

## DISJOINT 5-CYCLES IN A GRAPH

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### Abstract

We prove that if  $G$  is a graph of order  $5k$  and the minimum degree of  $G$  is at least  $3k$  then  $G$  contains  $k$  disjoint cycles of length 5.

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### 1. INTRODUCTION AND NOTATION

A set of graphs is said to be disjoint if no two of them have any common vertex. Corrádi and Hajnal [3] investigated the maximum number of disjoint cycles in a graph. They proved that if  $G$  is a graph of order at least  $3k$  with minimum degree at least  $2k$ , then  $G$  contains  $k$  disjoint cycles. In particular, when the order of  $G$  is exactly  $3k$ , then  $G$  contains  $k$  disjoint triangles. Erdős and Faudree [5] conjectured that if  $G$  is a graph of order  $4k$  with minimum degree at least  $2k$ , then  $G$  contains  $k$  disjoint cycles of length 4. This conjecture has been confirmed by Wang [8]. El-Zahar [4] conjectured that if  $G$  is a graph of order  $n = n_1 + n_2 + \cdots + n_k$  with  $n_i \geq 3$  ( $1 \leq i \leq k$ ) and the minimum degree of  $G$  is at least  $\lceil n_1/2 \rceil + \lceil n_2/2 \rceil + \cdots + \lceil n_k/2 \rceil$ , then  $G$  contains  $k$  disjoint cycles of lengths  $n_1, n_2, \dots, n_k$ , respectively. He proved this conjecture for  $k = 2$ . When  $n_1 = n_2 = \cdots = n_k = 3$ , this conjecture holds by Corrádi and Hajnal's result. When  $n_1 = n_2 = \cdots = n_k = 4$ , El-Zahar's conjecture reduces to the above conjecture of Erdős and Faudree. Abbasi [1] announced a solution to El-Zahar's conjecture for very large  $n$ .

In this paper, we develop a constructive method to show that El-Zahar's conjecture is true for all  $n = 5k$  with  $n_i = 5$  ( $1 \leq i \leq k$ ).

**Theorem 1.** *If  $G$  is a graph of order  $5k$  and the minimum degree of  $G$  is at least  $3k$ , then  $G$  contains  $k$  disjoint cycles of length 5.*

We shall use the terminology and notation from [2] except as indicated. Let  $G$  be a graph. Let  $u \in V(G)$ . The neighborhood of  $u$  in  $G$  is denoted by  $N(u)$ . Let  $H$  be a subgraph of  $G$  or a subset of  $V(G)$  or a sequence of distinct vertices of  $G$ . We define  $N(u, H)$  to be the set of neighbors of  $u$  contained in  $H$ , and let  $e(u, H) = |N(u, H)|$ . Clearly,  $N(u, G) = N(u)$  and  $e(u, G)$  is the degree of  $u$  in  $G$ . If  $X$  is a subgraph of  $G$  or a subset of  $V(G)$  or a sequence of distinct vertices of  $G$ , we define  $N(X, H) = \cup_u N(u, H)$  and  $e(X, H) = \sum_u e(u, H)$  where  $u$  runs over all the vertices in  $X$ . Let  $x$  and  $y$  be two distinct vertices. We define  $I(xy, H)$  to be  $N(x, H) \cap N(y, H)$  and let  $i(xy, H) = |I(xy, H)|$ . Let each of  $X_1, X_2, \dots, X_r$  be a subgraph of  $G$  or a subset of  $V(G)$ . We use  $[X_1, X_2, \dots, X_r]$  to denote the subgraph of  $G$  induced by the set of all the vertices that belong to at least one of  $X_1, X_2, \dots, X_r$ . We use  $C_i$  to denote a cycle of length  $i$  for all integers  $i \geq 3$ , and use  $P_j$  to denote a path of order  $j$  for all integers  $j \geq 1$ . For a cycle  $C$  of  $G$ , a chord of  $C$  is an edge of  $G - E(C)$  which joins two vertices of  $C$ , and we use  $\tau(C)$  to denote the number of chords of  $C$  in  $G$ . Furthermore, if  $x \in V(C)$ , we use  $\tau(x, C)$  to denote the number of chords of  $C$  that are incident with  $x$ . For each integer  $k \geq 3$ , a  $k$ -cycle is a cycle of length  $k$ . If  $S$  is a set of subgraphs of  $G$ , we write  $G \supseteq S$ .

For an integer  $k \geq 1$  and a graph  $G'$ , we use  $kG'$  to denote a set of  $k$  disjoint graphs isomorphic to  $G'$ . If  $G_1, \dots, G_r$  are  $r$  graphs and  $k_1, \dots, k_r$  are  $r$  positive integers, we use  $k_1G_1 \uplus \dots \uplus k_rG_r$  to denote a set of  $k_1 + \dots + k_r$  disjoint graphs which consist of  $k_1$  copies of  $G_1, \dots, k_{r-1}$  copies of  $G_{r-1}$  and  $k_r$  copies of  $G_r$ . For two graphs  $H_1$  and  $H_2$ , the union of  $H_1$  and  $H_2$  is still denoted by  $H_1 \cup H_2$  as usual, that is,  $H_1 \cup H_2 = (V(H_1) \cup V(H_2), E(H_1) \cup E(H_2))$ . Let each of  $Y$  and  $Z$  be a subgraph of  $G$ , or a subset of  $V(G)$ , or a sequence of distinct vertices of  $G$ . If  $Y$  and  $Z$  do not have any common vertices, we define  $E(Y, Z)$  to be the set of all the edges of  $G$  between  $Y$  and  $Z$ . Clearly,  $e(Y, Z) = |E(Y, Z)|$ . If  $C = x_1x_2 \dots x_rx_1$  is a cycle, then the operations on the subscripts of the  $x_i$ 's will be taken by modulo  $r$  in  $\{1, 2, \dots, r\}$ .

We use  $B$  to denote a graph of order 5 and size 6 such that  $B$  has two edge-disjoint triangles. We use  $F$  to denote a graph of order 5 and size 5 such that  $F$  has a vertex of degree 1 and a 4-cycle. Let  $F_1$  be the graph of order 5 obtained from  $F$  by adding a new edge to  $F$  such that the new edge joins the two vertices of  $F$  whose degrees in  $F$  are 2. Let  $F_2$  be the graph of order 5 and size 7 obtained from  $K_{2,3}$  by adding a new edge to  $K_{2,3}$  such that  $F_2$  has two adjacent vertices of degree 4. We use  $K_4^+$  to denote the graph of order 5 and size 7 such that  $K_4^+$  has a vertex of degree 1. Finally, we use  $K_5^-$  to denote a graph of order 5 with 9 edges.

Let  $\{H, L_1, \dots, L_t\}$  be a set of  $t+1$  disjoint subgraphs of  $G$  such that  $L_i \cong C_5$

for  $i = 1, \dots, t$ . We say that  $\{H, L_1, \dots, L_t\}$  is optimal if for any  $t + 1$  disjoint subgraphs  $H', L'_1, \dots, L'_t$  in  $[H, L_1, \dots, L_t]$  with  $H' \cong H$  and  $L'_i \cong C_5$  ( $1 \leq i \leq t$ ), we have that  $\sum_{i=1}^t \tau(L'_i) \leq \sum_{i=1}^t \tau(L_i)$ . Let  $L$  be a 5-cycle of  $G$  and  $H$  a subgraph of order 5 in  $G$ . We write  $H \geq L$  if  $H$  has a 5-cycle  $L'$  such that  $\tau(L') \geq \tau(L)$ . Moreover, if  $\tau(L') > \tau(L)$ , we write  $H > L$ .

Let  $L$  be a 5-cycle of  $G$ . Let  $u \in V(L)$  and  $x_0 \in V(G) - V(L)$ . We write  $x_0 \rightarrow (L, u)$  if  $[L - u + x_0] \supseteq C_5$ . Moreover, if  $[L - u + x_0] \geq L$  then we write  $x_0 \Rightarrow (L, u)$  and if  $[L - u + x_0] > L$  then we write  $x_0 \xrightarrow{a} (L, u)$ . In addition, if it does not hold that  $x_0 \xrightarrow{a} (L, u)$  then we write  $x_0 \xrightarrow{na} (L, u)$ . Clearly,  $x_0 \Rightarrow (L, u)$  when  $x_0 \xrightarrow{a} (L, u)$ . If  $x_0 \rightarrow (L, u)$  for all  $u \in V(L)$  then we write  $x_0 \rightarrow L$ . Similarly, we define  $x_0 \Rightarrow L$  and  $x_0 \xrightarrow{a} L$ . If  $[L - u + x_0] \supseteq B$ , we write  $x_0 \xrightarrow{z} (L, u)$ .

Let  $P$  be a path of order at least 2 or a sequence of at least two distinct vertices in  $G - V(L + x_0)$ . Let  $X$  be a subset of  $V(G) - V(L + x_0)$  with  $|X| \geq 2$ . We write  $x_0 \rightarrow (L, u; P)$  if  $x_0 \rightarrow (L, u)$  and  $u$  is adjacent to the two end vertices of  $P$ . In this case, if  $P$  is a path of order 4, then  $[x_0, L, P] \supseteq 2C_5$ . We write  $x_0 \rightarrow (L, u; X)$  if  $x_0 \rightarrow (L, u; xy)$  for some  $\{x, y\} \subseteq X$  with  $x \neq y$ . We write  $x_0 \rightarrow (L; P)$  if  $x_0 \rightarrow (L, u; P)$  for some  $u \in V(L)$  and  $x_0 \rightarrow (L; X)$  if  $x_0 \rightarrow (L, u; X)$  for some  $u \in V(L)$ . Similarly, we define the notation  $x_0 \xrightarrow{z} (L; P)$  and  $x_0 \xrightarrow{z} (L; X)$ . If it does not hold that  $x_0 \xrightarrow{z} (L; P)$ , we write  $x_0 \xrightarrow{nz} (L; P)$ . If it does not hold that  $x_0 \xrightarrow{z} (L; X)$ , we write  $x_0 \xrightarrow{nz} (L; X)$ .

## 2. SKETCH OF THE PROOF OF THEOREM 1 AND PRELIMINARY LEMMAS

### 2.1. Sketch of the proof of Theorem 1

Let  $G$  be a graph of order  $5k$  with minimum degree at least  $3k$ . Suppose, by way of contradiction, that  $G \not\supseteq kC_5$ . We may assume that  $G$  is maximal, i.e.,  $G + xy \supseteq kC_5$  for each pair of non-adjacent vertices  $x$  and  $y$  of  $G$ . Thus  $G \supseteq P_5 \uplus (k-1)C_5$ . Our first goal is to show that  $G \supseteq K_4^+ \uplus (k-1)C_5$ . This will be accomplished through a series of lemmas in Section 2.2. Say  $G \supseteq \{D, L_1, \dots, L_{k-1}\}$  with  $D \cong K_4^+$  and  $L_i \cong C_5$  ( $1 \leq i \leq k$ ). Let  $x_0 \in V(D)$  with  $e(x_0, D) = 1$  and let  $Q = D - x_0$ . We shall estimate the upper bound on  $2e(x_0, G) + e(Q, G) \geq 18k$ . This needs an estimation on each  $2e(x_0, L_i) + e(Q, L_i)$ . The idea is to show that if  $e(x_0, L_i)$  is increasing then  $e(Q, L_i)$  is decreasing for otherwise  $[D, L_i] \supseteq 2C_5$ , a contradiction. This is accomplished in Lemma 3.3. It turns out that  $2e(x_0, G) + e(Q, G) < 18k$ , a contradiction.

### 2.2. Preliminary lemmas

Our proof of Theorem 1 will use the following lemmas. Let  $G = (V, E)$  be a given graph in the following.

