(a_4a_5, T) \leq 3$ as $G_1 \not\supseteq 2C_5$. As $e(Q, L_1) \geq 17$, it follows that $e(a_1a_2a_3, Q) = 12$, $e(a_4a_5, T) = 3$ and $e(x_1, a_4a_5) = 2$. Then $[x_0, x_1, a_4, a_5, a_1] \supseteq C_5$ and $[T, a_2, a_3] \supseteq C_5$, a contradiction. Hence $e(x_1, a_3a_4) = 0$. As $e(Q, L_1) \geq 17$, $e(T, L_1) \geq 14$. This implies that $e(x_i, a_2a_5) = 2$ and $a_1x_j \in E$ for some $\{i, j\} \subseteq \{2, 3, 4\}$ with $i \neq j$. Consequently, $H \supseteq 2C_5$, a contradiction. ∎

We are now in the position to complete the proof of Theorem 1. Let $\mathcal{A}_r = \{L_t | e(x_0, L_t) = r, 1 \leq t \leq k-1\}$ for each $0 \leq r \leq 5$. Set $p_r = |\mathcal{A}_r|$ for each $0 \leq r \leq 5$. Clearly, $p_0 + p_1 + p_2 + p_3 + p_4 + p_5 = k - 1$. By Lemma 3.3, we obtain

$$
\begin{aligned}
e(x_0, G) &= e(x_0, D) + \sum_{r=0}^{5} \sum_{L_t \in \mathcal{A}_r} e(x_0, L_t) \\
&= 1 + p_1 + 2p_2 + 3p_3 + 4p_4 + 5p_5; \qquad (2)\\
e(D, G) &= e(D, D) + \sum_{r=0}^{5} \sum_{L_t \in \mathcal{A}_r} e(D, L_t) \\
&\leq 14 + 20p_0 + 17p_1 + 17p_2 + 15p_3 + 13p_4 + 10p_5. \qquad (3)
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
e(x_0, G) + e(D, G) &\leq 15 + 20p_0 + 18p_1 + 19p_2 + 18p_3 + 17p_4 + 15p_5 \\
&= 18k + 2p_0 + p_2 - p_4 - 3p_5 - 3. \qquad (4)
\end{aligned}
$$

As $3\sum_{r=0}^{5} p_r = 3k - 3$ and $e(x_0, G) \geq 3k$, we obtain, by using (2), the following

$$
\begin{aligned}
&1 + p_1 + 2p_2 + 3p_3 + 4p_4 + 5p_5 \\
&\geq 3 + 3p_0 + 3p_1 + 3p_2 + 3p_3 + 3p_4 + 3p_5. \qquad (5)
\end{aligned}
$$

This implies that $3p_0 + 2p_1 + p_2 - p_4 - 2p_5 + 2 \leq 0$. Thus $2p_0 + p_2 - p_4 - 3p_5 \leq -2$. Together with (4), we obtain $e(x_0, G) + e(D, G) \leq 18k - 5$. But by the degree condition on G, we have $e(x_0, G) + e(D, G) \geq 3k + 15k = 18k$, a contradiction. This proves Theorem 1.

References

[1] S. Abbasi, PhD Thesis (Rutgers University 1998).

[2] B. Bollobás, Extremal Graph Theory (Academic Press, London, 1978).

[3] K. Corrádi and A. Hajnal, *On the maximal number of independent circuits in a graph*, Acta Math. Acad. Sci. Hungar. **14** (1963) 423–439. doi:10.1007/BF01895727

[4] M.H. El-Zahar, *On circuits in graphs*, Discrete Math. **50** (1984) 227–230. doi:10.1016/0012-365X(84)90050-5

[5] P. Erdős, Some recent combinatorial problems, Technical Report, University of Bielefeld, Nov. 1990.

[6] B. Randerath, I. Schiermeyer and H. Wang, *On quadrilaterals in a graph*, Discrete Math. **203** (1999) 229–237. doi:10.1016/S0012-365X(99)00053-9

[7] H. Wang, *On quadrilaterals in a graph*, Discrete Math. **288** (2004) 149–166. doi:10.1016/j.disc.2004.02.020

[8] H. Wang, *Proof of the Erdős-Faudree conjecture on quadrilaterals*, Graphs and Combin. **26** (2010) 833–877.
doi:10.1007/s00373-010-0948-3

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