**Lemma 2.1.** *The following statements hold:*

- (a) *If  $P'$  and  $P''$  are two disjoint paths of  $G$  such that  $|V(P')| = 2$ ,  $2 \leq |V(P'')| \leq 3$  and  $e(P', P'') \geq 3$ , then  $[P', P''] \supseteq C_4$ .*
- (b) *If  $x$  and  $y$  are two distinct vertices and  $P$  is a path of order 3 in  $G$  such that  $\{x, y\} \cap V(P) = \emptyset$  and  $e(xy, P) \geq 5$ , then  $[x, y, P]$  contains a 5-cycle  $C$  such that  $\tau(C) \geq 2$ .*
- (c) *If  $D$  is a graph of order 5 with  $e(D) \geq 7$ , then  $D \supseteq C_5$ , unless  $D \cong K_4^+$  or  $D \cong F_2$ .*
- (d) *If  $R$  is a subset of  $V(G)$  and  $L$  is a 5-cycle of  $G - R$  such that  $|R| = 4$  and  $e(R, L) \geq 13$ , then  $u \rightarrow (L; R - \{u\})$  for some  $u \in R$ , or there exist two labellings  $R = \{y_1, y_2, y_3, y_4\}$  and  $L = b_1b_2b_3b_4b_5b_1$  such that  $N(y_1, L) = N(y_2, L) = \{b_1, b_2, b_3, b_4\}$ ,  $N(y_3, L) = \{b_1, b_5, b_4\}$  and  $N(y_4, L) = \{b_1, b_4\}$ .*

**Proof.** It is easy to check (a), (b) and (c). To prove (d), we suppose, for a contradiction, that  $u \not\rightarrow (L; R - \{u\})$  for all  $u \in R$ . Let  $R = \{y_1, y_2, y_3, y_4\}$  be such that  $e(y_1, L) \geq e(y_i, L)$  for all  $y_i \in R$ . As  $e(R, L) \geq 13$ ,  $e(y_1, L) \geq 4$  and there exists  $b \in V(L)$  such that  $e(b, R - \{y_1\}) \geq 2$ . If  $e(y_1, L) = 5$  then  $y_1 \rightarrow (L, b; R - \{y_1\})$ , a contradiction. Hence we may assume that  $L = b_1b_2b_3b_4b_5b_1$  and  $e(y_1, b_1b_2b_3b_4) = 4$ . Thus  $e(b_i, R - \{y_1\}) \leq 1$  for  $i \in \{2, 3, 5\}$ . Then  $6 \geq e(b_1b_4, R - \{y_1\}) \geq 13 - 4 - 3 = 6$ . It follows that  $e(b_1b_4, R - \{y_1\}) = 6$  and  $e(b_i, R - \{y_1\}) = 1$  for  $i \in \{2, 3, 5\}$ . W.l.o.g., say  $b_2y_2 \in E$ . Then  $e(b_3, y_3y_4) = 0$  as  $y_2 \not\rightarrow (L, b_3; R - \{y_2\})$ . Hence  $b_3y_2 \in E$ . W.l.o.g., say  $b_5y_3 \in E$ . Thus (d) holds. ■

**Lemma 2.2.** *Let  $D$  and  $L$  be disjoint subgraphs of  $G$  such that  $D \cong B$  and  $L \cong C_5$ . Say  $D = x_0x_1x_2x_0x_3x_4x_0$ . Suppose that  $e(D - x_0, L) \geq 13$ . Then  $[D, L] \supseteq 2C_5$ .*

**Proof.** Let  $H = [D, L]$ . On the contrary, suppose  $H \not\supseteq 2C_5$ . Then it is easy to see that

$$(1) \quad \begin{aligned} &x_i \not\rightarrow (L; x_jx_s) \text{ and } x_i \not\rightarrow (L; x_jx_t) \text{ for} \\ &\quad \{\{i, j\}, \{s, t\}\} = \{\{1, 2\}, \{3, 4\}\}. \end{aligned}$$

Let  $R = \{x_1, x_2, x_3, x_4\}$ . W.l.o.g., say  $e(x_1, L) \geq e(x_i, L)$  for all  $x_i \in R$ . Then  $e(x_1, L) \geq 4$ . First, assume that  $e(x_1, L) = 5$ . By (1),  $I(x_2x_3, L) = I(x_2x_4, L) = \emptyset$ . Thus  $e(x_2x_3, L) \leq 5$  and  $e(x_2x_4, L) \leq 5$ . Since  $e(R, L) \geq 13$ , it follows that  $e(x_4, L) \geq 3$  and  $e(x_3, L) \geq 3$ . As  $x_3 \not\rightarrow (L; x_1x_4)$ , we see that  $e(x_3, L) = 3$ . Similarly,  $e(x_4, L) = 3$ . Then  $e(x_2, L) = 2$ . As  $x_2 \not\rightarrow (L; x_1x_3)$ , we see that the two vertices of  $N(x_2, L)$  must be consecutive on  $L$ . Say  $N(x_2, L) = \{a_1, a_2\}$ . Then  $[x_0, x_1, x_2, a_1, a_2] \supseteq C_5$  and  $[x_3, x_4, a_3, a_4, a_5] \supseteq C_5$ , a contradiction. Therefore  $e(x_1, L) = 4$ . Say  $N(x_1, L) = \{a_1, a_2, a_3, a_4\}$ . By (1),  $I(x_2x_j, \{a_2, a_3, a_5\}) = \emptyset$  for  $j \in \{3, 4\}$ . Thus  $e(x_2x_j, L) \leq 7$  for  $j \in \{3, 4\}$  and so  $e(x_j, L) \geq 2$  for  $j \in \{3, 4\}$ .

First, assume  $e(x_2x_j, L) = 7$  for some  $j \in \{3, 4\}$ . Say  $e(x_2x_3, L) = 7$ . Then  $I(x_2x_3, L) = \{a_1, a_4\}$  and  $e(a_i, x_2x_3) = 1$  for  $i \in \{2, 3, 5\}$ . If  $e(x_4, a_2a_3) \geq 1$ , say w.l.o.g.  $x_4a_2 \in E$ , then  $[a_1, a_2, x_4, x_0, x_3] \supseteq C_5$  and so  $x_2a_5 \notin E$  as  $H \not\supseteq 2C_5$ . Consequently,  $x_3a_5 \in E$  and so  $H \supseteq 2C_5 = \{x_3a_5a_1a_2x_4x_3, x_1x_0x_2a_4a_3x_1\}$ , a contradiction. Hence  $e(x_4, a_2a_3) = 0$  and so  $e(x_4, a_1a_4) \geq 1$ . W.l.o.g., say  $x_4a_1 \in E$ . Then  $[x_3, x_4, a_1, a_5, a_4] \supseteq C_5$  and so  $e(x_2, a_2a_3) = 0$  as  $H \not\supseteq 2C_5$ . Thus  $e(x_3, a_2a_3) = 2$ . As  $e(x_3, L) \leq e(x_1, L) = 4$ ,  $x_3a_5 \notin E$ . Thus  $x_2a_5 \in E$ , and consequently,  $H \supseteq 2C_5 = \{x_3x_4a_1a_2a_3x_3, x_1x_0x_2a_5a_4x_1\}$ , a contradiction. Therefore  $e(x_2x_j, L) \leq 6$  for  $j \in \{3, 4\}$  and so  $e(x_j, L) \geq 3$  for  $j \in \{3, 4\}$ . Similarly, if  $e(x_3, L) = 4$  then  $e(x_1x_4, L) \leq 6$ , a contradiction. Hence  $e(x_3, L) = 3$ . Similarly,  $e(x_4, L) = 3$ . Then  $e(x_2, L) = 3$  as  $e(R, L) \geq 13$ . Assume  $x_2a_5 \in E$ . Then  $e(a_5, x_3x_4) = 0$  by (1). As  $e(x_3x_4, L) = 6$ , either  $e(x_3x_4, a_1a_2) \geq 3$  or  $e(x_3x_4, a_3a_4) \geq 3$ . Say w.l.o.g. the former holds. Then  $[x_3, x_0, x_4, a_1, a_2] \supseteq C_5$  and  $[x_1, x_2, a_5, a_4, a_3] \supseteq C_5$ , a contradiction. Hence  $x_2a_5 \notin E$ . As  $e(x_2, L) = 3$ , either  $e(x_2, a_1a_3) = 2$  or  $e(x_2, a_2a_4) = 2$ . W.l.o.g., say the former holds. As  $x_2 \not\rightarrow (L; x_1x_j)$  for  $j \in \{3, 4\}$ ,  $e(a_2, x_3x_4) = 0$ . As  $e(x_3x_4, L) = 6$ , either  $e(x_3x_4, a_3a_5) \geq 3$  or  $e(x_3x_4, a_1a_4) \geq 3$ . Thus either  $[x_3, x_4, a_3, a_4, a_5] \supseteq C_5$  or  $[x_3, x_4, a_4, a_5, a_1] \supseteq C_5$ . In each situation, we see that  $H \supseteq 2C_5$ , a contradiction. ■

**Lemma 2.3.** *Let  $P$  and  $L$  be disjoint subgraphs of  $G$  such that  $P \cong P_5$  and  $L \cong C_5$ . Suppose that  $\{P, L\}$  is optimal,  $e(P, L) \geq 16$  and  $[P, L] \not\supseteq 2C_5$ . Then  $[P, L] \supseteq F \uplus C_5$ .*

**Proof.** Say  $P = x_1x_2x_3x_4x_5$  with  $e(x_1, L) \geq e(x_5, L)$  and  $L = a_1a_2a_3a_4a_5a_1$ . Then  $e(x_1, L) \geq 1$ . Let  $H = [P, L]$ . On the contrary, suppose  $H \not\supseteq F \uplus C_5$ . Assume first that  $e(x_1, L) = 1$ . Say  $x_1a_1 \in E$ . As  $e(P, L) \geq 16$  and  $e(x_5, L) \leq 1$ ,  $e(x_2x_3x_4, L) \geq 14$ . Thus  $e(x_2, a_3a_4) \geq 1$ . W.l.o.g., say  $x_2a_3 \in E$ . Then  $[x_1, x_2, a_3, a_2, a_1] \supseteq C_5$ . As  $e(x_3x_4, L) \geq 14 - e(x_2, L) \geq 9$ ,  $e(x_3x_4, a_4a_5) \geq 3$ . By Lemma 2.1(a),  $[x_5, x_4, x_3, a_4, a_5] \supseteq F$  and so  $H \supseteq F \uplus C_5$ , a contradiction. Hence  $e(x_1, L) \geq 2$ .

As  $e(P, L) \geq 16$ ,  $I(x_2x_4, L) \neq \emptyset$  or  $I(x_3x_5, L) \neq \emptyset$ . Therefore  $x_1 \not\rightarrow L$  for otherwise  $H \supseteq F \uplus C_5$ . Hence  $e(x_1, L) \leq 4$ . We divide the proof into the following cases.

*Case 1.*  $e(x_1, L) = 4$ . Say  $N(x_1, L) = \{a_1, a_2, a_3, a_4\}$ . Then  $[L - a_i + x_1] \supseteq F$  for all  $a_i \in V(L)$ . Thus  $I(x_2x_5, L) = \emptyset$  as  $H \not\supseteq F \uplus C_5$ . As  $x_1 \not\rightarrow L$ ,  $\tau(a_5, L) = 0$ . Then  $x_1 \xrightarrow{a} (L, a_5)$ . By the optimality of  $\{P, L\}$ ,  $[P - x_1 + a_5] \not\supseteq P_5$  and so  $e(a_5, x_2x_5) = 0$  and  $e(a_5, x_3x_4) \leq 1$ . Thus  $e(x_2x_5, L) \leq 4$  and so  $e(x_3x_4, L) \geq 8$ . Suppose  $e(x_2, L) \geq 1$ . Then  $e(x_2, a_2a_4) \geq 1$  or  $e(x_2, a_1a_3) \geq 1$ . W.l.o.g., say the former holds. Then  $[x_1, x_2, a_2, a_3, a_4] \supseteq C_5$ . As  $H \not\supseteq F \uplus C_5$  and by Lemma 2.1(a), we see that  $e(x_3x_4, a_1a_5) \leq 2$ . It follows that  $e(x_3x_4, a_2a_3a_4) = 6$  and  $e(x_2x_5, L - a_5) = 4$ . Thus  $e(a_2, x_2x_5) > 0$ . Then  $[P - x_1 + a_2] \supseteq F$ . As  $x_1 \rightarrow$

$(L, a_2)$ ,  $H \supseteq F \uplus C_5$ , a contradiction. Hence  $e(x_2, L) = 0$ . Similarly, if  $e(x_5, L) = 4$  then  $e(x_4, L) = 0$  and so  $e(P, L) < 16$ , a contradiction. Hence  $e(x_5, L) \leq 3$  and so  $e(x_3x_4, L) \geq 9$ . As  $e(a_5, x_3x_4) \leq 1$ , it follows that  $e(x_3x_4, L - a_5) = 8$ ,  $e(a_5, x_3x_4) = 1$  and  $e(x_5, L) = 3$ . Then  $e(a_i, x_3x_5) = 2$  for some  $i \in \{2, 3\}$  and so  $H \supseteq F \uplus C_5$  as  $x_1 \rightarrow (L, a_i)$ , a contradiction.

*Case 2.*  $e(x_1, L) = 3$ . Then  $e(x_5, L) \leq 3$ . First, suppose that the three vertices in  $N(x_1, L)$  are not consecutive on  $L$ . Say  $N(x_1, L) = \{a_1, a_2, a_4\}$ . Clearly,  $I(x_2x_5, L) \subseteq \{a_4\}$  since  $H \not\supseteq 2C_5$  and  $H \not\supseteq F \uplus C_5$ . Hence  $e(x_2x_5, L) \leq 6$ . If  $x_2a_4 \in E$  then  $[x_1, x_2, a_1, a_5, a_4] \supseteq C_5$ . As  $H \not\supseteq F \uplus C_5$ ,  $e(x_3x_4, a_2a_3) \leq 2$ . Similarly,  $[x_1, x_2, a_2, a_3, a_4] \supseteq C_5$  and so  $e(x_3x_4, a_1a_5) \leq 2$ . Consequently,  $e(P, L) \leq 15$ , a contradiction. Hence  $x_2a_4 \notin E$ . Thus  $e(x_2x_5, L) \leq 5$  and so  $e(x_3x_4, L) \geq 8$ . If  $e(x_2, L) > 0$ , then  $[x_1, x_2, P'] \supseteq C_5$  where  $P' = L - \{a_i, a_{i+1}\}$  for some  $\{a_i, a_{i+1}\} \subseteq V(L)$ . As  $H \not\supseteq F \uplus C_5$ ,  $e(x_3x_4, a_i a_{i+1}) \leq 2$ . Consequently,  $e(x_3x_4, P') = 6$ ,  $e(x_3x_4, a_i a_{i+1}) = 2$  and  $e(x_2x_5, L) = 5$ . Hence  $e(a_t, x_2x_5) = 1$  for all  $a_t \in V(L)$ . Thus  $[P - x_1 + a_j] \supseteq F$  and  $x_1 \rightarrow (L, a_j)$  where  $a_j \in V(P') \cap \{a_3, a_5\}$ , a contradiction.

Therefore  $e(x_2, L) = 0$  and so  $e(x_3x_4, L) = 10$  and  $e(x_5, L) = 3$ . Consequently,  $H \supseteq 2C_5$  or  $H \supseteq F \uplus C_5$ , a contradiction. Therefore the three vertices in  $N(x_1, L)$  are consecutive on  $L$ . Say  $N(x_1, L) = \{a_1, a_2, a_3\}$ . Then  $I(x_2x_5, L) \subseteq \{a_1, a_3\}$  since  $H \not\supseteq 2C_5$  and  $H \not\supseteq F \uplus C_5$ . Thus  $e(x_2x_5, L) \leq 7$  and so  $e(x_3x_4, L) \geq 6$ . Assume  $e(x_2, a_4a_5) \geq 1$ . Say w.l.o.g.  $x_2a_4 \in E$ . Then  $[x_1, x_2, a_2, a_3, a_4] \supseteq C_5$  and  $[x_1, x_2, a_1, a_5, a_4] \supseteq C_5$ . As  $H \not\supseteq F \uplus C_5$  and by Lemma 2.1(a),  $e(x_3x_4, a_1a_5) \leq 2$  and  $e(x_3x_4, a_2a_3) \leq 2$ . It follows that  $e(x_2x_5, L) = 7$ ,  $e(x_3x_4, L) = 6$ ,  $e(a_4, x_3x_4) = 2$ , and  $e(x_2x_5, a_1a_3) = 4$ . Then  $[x_1, x_5, a_1, a_2, a_3] \supseteq C_5$  and  $[a_5, a_4, x_2, x_3, x_4] \supseteq F$ , a contradiction. Hence  $e(x_2, a_4a_5) = 0$  and so  $e(x_2, L) \leq 3$ . Thus  $e(x_3x_4, L) \geq 7$ . Assume  $e(x_2, a_1a_3) \geq 1$ . Then  $[x_1, x_2, a_1, a_2, a_3] \supseteq C_5$ . Then  $e(x_3x_4, a_4a_5) \leq 2$  as  $H \not\supseteq F \uplus C_5$ . Thus  $e(x_3x_4, a_1a_2a_3) \geq 5$ . As  $H \not\supseteq F \uplus C_5$  and  $x_1 \rightarrow (L, a_2)$ , we have  $e(a_2, x_2x_4) \leq 1$ . As  $e(P, L) \geq 16$ , it follows that  $e(a_2, x_2x_4) = 1$ ,  $e(x_3, a_1a_2a_3) = 3$ ,  $e(x_3x_4, a_4a_5) = 2$  and  $e(x_5, L) = 3$ . As  $H \not\supseteq F \uplus C_5$  and  $x_1 \rightarrow (L, a_2)$ , we see that  $x_5a_2 \notin E$ . Then  $e(x_5, a_4a_5) \geq 1$  and so  $[x_3, x_4, x_5, a_4, a_5] \supseteq F$ , a contradiction. Hence  $e(x_2, a_1a_3) = 0$  and so  $e(x_2, L) \leq 1$ . If  $e(x_5, L) = 3$  then we also have  $e(x_4, L) \leq 1$  by the symmetry and so  $e(P, L) \leq 13$ , a contradiction. Hence  $e(x_5, L) \leq 2$ . It follows that so  $e(x_3x_4, L) = 10$ ,  $e(x_2, L) = 1$  and  $e(x_5, L) = 2$ . Thus  $e(a_2, x_2x_4) = 2$  and so  $H \supseteq F \uplus C_5$ , a contradiction.

*Case 3.*  $e(x_1, L) = 2$ . Then  $e(x_5, L) \leq 2$  and  $e(x_3x_4, L) \geq 7$ . First, suppose that the two vertices in  $N(x_1, L)$  are not consecutive on  $L$ . Say  $N(x_1, L) = \{a_1, a_3\}$ . Assume  $e(x_2, a_1a_3) \geq 1$ . Then  $[x_1, x_2, a_1, a_2, a_3] \supseteq C_5$ . As  $H \not\supseteq F \uplus C_5$  and by Lemma 2.1(a),  $e(x_3x_4, a_4a_5) \leq 2$ . Hence  $e(x_3x_4, a_1a_2a_3) \geq 5$ . As  $x_1 \rightarrow (L, a_2)$  and  $H \not\supseteq F \uplus C_5$ ,  $e(a_2, x_2x_4) \leq 1$ . As  $e(P, L) \geq 16$ , it follows that  $e(a_2, x_2x_4) = 1$ ,  $e(x_5, L) = 2$ ,  $e(x_2, L - a_2) = 4$ ,  $e(x_3, a_1a_2a_3) = 3$  and

$e(x_3x_4, a_4a_5) = 2$ . As  $[x_3, x_4, x_5, a_4, a_5] \not\supseteq F$ ,  $e(x_5, a_4a_5) = 0$  by Lemma 2.1(a). As  $x_1 \rightarrow (L, a_2)$  and  $H \not\supseteq F \uplus C_5$ ,  $a_2x_5 \notin E$ . Thus  $e(x_5, a_1a_3) = 2$ . It follows that  $[x_1, x_2, a_1, a_5, a_4] \supseteq C_5$  and  $[x_3, x_4, x_5, a_3, a_2] \supseteq C_5$ , a contradiction. Hence  $e(x_2, a_1a_3) = 0$ . Thus  $e(x_3x_4, L) \geq 9$ . As  $e(x_3x_4, L) \leq 10$ ,  $e(x_2, L) \geq 2$  and so  $e(x_2, a_4a_5) \geq 1$ . Say w.l.o.g.  $x_2a_4 \in E$ . Then  $[x_1, x_2, a_4, a_5, a_1] \supseteq C_5$ . As  $H \not\supseteq F \uplus C_5$  and by Lemma 2.1(a),  $e(x_3x_4, a_2a_3) \leq 2$  and so  $e(x_3x_4, L) \leq 8$ , a contradiction. Therefore the two vertices in  $N(x_1, L)$  are consecutive on  $L$ . Say  $N(x_1, L) = \{a_1, a_2\}$ . Assume  $x_2a_4 \in E$ . Then  $[x_1, x_2, a_4, a_5, a_1] \supseteq C_5$  and  $[x_1, x_2, a_4, a_3, a_2] \supseteq C_5$ . Thus  $e(x_3x_4, a_2a_3) \leq 2$  and  $e(x_3x_4, a_1a_5) \leq 2$  since  $H \not\supseteq F \uplus C_5$ . Hence  $e(x_3x_4, L) \leq 6$ , a contradiction. Hence  $x_2a_4 \notin E$ . Thus  $e(x_3x_4, L) \geq 8$ . Assume  $e(x_2, a_3a_5) \geq 1$ . Say  $x_2a_3 \in E$ . Then  $[x_1, x_2, a_3, a_2, a_1] \supseteq C_5$  and so  $e(x_3x_4, a_4a_5) \leq 2$ . It follows that  $e(x_3x_4, a_1a_2a_3) = 6$ ,  $e(x_3x_4, a_4a_5) = 2$ ,  $e(x_2, L - a_4) = 4$  and  $e(x_5, L) = 2$ . As  $x_2a_5 \in E$  and by the symmetry, we also have  $e(x_3x_4, a_5a_1a_2) = 6$ . Then  $H \supseteq F \uplus C_5$ , a contradiction. Therefore  $e(x_2, a_3a_5) = 0$ . It follows that  $e(x_2, a_1a_2) = 2$ ,  $e(x_3x_4, L) = 10$  and  $e(x_5, L) = 2$ . Then  $H \supseteq F \uplus C_5$ , a contradiction. ■

**Lemma 2.4.** *Let  $D$  and  $L$  be disjoint subgraphs of  $G$  with  $D \cong F_2$  and  $L \cong C_5$ . Let  $R$  be the set of the three vertices of  $D$  with degree 2 in  $D$ . If  $e(R, L) \geq 10$ , then  $[D, L] \supseteq F_1 \uplus C_5$ .*

**Proof.** As  $e(R, L) \geq 10$ ,  $e(u, L) \geq 4$  for some  $u \in R$ . Thus  $u \rightarrow (L, v)$  for some  $v \in V(L)$  with  $e(v, R - \{u\}) \geq 1$ . Clearly,  $[D - u + v] \supseteq F_1$ . ■

**Lemma 2.5.** *Let  $D$  and  $L$  be disjoint subgraphs of  $G$  with  $D \cong F$  and  $L \cong C_5$ . Suppose that  $\{D, L\}$  is optimal and  $e(D, L) \geq 16$ . Then  $[D, L]$  contains one of  $F_1 \uplus C_5$ ,  $F_2 \uplus C_5$ ,  $B \uplus C_5$  and  $2C_5$ , or there exist two labellings  $D = x_0x_1x_2x_3x_4x_1$  and  $L = a_1a_2a_3a_4a_5a_1$  such that  $e(x_0, L) = 0$ ,  $e(x_1x_3, L) = 10$ ,  $N(x_2, L) = N(x_4, L) = \{a_1, a_2, a_4\}$ ,  $\tau(L) = 4$  and  $a_3a_5 \notin E$ .*

**Proof.** Say  $H = [D, L]$ . Suppose that  $H$  does not contain any of  $F_1 \uplus C_5$ ,  $F_2 \uplus C_5$ ,  $B \uplus C_5$  and  $2C_5$ . We shall prove that there exist two labellings of  $D$  and  $L$  satisfying the property in the lemma. Say  $D = x_0x_1x_2x_3x_4x_1$  and  $L = a_1a_2a_3a_4a_5a_1$ . Then  $x_2x_4 \notin E$ . Let  $Q = x_1x_2x_3x_4x_1$ . If  $e(x_0, L) \geq 4$ , then for each  $a_i \in V(L)$ ,  $[L - a_i + x_0] \supseteq C_5$  or  $[L - a_i + x_0] \supseteq F_1$ . Thus  $[Q + a_i] \not\supseteq C_5$  and so  $e(a_i, Q) \leq 2$  for each  $a_i \in V(L)$ . Consequently,  $e(D, L) \leq 15$ , a contradiction. Therefore  $e(x_0, L) \leq 3$ . We divide the proof into the following cases.

*Case 1.*  $e(x_0, L) = 0$ . First, suppose that  $e(x_2, L) \geq 4$  or  $e(x_4, L) \geq 4$ . Say,  $\{a_1, a_2, a_3, a_4\} \subseteq N(x_2, L)$ . Assume  $e(x_1, a_2a_3) \geq 1$ . Say w.l.o.g.  $x_1a_2 \in E$ .

Then  $[x_0, x_1, x_2, a_2, a_1] \supseteq F_1$  and  $[x_0, x_1, x_2, a_2, a_3] \supseteq F_1$ . As  $H \not\supseteq F_1 \uplus C_5$ , we see that  $e(x_3x_4, a_3a_5) \leq 2$  and  $e(x_3x_4, a_1a_4) \leq 2$ . As  $e(Q, L) \geq 16$ , it follows that  $e(x_1x_2, L) = 10$  and  $e(a_2, x_3x_4) = 2$ . Thus  $[x_0, x_1, a_2, x_3, x_4] \supseteq F_1$  and  $x_2 \rightarrow$

$(L, a_2)$ , a contradiction. Hence  $e(x_1, a_2a_3) = 0$ . As  $e(x_1, L) \geq 1$ , this argument implies that  $e(x_2, L) \neq 5$ . Similarly,  $e(x_4, L) \neq 5$ . As  $e(Q, L) \geq 16$ , it follows that  $e(x_1, a_1a_5a_4) = 3$ ,  $e(x_3, L) = 5$  and  $e(x_4, L) = 4$ . Then  $[x_0, x_1, x_2, a_1, a_2] \supseteq F_1$  and  $[x_3, x_4, a_3, a_4, a_5] \supseteq C_5$ , a contradiction. Hence  $e(x_2, L) \leq 3$  and  $e(x_4, L) \leq 3$ . Consequently,  $e(x_1x_3, L) = 10$ ,  $e(x_2, L) = e(x_4, L) = 3$ . Then  $x_2$  is adjacent two consecutive vertices of  $L$ . Say w.l.o.g.  $e(x_2, a_1a_2) = 2$ . Then  $[x_0, x_1, x_2, a_1, a_2] \supseteq F_1$ . Thus  $e(x_4, a_3a_5) = 0$  as  $H \not\supseteq F_1 \uplus C_5$ . Hence  $e(x_4, a_1a_2a_4) = 3$ . Similarly,  $e(x_2, a_1a_2a_4) = 3$ . Clearly,  $[D - x_3 + a_i] \supseteq F$  for  $i \in \{1, 2\}$ . As  $\{D, L\}$  is optimal,  $x_3 \xrightarrow{na} (L, a_i)$  for  $i \in \{1, 2\}$ . This implies that  $\tau(a_1, L) = \tau(a_2, L) = 2$ . As  $[x_0, x_1, x_2, a_1, a_2] \supseteq F_1$ ,  $[x_3, x_4, a_3, a_4, a_5] \not\supseteq C_5$ . This implies that  $a_3a_5 \notin E$ . Therefore these two labellings satisfy the property described in the lemma.

*Case 2.*  $e(x_0, L) = 1$ . Then  $e(Q, L) \geq 15$ . Say  $x_0a_1 \in E$ . First, suppose  $e(x_1, a_3a_4) \geq 1$ . Say w.l.o.g.  $x_1a_3 \in E$ . Then  $[x_1, x_0, a_1, a_2, a_3] \supseteq C_5$ . By Lemma 2.1(c), we have  $e(a_4a_5, x_2x_3x_4) \leq 3$  since  $H \not\supseteq 2C_5$ ,  $H \not\supseteq F_1 \uplus C_5$  and  $H \not\supseteq F_2 \uplus C_5$ . Thus  $e(a_4a_5, Q) \leq 5$ . Similarly, if  $x_1a_4 \in E$  then  $e(a_2a_3, Q) \leq 5$  and so  $e(Q, L) \leq 14$ , a contradiction. Hence  $x_1a_4 \notin E$ . Thus  $e(a_4a_5, Q) \leq 4$  and so  $e(a_1a_2a_3, Q) \geq 11$ . This implies that if  $e(a_2, x_1x_3) = 2$  then there is a choice  $\{i, j\} = \{2, 4\}$  such that  $e(x_i, a_1a_3) = 2$  and  $e(a_2, x_1x_jx_3) = 3$ . Thus  $[x_0, x_1, x_j, x_3, a_2] \supseteq F_1$  and  $x_i \rightarrow (L, a_2)$ , a contradiction. Hence  $e(a_2, x_1x_3) = 1$ ,  $e(a_1a_3, Q) = 8$ ,  $e(a_2, x_2x_4) = 2$  and  $e(a_4a_5, Q) = 4$  with  $a_5x_1 \in E$ . Consequently,  $[a_4, a_5, a_1, x_0, x_1] \supseteq F_1$  and  $[a_2, a_3, x_2, x_3, x_4] \supseteq C_5$ , a contradiction. Therefore  $e(x_1, a_3a_4) = 0$ .

Next, suppose  $e(x_1, a_1a_5) = 2$  or  $e(x_1, a_1a_2) = 2$ . Say w.l.o.g.  $e(x_1, a_1a_5) = 2$ . Then  $[a_4, a_5, a_1, x_0, x_1] \supseteq F_1$ . Thus  $e(a_2a_3, x_2x_4) \leq 2$ . Hence  $e(a_2a_3, Q) \leq 5$  and so  $e(a_1a_5a_4, x_2x_3x_4) \geq 8$ . This implies that if  $x_3a_5 \in E$  then there is a choice  $\{i, j\} = \{2, 4\}$  such that  $e(a_5, x_1x_ix_3) = 3$ ,  $e(x_j, a_1a_4) = 2$  and consequently,  $H \supseteq F_1 \uplus C_5$ , a contradiction. Hence  $a_5x_3 \notin E$  and it follows that  $e(a_1, x_2x_3x_4) = 3$ ,  $e(a_5, x_2x_4) = 2$ ,  $e(a_4, x_2x_3x_4) = 3$ ,  $e(a_2a_3, Q) = 5$  with  $a_2x_1 \in E$ . Then  $[a_3, a_2, a_1, x_0, x_1] \supseteq F_1$  and  $[a_4, a_5, x_2, x_3, x_4] \supseteq C_5$ , a contradiction. Therefore  $e(x_1, a_1a_5) \leq 1$  and  $e(x_1, a_1a_2) \leq 1$ . Thus  $e(x_1, L) \leq 2$ . Assume that  $a_1x_3 \in E$ . Then  $x_2 \not\rightarrow (L, a_1)$  as  $H \not\supseteq 2C_5$ . Hence  $e(x_2, a_2a_5) \leq 1$ , and similarly,  $e(x_4, a_2a_5) \leq 1$ . As  $e(Q, L) \geq 15$ , it follows that  $e(x_1, a_2a_5) = 2$ ,  $e(x_3, L) = 5$ ,  $e(x_2x_4, a_1a_3a_4) = 6$  and  $e(x_2, a_2a_5) = e(x_4, a_2a_5) = 1$ . Say w.l.o.g.  $a_5x_4 \in E$ . Then  $[D - x_2 + a_5] \supseteq F_1$  and  $x_2 \rightarrow (L, a_5)$ , a contradiction. Therefore  $a_1x_3 \notin E$ . If  $x_1a_1 \in E$  then  $e(x_1, a_2a_5) = 0$  and so  $e(a_1, Q - x_3) + e(L - a_1, Q - x_1) \geq 15$ . Then  $[D - x_2 + a_1] \supseteq F_1$  and  $x_2 \rightarrow (L, a_1)$ , a contradiction. Hence  $N(x_1, L) \subseteq \{a_2, a_5\}$ . As  $e(Q, L) \geq 15$ ,  $e(a_2a_5, x_2x_4) \geq 3$  and  $e(a_2a_4, x_3x_i) \geq 3$  for  $i \in \{2, 4\}$ . Say w.l.o.g.  $x_2a_5 \in E$ . Then  $[x_0, x_1, x_2, a_5, a_1] \supseteq C_5$  and  $[x_3, x_4, a_2, a_3, a_4] \supseteq C_5$ , a contradiction.

*Case 3.*  $N(x_0, L) = \{a_i, a_{i+2}\}$  for some  $i \in \{1, 2, 3, 4, 5\}$ . Say  $N(x_0, L) = \{a_1, a_3\}$ . Then  $e(Q, L) \geq 14$ . As  $H \not\supseteq 2C_5$ ,  $e(a_2, Q) \leq 2$ . We claim that



$e(x_1, a_1a_3) = 0$ . On the contrary, say  $e(x_1, a_1a_3) \geq 1$ . Then  $[x_0, x_1, a_1, a_2, a_3] \supseteq C_5$ . Since  $H \not\supseteq 2C_5$ ,  $H \not\supseteq F_1 \uplus C_5$  and  $H \not\supseteq F_2 \uplus C_5$ , we see that  $e(a_4a_5, x_2x_3x_4) \leq 3$  by Lemma 2.1(c). Thus  $e(a_4a_5, Q) \leq 5$  and  $e(a_1a_3, Q) \geq 14 - e(a_2, Q) - e(a_4a_5, Q) \geq 7$ . As  $e(a_1a_3, Q) \leq 8$ , it follows that either  $e(a_1, Q) = 4$  and  $x_1a_5 \in E$  or  $e(a_3, Q) = 4$  and  $x_1a_4 \in E$ . Say w.l.o.g. the former holds. Then  $[D - x_3 + a_1] \supseteq F_2$ ,  $[x_0, x_1, a_1, a_5, a_4] \supseteq F_1$  and  $[x_0, x_1, a_1, a_5, x_i] \supseteq F_2$  for  $i \in \{2, 4\}$ . Furthermore, if  $x_1a_2 \in E$  then  $[x_0, x_1, a_1, a_5, a_2] \supseteq F_2$  and  $[x_0, x_1, a_1, a_2, x_i] \supseteq F_2$  for  $i \in \{2, 4\}$ . Assume for the moment that  $e(a_3, x_2x_4) = 2$ . Then we see that  $e(a_2, x_2x_4) = 0$  as  $H \not\supseteq F_1 \uplus C_5$ . If  $x_1a_2 \in E$ , then  $e(a_4, x_2x_4) = 0$  as  $H \not\supseteq F_2 \uplus C_5$  and for the same reason,  $[a_3, a_4, a_5, x_3, x_i] \not\supseteq C_5$  for  $i \in \{2, 4\}$ . This implies that  $x_3a_5 \notin E$  and so  $e(a_5, x_2x_4) \geq 1$  since  $8 \geq e(a_1a_3, Q) \geq 14 - e(a_2, Q) - e(a_4a_5, Q) \geq 7$ . Thus  $x_3a_3 \notin E$  since  $[a_3, a_4, a_5, x_3, x_i] \not\supseteq C_5$  for  $i \in \{2, 4\}$ . It follows that  $\{a_3x_1, x_3a_4\} \subseteq E$ . Consequently,  $[a_1, a_5, a_4, x_2, x_3] \supseteq C_5$  and  $[x_0, x_1, x_4, a_2, a_3] \supseteq F_2$ , a contradiction. Hence  $x_1a_2 \notin E$ . As  $e(Q, L) \geq 14$ , it follows that  $a_2x_3 \in E$ ,  $e(a_1a_3, Q) = 8$ ,  $e(x_1, a_4a_5) = 2$  and  $e(a_4a_5, x_2x_3x_4) = 3$ . Say w.l.o.g.  $a_4x_2 \in E$ . Then  $[a_2, a_3, a_4, x_2, x_3] \supseteq C_5$  and so  $H \supseteq F_2 \uplus C_5$ , a contradiction. Hence  $e(a_3, x_2x_4) \leq 1$ . It follows that  $e(a_3, x_2x_4) = 1$ ,  $e(a_3, x_1x_3) = 2$ ,  $e(a_2, Q) = 2$  and  $e(a_4a_5, Q) = 5$  with  $e(x_1, a_4a_5) = 2$ . Thus  $[x_0, x_1, a_5, a_4, a_3] \supseteq C_5$  and so  $e(a_2, x_1x_3) = 2$  as  $H \not\supseteq 2C_5$ . Say w.l.o.g.  $a_3x_2 \in E$ . As  $H \not\supseteq F_2 \uplus C_5$ , we see that  $[x_2, x_3, a_5, a_4, a_3] \not\supseteq C_5$  and  $[a_3, a_4, x_2, x_3, x_4] \not\supseteq C_5$ . This implies that  $e(a_5, x_2x_3) = 0$  and  $a_4x_4 \notin E$ . As  $e(a_4a_5, x_2x_3x_4) = 3$ , it follows that  $[a_4, a_5, x_2, x_3, x_4] \supseteq C_5$  and so  $H \supseteq 2C_5$ , a contradiction. Therefore  $e(x_1, a_1a_3) = 0$ . Assume  $e(x_1, a_4a_5) = 0$ . As  $e(Q, L) \geq 14$ , it follows that  $e(x_2x_3x_4, L - a_2) = 12$  and  $e(a_2, Q) = 2$ . Thus  $[x_2, x_3, x_4, a_4, a_5] \supseteq K_5^-$ . As  $[x_1, x_0, a_1, a_2, a_3] \supseteq F$ , we have  $\tau(L) \geq 4$  by the optimality of  $\{D, L\}$ . Consequently,  $x_0 \rightarrow (L, a_r)$  for some  $r \in \{4, 5\}$  and so  $H \supseteq 2C_5$  as  $[Q + a_r] \supseteq C_5$ , a contradiction. Hence  $e(x_1, a_4a_5) \geq 1$ . Say w.l.o.g.  $x_1a_5 \in E$ . Then  $[x_0, x_1, a_5, a_4, a_3] \supseteq C_5$ . Since  $H \not\supseteq 2C_5$ ,  $H \not\supseteq F_1 \uplus C_5$  and  $H \not\supseteq F_2 \uplus C_5$ , we see that  $e(a_1a_2, x_2x_3x_4) \leq 3$  by Lemma 2.1(c). Thus  $e(a_1a_2, Q) \leq 4$  and so  $e(a_3a_4a_5, Q) \geq 10$ . Hence  $e(a_4a_5, Q) \geq 7$ . As above, we shall have that  $[x_2, x_3, x_4, a_4, a_5] \not\supseteq K_5^-$ . This implies that  $e(a_4a_5, x_2x_3x_4) \neq 6$ . Thus  $e(a_4a_5, x_2x_3x_4) = 5$ ,  $e(x_1, a_4a_5) = 2$ ,  $e(a_3, x_2x_3x_4) = 3$  and  $e(a_1a_2, Q) = 4$ . Similarly, we shall have  $e(a_1, x_2x_3x_4) = 3$  as  $[x_0, x_1, a_4, a_5, a_1] \supseteq C_5$ . As  $e(a_4a_5, x_2x_3x_4) = 5$ , we may assume w.l.o.g. that  $e(a_4, x_2x_3x_4) = 3$ . Thus  $[a_3, a_4, x_2, x_3, x_4] \supseteq K_5^-$  and  $[a_2, a_1, a_5, x_1, x_0] \supseteq F$ . By the optimality of  $\{D, L\}$ , we shall have  $\tau(L) \geq 4$ . Thus  $x_0 \rightarrow (L, a_r)$  for some  $r \in \{4, 5\}$  and so  $H \supseteq 2C_5$ , a contradiction.

*Case 4.*  $N(x_0, L) = \{a_i, a_{i+1}\}$  for some  $i \in \{1, 2, 3, 4, 5\}$ . Say,  $N(x_0, L) = \{a_1, a_2\}$ . First, suppose that  $x_1a_4 \in E$ . Then  $[x_0, x_1, a_4, a_5, a_1] \supseteq C_5$  and  $[x_0, x_1, a_4, a_3, a_2] \supseteq C_5$ . Since  $H \not\supseteq 2C_5$ ,  $H \not\supseteq F_1 \uplus C_5$  and  $H \not\supseteq F_2 \uplus C_5$ , we see that  $e(a_2a_3, Q - x_1) \leq 3$  and  $e(a_1a_5, Q - x_1) \leq 3$  by Lemma 2.1(c). As  $e(Q, L) \geq 14$ , it follows that  $e(x_1, L) = 5$ ,  $e(a_4, Q) = 4$ ,  $e(a_2a_3, Q - x_1) = 3$  and

$e(a_1a_5, Q - x_1) = 3$ . Then  $[x_0, x_1, a_5, a_1, a_2] \supseteq C_5$  and so  $e(a_3a_4, Q - x_1) \leq 3$ . Thus  $e(a_3, Q - x_1) = 0$  as  $e(a_4, Q - x_1) = 3$ . Similarly,  $e(a_5, Q - x_1) = 0$ . Thus  $e(a_1a_2, Q - x_1) = 6$ . Then  $[a_1, x_2, x_3, a_4, a_5] \supseteq C_5$  and  $[a_3, a_2, x_0, x_1, x_4] \supseteq F_2$ , a contradiction. Hence  $x_1a_4 \notin E$ .

Next, suppose  $e(x_3, a_1a_2) = 2$ . Then  $e(x_i, a_1a_3) \leq 1$  and  $e(x_i, a_2a_5) \leq 1$  for each  $i \in \{2, 4\}$  as  $H \not\supseteq 2C_5$ . Thus  $e(x_2x_4, L - a_4) \leq 4$  and so  $e(x_1, L - a_4) + e(x_3, L) + e(a_4, x_2x_4) \geq 10$ . Then  $e(x_1, a_1a_2) \geq 1$ . Thus  $[x_i, x_1, x_0, a_1, a_2] \supseteq F_1$  for  $i \in \{2, 4\}$ . Clearly,  $e(x_3, a_3a_5) \geq 1$ . Assume  $e(x_3, a_3a_5) = 2$ . Then  $e(x_2x_4, a_3a_5) = 0$  as  $H \not\supseteq F_1 \uplus C_5$ . If  $e(a_4, x_2x_4) = 1$ , then  $e(x_1, L - a_4) = 4$ ,  $e(x_3, L) = 5$  and  $e(x_2x_4, a_1a_2) = 4$ . Thus  $[x_0, x_1, x_4, a_2, a_3] \supseteq F_2$  and  $[x_3, a_4, a_5, a_1, x_2] \supseteq C_5$ , a contradiction. Hence  $e(a_4, x_2x_4) = 2$ . If  $x_3a_4 \in E$  then  $[x_2, x_3, x_4, a_4, a_i] \supseteq F_2$  for  $i \in \{3, 5\}$ . As  $e(x_1, a_3a_5) \geq 1$ , we see that  $H \supseteq F_2 \uplus C_5$ , a contradiction. Thus  $x_3a_4 \notin E$ ,  $e(x_1, L - a_4) = 4$ ,  $e(x_3, L - a_4) = 4$ ,  $e(a_4, x_2x_4) = 2$  and  $e(x_2x_4, a_1a_2) = 4$ . Thus  $[x_0, x_1, x_4, a_2, a_3] \supseteq F_2$  and  $[x_3, a_1, a_5, a_4, x_2] \supseteq C_5$ , a contradiction. We conclude that  $e(x_3, a_3a_5) = 1$ . Thus  $e(x_1, L - a_4) = 4$ ,  $e(x_3, L) = 4$  and  $e(a_4, x_2x_4) = 2$ . Say w.l.o.g.  $x_3a_5 \in E$ . Then  $[x_2, x_4, a_5, a_4, x_3] \supseteq F_2$  and  $[x_0, x_1, a_1, a_2, a_3] \supseteq C_5$ , a contradiction. Therefore  $e(x_3, a_1a_2) \leq 1$ . Next, suppose that  $e(x_2, a_1a_2) \geq 1$  and  $e(x_4, a_1a_2) \geq 1$ . Then  $[x_i, x_1, x_0, a_1, a_2] \supseteq C_5$  for  $i \in \{2, 4\}$ . Since  $H \not\supseteq 2C_5$ ,  $H \not\supseteq F_1 \uplus C_5$  and  $H \not\supseteq F_2 \uplus C_5$ , we see that  $e(x_3x_i, a_3a_4a_5) \leq 3$  for  $i \in \{2, 4\}$  by Lemma 2.1(c). Furthermore, if for some  $i \in \{2, 4\}$ , say  $i = 2$ , we have  $e(x_2, a_3a_4a_5) = 3$ , then  $[x_2, a_3, a_4, a_5, a_j] \supseteq F_1$  for  $j \in \{1, 2\}$  and so  $e(x_3, a_1a_2) = 0$  since  $H \not\supseteq C_5 \uplus F_1$ . Consequently,  $e(x_1, L - a_4) = 4$ ,  $e(x_2x_4, L) = 10$  and so  $H \supseteq 2C_5$ , a contradiction. Therefore if  $e(x_3, a_3a_4a_5) = 0$  then  $e(x_i, a_3a_4a_5) \leq 2$  for  $i \in \{2, 4\}$ . Together with  $x_1a_4 \notin E$  and  $e(x_3, a_1a_2) \leq 1$ , we see that if  $e(x_3, a_3a_4a_5) = 0$  or  $e(x_3, a_3a_4a_5) > 1$  then  $e(Q, L) \leq 13$ , a contradiction. Hence  $e(x_3, a_3a_4a_5) = 1$ . It follows that  $e(x_1, L - a_4) = 4$ ,  $e(x_3, a_1a_2) = 1$ ,  $e(x_2x_4, a_1a_2) = 4$ ,  $e(x_2, a_3a_4a_5) = 2$  and  $e(x_4, a_3a_4a_5) = 2$ . If  $e(x_3, a_3a_5) = 1$ , then either  $[x_2, x_3, a_3, a_4, a_5] \supseteq C_5$  or  $[x_2, x_3, a_3, a_4, a_5] \supseteq F_1$ , and consequently,  $H \supseteq C_5 \uplus F_1$ , a contradiction. Hence  $x_3a_4 \in E$ . Then we see that  $[x_2, x_3, a_4, a_5, a_1] \supseteq C_5$  and  $[x_0, x_1, x_4, a_2, a_3] \supseteq F_2$ , a contradiction. Therefore either  $e(x_2, a_1a_2) = 0$  or  $e(x_4, a_1a_2) = 0$ . Say w.l.o.g.  $e(x_4, a_1a_2) = 0$ .

Finally, if  $e(x_2, a_1a_2) \geq 1$  then, as above, we would have  $e(x_3x_4, a_3a_4a_5) \leq 3$  and so  $e(Q, L) \leq 13$ , a contradiction. Hence  $e(x_2, a_1a_2) = 0$ . As  $e(Q, L) \geq 14$ , it follows that  $e(x_1, L - a_4) = 4$ ,  $e(x_3, L - a_i) = 4$  for some  $i \in \{1, 2\}$  and  $e(x_2x_4, a_3a_4a_5) = 6$ . As  $[x_2, x_3, x_4, a_4, a_5] \supseteq C_5$ , we see  $H \supseteq 2C_5$ , a contradiction.

*Case 5.*  $N(x_0, L) = \{a_i, a_{i+1}, a_{i+2}\}$  for some  $i \in \{1, 2, 3, 4, 5\}$ .

Say  $N(x_0, L) = \{a_1, a_2, a_3\}$ . Then for each  $i \in \{2, 4, 5\}$ ,  $[L - a_i + x_0] \supseteq C_5$  or  $[L - a_i + x_0] \supseteq F_1$  and so  $e(a_i, Q) \leq 2$ . Thus  $e(a_1a_3, Q) \geq 7$ . Hence  $[Q + a_i] \supseteq C_5$  for each  $i \in \{1, 3\}$ . Therefore  $[L - a_i + x_0] \not\supseteq C_5$  and  $[L - a_i + x_0] \not\supseteq B$  for each  $i \in \{1, 3\}$ . This implies that  $\tau(L) \leq 1$ . As  $e(a_1a_3, Q) \leq 8$ ,  $e(a_4a_5, Q) \geq 3$ . Say

w.l.o.g.  $e(a_5, Q) = 2$ . As  $[Q + a_5] \not\supseteq C_5$ ,  $N(a_5, Q) = \{x_2, x_4\}$  or  $N(a_5, Q) = \{x_1, x_3\}$ . First, assume  $N(a_5, Q) = \{x_2, x_4\}$ . Then  $[a_4, a_5, x_2, x_3, x_4] \supseteq F$ . As  $e(a_1a_3, Q) \geq 7$ ,  $e(x_1, a_1a_3) \geq 1$  and so  $[x_0, x_1, a_1, a_2, a_3] \supseteq C' \cong C_5$  with  $\tau(C') \geq 2$ , contradicting the optimality of  $\{D, L\}$ . Hence  $N(a_5, Q) = \{x_1, x_3\}$ . Then  $[a_4, a_5, x_1, x_i, x_3] \supseteq F$  for each  $i \in \{2, 4\}$ . By the optimality of  $\{D, L\}$  and Lemma 2.1(b), we get  $e(x_i, a_1a_3) \leq 1$  for each  $i \in \{2, 4\}$  and so  $e(a_1a_3, Q) \leq 6$ , a contradiction.

*Case 6.*  $N(x_0, L) = \{a_i, a_{i+1}, a_{i+3}\}$  for some  $i \in \{1, 2, 3, 4, 5\}$ .

Say  $N(x_0, L) = \{a_1, a_2, a_4\}$ . Clearly,  $x_0 \rightarrow (L, a_3)$  and  $x_0 \rightarrow (L, a_5)$ . Thus  $e(a_3, Q) \leq 2$  and  $e(a_5, Q) \leq 2$  for otherwise  $H \supseteq 2C_5$ . As  $H \not\supseteq 2C_5$ , we see that  $x_0 \not\rightarrow L$  and so  $a_3a_5 \notin E$ . As  $e(Q, L) \geq 13$ ,  $e(a_3a_5, Q) \geq 1$ . Say w.l.o.g.  $e(a_5, Q) \geq 1$ . Then  $[Q + a_5] \supseteq F$ . By the optimality of  $\{D, L\}$ ,  $\tau(L) \geq \tau(x_0a_1a_2a_3a_4x_0)$ . This implies that  $a_2a_5 \in E$ . Similarly, if  $e(a_3, Q) \geq 1$  then  $a_1a_3 \in E$ . Assume  $a_1a_3 \notin E$ . Then  $e(a_3, Q) = 0$  and so  $e(a_1a_2a_4, Q) \geq 11$ . Then  $e(a_r, Q) = 4$  for some  $r \in \{1, 2\}$  and  $[L - a_r + x_0] \supseteq F$ . As  $\tau(a_rx_1x_2x_3x_4a_r) \geq 3$ , it follows that  $\tau(L) = 3$  and so  $\{a_1a_4, a_2a_4\} \subseteq E$ . Thus  $[L - a_1 + x_0] \supseteq F_2$  and  $[Q + a_1] \supseteq C_5$ , a contradiction. Therefore  $a_1a_3 \in E$ . Thus  $[L - a_4 + x_0] \supseteq F_2$ . Hence  $[Q + a_4] \not\supseteq C_5$  and so  $e(a_4, Q) \leq 2$ . Consequently,  $e(a_1a_2, Q) \geq 7$  and so  $[Q + a_i] \supseteq C_5$  for each  $i \in \{1, 2\}$ . Hence  $a_1a_4 \notin E$  and  $a_2a_4 \notin E$  for otherwise  $H \supseteq F_2 \uplus C_5$ . Hence  $\tau(L) = 2$ . By the optimality of  $\{D, L\}$ ,  $[Q + a_i] \not\supseteq C$  with  $C \cong C_5$  and  $\tau(C) \geq 3$  for each  $i \in \{1, 2\}$ . This implies that  $e(a_i, Q) \leq 3$  for each  $i \in \{1, 2\}$  and therefore  $e(a_1a_2, Q) \leq 6$ , a contradiction. ■

**Lemma 2.6.** *Let  $D$ ,  $L_1$  and  $L_2$  be disjoint subgraphs of  $G$  with  $D \cong F$  and  $L_1 \cong L_2 \cong C_5$ . Suppose that  $L_1 = a_1a_2a_3a_4a_5a_1$ ,  $V(D) = \{x_0, x_1, x_2, x_3, x_4\}$  and  $E(D) = \{x_0x_1, x_1x_2, x_2x_3, x_3x_4, x_4x_1\}$  such that  $e(x_0, L_1) = 0$ , and  $e(x_1x_3, L_1) = 10$ ,  $N(x_2, L_1) = N(x_4, L_1) = \{a_1, a_2, a_4\}$ ,  $\tau(L_1) = 4$  and  $a_3a_5 \notin E$ . Suppose that  $e(x_0x_2a_3a_5, L_2) \geq 13$ . Then  $[D, L_1, L_2]$  contains either of  $F_1 \uplus 2C_5$  or  $3C_5$ .*

**Proof.** For the proof, we may assume that none of  $x_0x_3, x_1x_3$  and  $x_2x_4$  is an edge as they will not be used in the proof. Set  $G_1 = [D, L_1]$ ,  $G_2 = [G_1, L_2]$  and  $R = \{x_0, x_2, a_3, a_5\}$ . It is easy to see that for any permutation  $f$  of  $\{x_2, a_3, a_5\}$ , we can extend  $f$  to be an automorphism of  $G_1$  such that every vertex of  $G_1 - \{x_2, a_3, a_5\}$  is fixed under  $f$ . Therefore  $x_2, a_3$  and  $a_5$  are in the symmetric position in the following argument. On the contrary, suppose that  $G_2 \not\supseteq F_1 \uplus 2C_5$  and  $G_2 \not\supseteq 3C_5$ . It is easy to check that if  $u \rightarrow (L_2; R - \{u\})$  for some  $u \in R$  then  $G_2 \supseteq F_1 \uplus 2C_5$  or  $G_2 \supseteq 3C_5$ . Therefore  $u \not\rightarrow (L_2; R - \{u\})$  for each  $u \in R$ . By Lemma 2.1(d), there exist two labellings  $R = \{y_1, y_2, y_3, y_4\}$  and  $L_2 = b_1b_2b_3b_4b_5b_1$  such that  $e(y_1y_2, b_1b_2b_3b_4) = 8$ ,  $e(y_3, b_1b_5b_4) = 3$  and  $e(y_4, b_1b_4) = 2$ . If  $x_0 \in \{y_1, y_2\}$ , we may assume that  $\{y_1, y_2\} = \{x_0, x_2\}$ . Then  $[x_0, x_1, x_2, b_2, b_3] \supseteq C_5$ ,  $[a_3, a_5, b_1, b_5, b_4] \supseteq C_5$  and  $[x_3, x_4, a_1, a_2, a_4] \supseteq C_5$ , a contradiction. Hence  $x_0 \notin$

$\{y_1, y_2\}$ . Say w.l.o.g. that  $\{y_1, y_2\} = \{a_3, a_5\}$ . Thus  $[a_3, a_4, a_5, b_2, b_3] \supseteq C_5$ ,  $[x_0, x_2, b_1, b_5, b_4] \supseteq C_5$  and  $[x_1, x_4, x_3, a_1, a_2] \supseteq C_5$ , a contradiction. ■

**Lemma 2.7.** *Let  $D$  and  $L$  be disjoint subgraphs of  $G$  with  $D \cong K_4^+$  and  $L \cong B$ . Let  $R$  be the set of the four vertices of  $L$  with degree 2 in  $L$ . Suppose that  $e(D, R) \geq 13$ . Then either  $[D, L] \supseteq K_4^+ \uplus C_5$  or  $[D, L] \supseteq 2C_5$  or  $[D, L] \supseteq B \uplus C_5$ .*

**Proof.** Say  $H = [D, L]$ . On the contrary, suppose that  $H$  contains none of  $K_4^+ \uplus C_5$ ,  $2C_5$  and  $B \uplus C_5$ . Say  $V(D) = \{x_0, x_1, x_2, x_3, x_4\}$  with  $e(x_0, D) = 1$  and  $x_0x_1 \in E$ . Let  $Q = [x_1, x_2, x_3, x_4]$ . Say  $L = a_0a_1a_2a_0a_3a_4a_0$ . Then  $Q \cong K_4$  and  $R = \{a_1, a_2, a_3, a_4\}$ . If  $e(x_0, R) \geq 3$ , say w.l.o.g.  $e(x_0, a_1a_2a_3) = 3$ , then  $[L - a_i + x_0] \supseteq C_5$  and so  $Q + a_i \not\supseteq C_5$  for each  $i \in \{1, 2, 4\}$ . Consequently,  $e(a_i, Q) \leq 1$  for all  $i \in \{1, 2, 4\}$  and so  $e(D, R) \leq 11$ , a contradiction. Hence  $e(x_0, R) \leq 2$ . Suppose that  $e(x_0, R) = 2$ . Then  $e(R, Q) \geq 11$ . First, assume  $e(x_0, a_1a_2) = 1$  and  $e(x_0, a_3a_4) = 1$ . Say w.l.o.g.  $e(x_0, a_1a_3) = 2$ . Then  $e(a_2, Q) \leq 1$  and  $e(a_4, Q) \leq 1$  as  $H \not\supseteq 2C_5$ . Consequently,  $e(R, Q) \leq 10$ , a contradiction. Therefore we may assume w.l.o.g. that  $e(x_0, a_1a_2) = 2$ . We claim  $e(x_1, a_1a_2) = 0$ . To see this, suppose  $e(x_1, a_1a_2) \geq 1$ . Then  $[x_0, x_1, a_1, a_2, a_0] \supseteq C_5$ . Thus  $e(a_3a_4, x_2x_3x_4) \leq 2$  for otherwise  $[a_3, a_4, x_2, x_3, x_4] \supseteq C_5$  or  $[a_3, a_4, x_2, x_3, x_4] \supseteq K_4^+$ . Thus  $e(a_3a_4, Q) \leq 4$  and so  $e(a_1a_2, Q) \geq 7$ . Say w.l.o.g.  $e(a_1, Q) = 4$ . Then  $[D - x_i + a_1] \supseteq K_4^+$  for each  $i \in \{2, 3, 4\}$  and so  $[L - a_1 + x_i] \not\supseteq C_5$  for each  $i \in \{2, 3, 4\}$ . Thus  $I(a_2a_3, Q - x_1) = \emptyset$  and so  $e(a_2a_3, Q) \leq 5$ . Hence  $e(a_4, Q) \geq 2$ . Similarly,  $e(a_3, Q) \geq 2$ . It follows that  $[a_3, a_4, x_2, x_3, x_4] \supseteq C_5$  or  $[a_3, a_4, x_2, x_3, x_4] \supseteq B$ , a contradiction. This shows that  $e(x_1, a_1a_2) = 0$ . Suppose  $e(a_1, Q - x_1) = 3$  or  $e(a_2, Q - x_1) = 3$ . Then  $[x_0, x_1, x_i, a_1, a_2] \supseteq C_5$  for each  $i \in \{2, 3, 4\}$ . Thus  $[x_i, x_j, a_0, a_3, a_4] \not\supseteq C_5$  and  $[x_i, x_j, a_0, a_3, a_4] \not\supseteq B$  for each  $2 \leq i < j \leq 4$ . This implies that  $e(a_3a_4, Q - x_1) \leq 2$ . Hence  $e(a_1a_2, Q) \geq 7$  and so  $e(x_1, a_1a_2) \geq 1$ , a contradiction. Hence  $e(a_i, Q - x_1) \leq 2$  for each  $i \in \{1, 2\}$  and so  $e(a_3a_4, Q) \geq 7$ . Say w.l.o.g.  $e(a_4, Q) = 4$ . Then  $[D - x_i + a_4] \supseteq K_4^+$  for each  $i \in \{2, 3, 4\}$  and therefore  $I(a_1a_3, Q - x_1) = \emptyset$  as  $H \not\supseteq K_4^+ \uplus C_5$ . Thus  $e(a_1a_3, Q) \leq 4$  and so  $e(a_2, Q) \geq 3$ , a contradiction. Next, suppose  $e(x_0, R) = 1$ . Then  $e(Q, R) \geq 12$ . Say  $x_0a_1 \in E$ . Suppose  $e(x_1, a_1a_2) \geq 1$ . Then  $[x_0, x_1, a_1, a_2, a_0] \supseteq C_5$  or  $[x_0, x_1, a_1, a_2, a_0] \supseteq B$ . Thus  $[x_2, x_3, x_4, a_3, a_4] \not\supseteq C_5$ . This implies that  $e(a_3a_4, Q - x_1) \leq 3$ . Thus  $e(a_3a_4, Q) \leq 5$  and so  $e(a_1a_2, Q) \geq 7$ . Thus  $[D - x_i + a_1] \supseteq C_5$  for all  $i \in \{2, 3, 4\}$ . As  $H \not\supseteq 2C_5$ ,  $I(a_2a_3, Q - x_1) = \emptyset$  and  $I(a_2a_4, Q - x_1) = \emptyset$ . Hence  $e(a_2a_3, Q) \leq 5$  and so  $e(a_4, Q) \geq 3$ . Then  $I(a_2a_4, Q - x_1) \neq \emptyset$ , a contradiction. Hence  $e(x_1, a_1a_2) = 0$ . Thus  $e(a_1a_2, Q) \leq 6$  and  $e(a_3a_4, Q) \geq 6$ . Then  $[x_i, x_j, a_3, a_4, a_0] \supseteq C_5$  for some  $2 \leq i < j \leq 4$ . Say  $\{i, j, k\} = \{2, 3, 4\}$ . Then  $a_2x_k \notin E$  as  $H \not\supseteq 2C_5$ . Therefore  $e(a_1a_2, Q) \leq 5$  and so  $e(a_3a_4, Q) \geq 7$ . Thus  $[x_r, x_t, a_3, a_4, a_0] \supseteq C_5$  for all  $2 \leq r < t \leq 4$ . Therefore  $e(a_2, Q - x_1) = 0$  as  $H \not\supseteq 2C_5$ . Consequently,  $e(Q, R) \leq 11$ , a contradiction.

Finally, suppose  $e(x_0, R) = 0$ . As  $e(R, Q) \geq 13$ ,  $e(a_i, Q) = 4$  for some  $a_i \in R$ .

Say  $e(a_1, Q) = 4$ . Then  $I(a_2a_3, Q - x_1) = \emptyset$  as  $H \not\supseteq K_4^+ \uplus C_5$ . Thus  $e(a_4, Q) = 4$  as  $e(R, Q) \geq 13$ . Similarly,  $e(a_3, Q) = 4$ . Then we readily see that  $H \supseteq K_4^+ \uplus C_5$ , a contradiction. ■

**Lemma 2.8.** *Let  $B_1$  and  $B_2$  be disjoint subgraphs of  $G$  such that  $B_1 \cong B$  and  $B_2 \cong B$ . Let  $R$  be the set of the four vertices of  $B_1$  with degree 2 in  $B_1$ . Suppose that  $e(R, B_2) \geq 13$ . Then  $[B_1, B_2] \supseteq 2C_5$  or  $[B_1, B_2] \supseteq B \uplus C_5$ .*

**Proof.** On the contrary, suppose that  $[B_1, B_2] \not\supseteq 2C_5$  and  $[B_1, B_2] \not\supseteq B \uplus C_5$ . Say  $B_1 = a_0a_1a_2a_0a_3a_4a_0$  and  $B_2 = b_0b_1b_2b_0b_3b_4b_0$ . Then  $R = \{a_1, a_2, a_3, a_4\}$  and  $e(R, B_2 - b_0) \geq 9$ . This implies that  $e(a_i a_{i+1}, b_j b_{j+1}) \geq 3$  for some  $i \in \{1, 3\}$  and  $j \in \{1, 3\}$ . Say w.l.o.g.  $e(a_1a_2, b_1b_2) \geq 3$ . Then  $[a_1, a_2, b_0, b_1, b_2] \supseteq C_5$  and  $[b_1, b_2, a_0, a_1, a_2] \supseteq C_5$ .

Therefore  $[a_0, a_3, a_4, b_3, b_4] \not\supseteq C_5$ ,  $[a_0, a_3, a_4, b_3, b_4] \not\supseteq B$ ,  $[b_0, b_3, b_4, a_3, a_4] \not\supseteq C_5$  and  $[b_0, b_3, b_4, a_3, a_4] \not\supseteq B$ . This implies that  $e(a_3a_4, b_3b_4) \leq 1$  and  $e(b_0, a_3a_4) \leq 1$ . If  $e(a_1a_2, b_3b_4) \geq 3$ , then we also have that  $e(a_3a_4, b_1b_2) \leq 1$  and it follows that  $e(a_1a_2, B_2) = 10$  and  $e(a_3a_4, b_3b_4) = 1$  as  $e(R, B_2) \geq 13$ . Consequently,  $[B_2 - b_r + a_1] \supseteq C_5$  and  $[B_1 - a_1 + b_r] \supseteq C_5$  where  $r \in \{3, 4\}$  with  $e(b_r, a_3a_4) = 1$ , a contradiction. Hence  $e(a_1a_2, b_3b_4) \leq 2$ . Suppose  $e(a_3a_4, b_1b_2) \geq 3$ . Similarly, we shall have  $e(a_1a_2, b_3b_4) \leq 1$ ,  $e(b_0, a_1a_2) \leq 1$  and so  $e(R, B_2) \leq 12$ , a contradiction. Therefore,  $e(a_3a_4, b_1b_2) \leq 2$ . Thus  $e(a_3a_4, B_2) \leq 4$  and so  $e(a_1a_2, B_2) \geq 9$ . Consequently,  $e(a_1a_2, b_3b_4) \geq 3$ , a contradiction. ■

**Lemma 2.9.** *Let  $D$  and  $L$  be disjoint subgraphs of  $G$  with  $D \cong F_1$  and  $L \cong C_5$ . Suppose that  $\{D, L\}$  is optimal and  $e(D, L) \geq 16$ . Then  $[D, L]$  contains one of  $K_4^+ \uplus C_5$ ,  $K_4^+ \uplus B$ ,  $2C_5$  and  $B \uplus C_5$ , or there exist two labellings  $L = a_1a_2a_3a_4a_5a_1$  and  $V(D) = \{x_0, x_1, x_2, x_3, x_4\}$  with  $E(D) = \{x_0x_1, x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_2x_4\}$  such that  $e(x_0, L) = 0$ ,  $e(a_1a_2a_4, D - x_0) = 12$ ,  $N(a_3, D) = N(a_5, D) = \{x_2, x_4\}$ ,  $\tau(L) = 4$  and  $a_3a_5 \notin E$ .*

**Proof.** Say  $H = [D, L]$ . Say that  $H$  does not contain any of  $K_4^+ \uplus C_5$ ,  $K_4^+ \uplus B$ ,  $2C_5$  and  $B \uplus C_5$ .

Let  $V(D) = \{x_0, x_1, x_2, x_3, x_4\}$ ,  $E(D) = \{x_0x_1, x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_2x_4\}$  and  $L = a_1a_2a_3a_4a_5a_1$ . Set  $Q = [x_1, x_2, x_3, x_4]$ . Since  $H \not\supseteq 2C_5$  and  $H \not\supseteq B \uplus C_5$ , we see that for each  $a_i \in V(L)$ , if  $x_0 \rightarrow (L, a_i)$  or  $x_0 \xrightarrow{z} (L, a_i)$  then  $e(a_i, Q) \leq 2$ . Thus  $x_0 \not\rightarrow L$  for otherwise  $e(D, L) \leq 15$ . Hence  $e(x_0, L) \leq 4$ .

Assume  $e(x_0, L) = 4$ . Say  $e(x_0, a_1a_2a_3a_4) = 4$ . As  $x_0 \not\rightarrow L$ ,  $\tau(a_5, L) = 0$ . Clearly,  $e(a_i, Q) \leq 2$  for each  $i \in \{2, 3, 5\}$  since  $H \not\supseteq 2C_5$ . Thus  $e(a_1a_4, Q) \geq 6$ . Say  $e(a_1, Q) \geq 3$ . Then  $[Q + a_1] \supseteq C$  with  $C \cong C_5$  and  $\tau(C) \geq 3$ . Then  $a_2a_4 \notin E$  for otherwise  $[L - a_1 + x_0] \supseteq K_4^+$ . Thus  $\tau(L) \leq 2$ . As  $[L - a_1 + x_0] \supseteq F_1$ , we see that  $2 \geq \tau(L) \geq \tau(C) \geq 3$  by the optimality of  $\{D, L\}$ , a contradiction. Therefore  $e(x_0, L) \leq 3$  and so  $e(Q, L) \geq 13$ . Set  $T = x_2x_3x_4x_2$ . We divide the proof into the following six cases.

*Case 1.*  $N(x_0, L) = \{a_i, a_{i+1}, a_{i+2}\}$  for some  $i \in \{1, 2, 3, 4, 5\}$ .

Say  $N(x_0, L) = \{a_1, a_2, a_3\}$ . Then  $Q + a_2 \not\supseteq C_5$  and so  $e(a_2, Q) \leq 2$ . As  $x_0 \not\rightarrow L$ , we see that  $\tau(a_2, L) \leq 1$ . If  $\{a_1a_4, a_3a_5\} \subseteq E$  then  $x_0 \rightarrow (L, a_i)$  or  $x_0 \xrightarrow{z} (L, a_i)$  and so  $e(a_i, Q) \leq 2$  for each  $a_i \in V(L)$ . Consequently,  $e(Q, L) \leq 10$ , a contradiction. Hence  $a_1a_4 \notin E$  or  $a_3a_5 \notin E$ . Thus  $\tau(L) \leq 3$ . Suppose  $\tau(a_2, L) = 1$ . Say w.l.o.g.  $a_2a_4 \in E$ . Then  $x_0 \rightarrow (L, a_i)$  for  $i \in \{3, 5\}$ . Thus  $e(a_i, Q) \leq 2$  for  $i \in \{3, 5\}$ . As  $e(Q, L) \geq 13$ ,  $e(a_1a_4, Q) \geq 7$ . Thus  $[Q + a_r]$  contains a 5-cycle with at least 4 chords, where  $e(a_r, Q) = 4$  with  $r \in \{1, 4\}$ . As  $[L - a_r + x_0] \supseteq F_1$  and by the optimality of  $\{D, L\}$ , we have  $\tau(L) \geq 4$ , a contradiction. Hence  $\tau(a_2, L) = 0$ . Suppose  $a_1a_3 \in E$ . Then  $[L - a_i + x_0] \supseteq K_4^+$  for each  $i \in \{4, 5\}$ . As  $H \not\supseteq K_4^+ \uplus C_5$ ,  $e(a_i, Q) \leq 2$  for  $i \in \{4, 5\}$ . As  $e(Q, L) \geq 13$ ,  $e(a_1a_3, Q) \geq 7$  and  $e(a_4a_5, Q) \geq 3$ . Say w.l.o.g.  $e(a_5, Q) = 2$ . As  $[Q + a_5] \not\supseteq C_5$ ,  $e(a_5, x_2x_4) = 2$ . As  $e(x_1, a_1a_3) \geq 1$ ,  $[x_1, x_0, a_1, a_2, a_3] \supseteq C_5$ . Thus  $e(a_4, T) = 0$  as  $H \not\supseteq 2C_5$ . It follows that  $e(a_1a_3, Q) = 8$  and  $a_4x_1 \in E$ . Consequently,  $H \supseteq 2C_5$ , a contradiction. Hence  $a_1a_3 \notin E$  and so  $\tau(L) \leq 1$ . Since  $[L - a_i + x_0] \supseteq F_1$  for each  $i \in \{4, 5\}$ , we see that  $[Q + a_i]$  does not contain a 5-cycle with at least 2 chords for each  $i \in \{4, 5\}$  by the optimality of  $\{D, L\}$ . This implies that for each  $i \in \{4, 5\}$ ,  $e(a_i, Q) \leq 2$  and if  $e(a_i, Q) = 2$  then  $e(a_i, x_2x_4) = 2$ . Similar to the above, we see that  $H \supseteq 2C_5$ , a contradiction.

*Case 2.*  $N(x_0, L) = \{a_i, a_{i+1}, a_{i+3}\}$  for some  $i \in \{1, 2, 3, 4, 5\}$ .

Say  $N(x_0, L) = \{a_1, a_2, a_4\}$ . Then for each  $i \in \{3, 5\}$ ,  $x_0 \rightarrow (L, a_i)$  and so  $e(a_i, Q) \leq 2$ . Thus  $e(a_1a_2a_4, Q) \geq 13 - e(a_3a_5, Q) \geq 9$ . Suppose that  $e(a_3, Q) = 2$  or  $e(a_5, Q) = 2$ . Say w.l.o.g.  $e(a_5, Q) = 2$ . Then  $e(a_5, x_2x_4) = 2$  as  $[Q + a_5] \not\supseteq C_5$ . If  $a_3x_3 \in E$  then  $[a_3, a_4, a_5, x_3, x_i] \supseteq C_5$  for  $i \in \{2, 4\}$  and so  $e(x_i, a_1a_2) = 0$  for  $i \in \{2, 4\}$  since  $H \not\supseteq 2C_5$ . Consequently,  $e(a_1a_2a_4, Q) \leq 8$ , a contradiction. Hence  $a_3x_3 \notin E$ . If  $a_3x_1 \in E$  then  $[x_1, x_0, a_1, a_2, a_3] \supseteq C_5$  and so  $e(a_4, T) = 0$  as  $H \not\supseteq 2C_5$ . Thus  $e(a_1a_2a_4, Q) = 9$  and so  $e(a_3, Q) = 2$ . Consequently,  $[Q + a_3] \supseteq C_5$ , a contradiction. Hence  $N(a_3, Q) \subseteq \{x_2, x_4\}$ . If  $e(x_1, a_2a_4) \geq 1$  then  $[x_1, x_0, a_2, a_3, a_4] \supseteq C_5$  and so  $e(a_1, T) = 0$  as  $H \not\supseteq 2C_5$ . It follows that  $e(a_3, x_2x_4) = 2$  and  $e(a_2a_4, Q) = 8$ . Consequently,  $H \supseteq 2C_5$ , a contradiction. Hence  $e(x_1, a_2a_4) = 0$ . Thus  $e(a_2a_4, T) \geq 5$  as  $e(a_1a_2a_4, Q) \geq 9$ . Hence  $[x_3, x_4, a_2, a_3, a_4] \supseteq C_5$  and  $[x_0, x_1, x_2, a_5, a_1] \supseteq C_5$ , a contradiction.

Therefore  $e(a_3, Q) \leq 1$  and  $e(a_5, Q) \leq 1$ . Then  $e(a_1a_2a_4, Q) \geq 11$ . Thus  $e(a_1a_2, Q) \geq 7$ . Say w.l.o.g.  $e(a_1, Q) = 4$ . Then  $[a_5, a_1, x_2, x_3, x_4] \supseteq K_4^+$ . As  $e(x_1, a_2a_4) \geq 1$ ,  $[x_1, x_0, a_2, a_3, a_4] \supseteq C_5$  and so  $H \supseteq K_4^+ \uplus C_5$ , a contradiction.

*Case 3.*  $N(x_0, L) = \{a_i, a_{i+1}\}$  for some  $i \in \{1, 2, 3, 4, 5\}$ . In this case,  $e(Q, L) \geq 14$ . Say  $e(x_0, a_1a_2) = 2$ . Suppose  $x_1a_4 \in E$ . Then  $[x_1, x_0, a_1, a_5, a_4] \supseteq C_5$ . As  $H$  does not contain one of  $2C_5$  and  $K_4^+ \uplus C_5$ , we see that  $e(a_2a_3, T) \leq 2$ . Similarly,  $e(a_1a_5, T) \leq 2$  as  $[x_1, x_0, a_2, a_3, a_4] \supseteq C_5$ . Thus  $e(Q, L) \leq 12$ , a contradiction. Hence  $x_1a_4 \notin E$ . Next, suppose that  $e(x_1, a_3a_5) \geq 1$ . Say w.l.o.g.  $x_1a_3 \in E$ . Then  $[x_1, x_0, a_1, a_2, a_3] \supseteq C_5$ . As  $H$  does not contain one of  $2C_5$ ,

$B \uplus C_5$  and  $K_4^+ \uplus C_5$ , we have that  $e(a_4a_5, T) \leq 2$  and either  $e(a_4, T) = 0$  or  $e(a_5, T) = 0$ . If we also have  $x_1a_5 \in E$  then  $e(a_3a_4, T) \leq 2$  and either  $e(a_4, T) = 0$  or  $e(a_3, T) = 0$ . Consequently, it follows, as  $e(Q, L) \geq 14$ , that  $e(a_5, T) = 2$ ,  $e(a_3, T) = 2$ ,  $e(a_4, T) = 0$  and  $e(a_1a_2, Q) = 8$ . Then  $x_i \rightarrow (L, a_1)$  for some  $x_i \in V(T)$  with  $e(x_i, a_2a_5) = 2$  and so  $H \supseteq 2C_5$ , a contradiction. Hence  $x_1a_5 \notin E$ . Thus  $e(a_1a_2a_3, Q) \geq 12$ . Then  $x_3 \rightarrow (L, a_2)$  and so  $H \supseteq 2C_5$ , a contradiction. We conclude that  $e(x_1, a_3a_4a_5) = 0$ .

As  $e(Q, L) \geq 14$ ,  $e(x_2x_4, a_1a_2) \geq 1$ . Say w.l.o.g.  $e(x_2, a_1a_2) \geq 1$ . Then  $[x_2, x_1, x_0, a_1, a_2] \supseteq C_5$ . As  $H \not\supseteq 2C_5$  and by Lemma 2.1(c),  $e(x_3x_4, a_3a_4a_5) \leq 4$ . Thus  $e(a_3a_4a_5, Q) \leq 7$ . Hence  $e(a_1a_2, Q) \geq 7$ . Say w.l.o.g.  $e(a_1, Q) = 4$ . Then  $x_i \not\rightarrow (L, a_1)$  for each  $x_i \in V(T)$  since  $H \not\supseteq 2C_5$ . This implies that  $I(a_2a_5, T) = \emptyset$  and so  $e(a_2a_5, Q) \leq 4$ . Consequently,  $e(a_3a_4, T) = 6$  as  $e(Q, L) \geq 14$ . Thus  $[a_5, a_4, a_3, x_3, x_4] \supseteq K_4^+$  and  $[x_2, x_1, x_0, a_2, a_1] \supseteq C_5$ , a contradiction.

*Case 4.*  $N(x_0, L) = \{a_i, a_{i+2}\}$  for some  $i \in \{1, 2, 3, 4, 5\}$ . Say,  $N(x_0, L) = \{a_1, a_3\}$ . The  $e(a_2, Q) \leq 2$  as  $H \not\supseteq 2C_5$ . First, suppose  $e(x_1, a_1a_3) \geq 1$ . Then  $[x_1, x_0, a_1, a_2, a_3] \supseteq C_5$  and therefore  $e(a_4a_5, T) \leq 2$ . Thus  $e(a_1a_3, Q) \geq 14 - 2 - 2 - e(x_1, a_4a_5) \geq 8$ . It follows that  $e(a_1a_3, Q) = 8$ ,  $e(a_2, Q) = 2$ ,  $e(a_4a_5, T) = 2$  and  $e(x_1, a_4a_5) = 2$ . Consequently,  $H \supseteq 2C_5$ , a contradiction. Hence  $e(x_1, a_1a_3) = 0$ . Next, suppose  $e(x_1, a_4a_5) \geq 1$ . Say w.l.o.g.  $x_1a_4 \in E$ . Then  $[x_1, x_0, a_1, a_5, a_4] \supseteq C_5$  and so  $e(a_2a_3, T) \leq 2$ . Thus  $e(a_1a_5a_4, Q) \geq 14 - 3 = 11$ . It follows that  $e(a_4a_5, Q) = 8$ ,  $e(a_1, T) = 3$ ,  $x_1a_2 \in E$  and  $e(a_2a_3, T) = 2$ . Then  $[D - x_1 + a_1] \supseteq K_4^+$  and  $[L - a_1 + x_1] \supseteq C_5$ , a contradiction. Hence  $e(x_1, a_4a_5) = 0$ . As  $e(Q, L) \geq 14$ , it follows that  $e(T, L - a_2) = 12$  and  $e(a_2, Q) = 2$ . Then we readily see that  $H \supseteq 2C_5$ , a contradiction.

*Case 5.*  $e(x_0, L) = 1$ . Then  $e(Q, L) \geq 15$ . Say  $x_0a_1 \in E$ . First, suppose  $e(x_1, a_3a_4) \geq 1$ . Say w.l.o.g.  $x_1a_3 \in E$ . Then  $[x_1, x_0, a_1, a_2, a_3] \supseteq C_5$ . Thus  $e(a_4a_5, T) \leq 2$  and so  $e(a_4a_5, Q) \leq 4$ . If we also have  $x_1a_4 \in E$  then  $e(a_2a_3, T) \leq 2$  as  $[x_1, x_0, a_1, a_5, a_4] \supseteq C_5$ . But then we obtain  $e(Q, L) \leq 12$ , a contradiction. Hence  $x_1a_4 \notin E$ . As  $e(Q, L) \geq 15$ , it follows that  $e(a_1a_2a_3, Q) = 12$ ,  $e(a_4a_5, T) = 2$  and  $x_1a_5 \in E$ . Then  $[a_4, a_5, x_1, x_0, a_1] \supseteq F_1$  and  $[T, a_2, a_3] \supseteq K_5$ . By the optimality of  $\{D, L\}$ ,  $[L] \cong K_5$  and so  $H \supseteq 2C_5$ , a contradiction. Hence  $e(x_1, a_3a_4) = 0$ . Then  $e(a_2a_5, Q) \geq 15 - e(a_1a_3a_4, Q) \geq 15 - 10 = 5$ . Thus  $e(x_2x_4, a_2a_5) \geq 1$ . Say w.l.o.g.  $x_2a_5 \in E$ . Then  $[x_0, x_1, x_2, a_5, a_1] \supseteq C_5$ . As  $H \not\supseteq 2C_5$ ,  $e(a_2a_4, x_3x_4) \leq 2$ . Clearly,  $e(a_2a_3a_4, x_1x_2) \leq 4$ . Then  $e(a_1a_5, Q) \geq 15 - 6 - e(a_3, x_3x_4) \geq 7$  and so  $e(a_1, T) \geq 2$ . Suppose that  $a_1x_3 \in E$ . Then  $x_i \not\rightarrow (L, a_1)$  for all  $x_i \in V(T)$  for otherwise  $H \supseteq 2C_5$ . This implies that  $I(a_2a_5, T) = \emptyset$ . As  $x_2a_5 \in E$ ,  $x_2a_2 \notin E$  and so  $e(a_2a_3a_4, x_1x_2) \leq 3$ . As  $e(Q, L) \geq 15$ , it follows that  $e(a_1a_5, Q) = 8$ ,  $e(a_2a_3a_4, x_3x_4) = 4$  and so  $e(x_3x_4, a_3a_4) = 4$ . Thus  $[a_2, a_3, a_4, x_3, x_4] \supseteq K_4^+$  and so  $H \supseteq K_4^+ \uplus C_5$ , a contradiction. Hence  $a_1x_3 \notin E$ . Thus  $e(a_1a_5, Q) = 7$ . It follows that  $e(a_1, Q - x_3) = 3$ ,  $e(a_5, Q) = 4$ ,  $e(a_2a_4, x_3x_4) = 2$ ,  $e(a_3, x_3x_4) = 2$ ,  $e(x_2, a_3a_4) = 2$  and  $e(a_2, x_1x_2) = 2$ . Then

$[x_2, x_1, x_0, a_1, a_2] \supseteq C_5$  and  $[a_5, a_4, a_3, x_3, x_4] \supseteq C_5$ , a contradiction.

*Case 6.*  $e(x_0, L) = 0$ . As  $H \not\supseteq K_4^+ \uplus C_5$ , we see that for each  $a_i \in V(L)$ , if  $e(a_i, Q - x_3) = 3$  then  $x_3 \not\rightarrow (L, a_i)$ . Since  $e(a_i, Q) = 4$  for some  $a_i \in V(L)$  as  $e(Q, L) \geq 16$ , it follows that  $x_3 \not\rightarrow L$  and so  $e(x_3, L) \leq 4$ . First, suppose  $e(x_3, L) = 4$ . Say  $e(x_3, L - a_5) = 4$ . Then  $e(a_i, Q - x_3) \leq 2$  for each  $i \in \{2, 3, 5\}$ . As  $e(Q, L) \geq 16$ , it follows that  $e(a_i, Q - x_3) = 2$  for  $i \in \{2, 3, 5\}$  and  $e(a_1a_4, Q - x_3) = 6$ . If  $x_1a_5 \in E$ , then  $e(a_5, x_1x_2) = 2$  or  $e(a_5, x_1x_4) = 2$ . Say w.l.o.g.  $e(a_5, x_1x_2) = 2$ . Then  $[x_0, x_1, x_2, a_1, a_5] \supseteq K_4^+$  and  $[x_3, x_4, a_2, a_3, a_4] \supseteq C_5$ , a contradiction. Hence  $e(a_5, x_2x_4) = 2$ . Then  $[D - x_3 + a_5] \supseteq F_1$ . By the optimality of  $\{D, L\}$ ,  $\tau(L) \geq \tau(x_3a_1a_2a_3a_4x_3)$ . This implies that  $\tau(a_5, L) = 2$  and so  $x_3 \rightarrow (L, a_1)$ , a contradiction.

Next, suppose that  $e(x_3, L) = 3$  and  $N(x_3, L) = \{a_i, a_{i+1}, a_{i+3}\}$  for some  $i \in \{1, 2, 3, 4, 5\}$ . Say  $N(x_3, L) = \{a_1, a_2, a_4\}$ . Then  $e(a_3, Q - x_3) \leq 2$  and  $e(a_5, Q - x_3) \leq 2$ . As  $e(Q, L) \geq 16$ , it follows that  $e(a_1a_2a_4, Q - x_3) = 9$ ,  $e(a_3, Q - x_3) = 2$  and  $e(a_5, Q - x_3) = 2$ . If  $e(x_1, a_3a_5) \geq 1$ , then we may assume w.l.o.g. that  $e(a_3, x_1x_2) = 2$ . Consequently,  $[x_0, x_1, x_2, a_2, a_3] \supseteq K_4^+$  and  $[x_3, x_4, a_1, a_5, a_4] \supseteq C_5$ , a contradiction. Hence  $e(a_3a_5, x_2x_4) = 4$ . Clearly,  $[x_0, x_1, x_2, a_2, a_3] \supseteq F_1$  and  $\tau(x_4x_3a_1a_5a_4x_4) \geq 3$ . Thus  $\tau(L) \geq 3$  by the optimality of  $\{D, L\}$ . As  $x_3 \not\rightarrow (L, a_1)$ ,  $a_3a_5 \notin E$ . Thus  $a_1a_4 \in E$  or  $a_2a_4 \in E$ . Say w.l.o.g.  $a_1a_4 \in E$ . Then  $\tau(x_4x_3a_1a_5a_4x_4) = 4$ . Thus  $\tau(L) = 4$  and so the lemma holds.

Next, suppose that  $N(x_3, L) = \{a_i, a_{i+1}, a_{i+2}\}$  for some  $i \in \{1, 2, 3, 4, 5\}$ . Say  $N(x_3, L) = \{a_1, a_2, a_3\}$ . Then  $e(a_2, Q - x_3) \leq 2$ . As  $e(D, L) \geq 16$ , either  $e(a_1a_5, Q - x_3) = 6$  or  $e(a_3a_4, Q - x_3) = 6$ . Say w.l.o.g.  $e(a_1a_5, Q - x_3) = 6$ . Then  $[x_0, x_1, x_i, a_1, a_5] \supseteq K_4^+$  and so  $[x_3, x_j, a_2, a_3, a_4] \not\supseteq C_5$  for each  $\{i, j\} = \{2, 4\}$ . This implies that  $e(a_4, x_2x_4) = 0$  and so  $e(D, L) \leq 15$ , a contradiction.

Next, suppose that  $e(x_3, L) = 2$  and  $N(x_3, L) = \{a_i, a_{i+2}\}$  for some  $i \in \{1, 2, 3, 4, 5\}$ . Say  $N(x_3, L) = \{a_1, a_3\}$ . Then  $e(a_2, Q - x_3) \leq 2$ . As  $e(Q, L) \geq 16$ , it follows that  $e(L - a_2, Q - x_3) = 12$  and  $e(a_2, Q - x_3) = 2$ . Then  $[x_0, x_1, x_2, a_4, a_5] \supseteq K_4^+$  and  $[x_3, x_4, a_1, a_2, a_3] \supseteq C_5$ , a contradiction.

Next, suppose that  $e(x_3, L) = 2$  and  $N(x_3, L) = \{a_i, a_{i+1}\}$  for some  $i \in \{1, 2, 3, 4, 5\}$ . Say  $N(x_3, L) = \{a_1, a_2\}$ . As  $e(Q, L) \geq 16$ , either  $e(a_1a_5, Q - x_3) = 6$  or  $e(a_2a_3, Q - x_3) = 6$ . Say w.l.o.g.  $e(a_1a_5, Q - x_3) = 6$ . Then  $[x_0, x_1, x_i, a_1, a_5] \supseteq K_4^+$  and so  $[x_j, x_3, a_2, a_3, a_4] \not\supseteq C_5$  for each  $\{i, j\} = \{2, 4\}$ . This implies that  $e(a_4, x_2x_4) = 0$ . Consequently,  $e(Q, L) \leq 15$ , a contradiction.

Finally, we have  $e(x_3, L) = 1$ . Then  $e(L, Q - x_3) = 15$ , clearly,  $H \supseteq K_4^+ \uplus C_5$ , a contradiction.  $\blacksquare$

**Lemma 2.10.** *Let  $D$ ,  $L_1$  and  $L_2$  be disjoint subgraphs of  $G$  with  $D \cong F_1$  and  $L_1 \cong L_2 \cong C_5$ . Suppose that  $L_1 = a_1a_2a_3a_4a_5a_1$ ,  $V(D) = \{x_0, x_1, x_2, x_3, x_4\}$  and  $E(D) = \{x_0x_1, x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_2x_4\}$  such that*



$e(x_0, L_1) = 0$ ,  $e(a_1a_2a_4, D - x_0) = 12$ ,  $N(a_3, D) = N(a_5, D) = \{x_2, x_4\}$ ,  $\tau(L_1) = 4$  and  $a_3a_5 \notin E$ . Suppose that  $\{D, L_1, L_2\}$  is optimal and  $e(x_0x_3a_3a_5, L_2) \geq 13$ . Then  $[D, L_1, L_2]$  contains either  $K_4^+ \uplus 2C_5$  or  $3C_5$ .

**Proof.** Let  $G_1 = [D, L_1]$ ,  $G_2 = [D, L_1, L_2]$  and  $R = \{x_0, x_3, a_3, a_5\}$ . On the contrary, suppose that  $G_2$  does not contain any of  $K_4^+ \uplus 2C_5$  and  $3C_5$ . It is easy to see that for any permutation  $f$  of  $\{x_3, a_3, a_5\}$ , we can extend  $f$  to be an automorphism of  $G_1$  such that any vertex in  $G_1 - \{x_3, a_3, a_5\}$  is fixed under  $f$ . Thus  $x_3$ ,  $a_3$  and  $a_5$  are in the symmetric position in the following argument. It is easy to check that if  $u \rightarrow (L_2; R - \{u\})$  for some  $u \in R$ , then  $G_2 \supseteq K_4^+ \uplus 2C_5$  or  $G_2 \supseteq 3C_5$ . Thus  $u \not\rightarrow (L_2; R - \{u\})$  for each  $u \in R$ . By Lemma 2.1(d), there exist two labellings  $R = \{y_1, y_2, y_3, y_4\}$  and  $L_2 = b_1b_2b_3b_4b_5b_1$  such that  $e(y_1y_2, b_1b_2b_3b_4) = 8$ ,  $e(y_3, b_1b_5b_4) = 3$  and  $e(y_4, b_1b_4) = 2$ . If  $x_0 \in \{y_1, y_2\}$ , we may assume w.l.o.g. that  $\{x_0, x_3\} = \{y_1, y_2\}$ . Then  $[G_1 - x_0 + b_5] \supseteq F_1 \uplus K_5^-$ . By the optimality of  $\{D, L_1, L_2\}$ ,  $x_0 \xrightarrow{na} (L_2, b_5)$ . This implies that  $\tau(b_5, L_2) = 2$ . Thus  $x_0 \rightarrow (L_2, b_1; R - \{x_0\})$ , a contradiction. Hence  $x_0 \notin \{y_1, y_2\}$ . W.l.o.g., say  $\{a_3, a_5\} = \{y_1, y_2\}$ . Then  $[a_3, a_4, a_5, b_2, b_3] \supseteq C_5$ ,  $[x_0, x_3, b_1, b_5, b_4] \supseteq C_5$  and  $[x_2, x_1, x_4, a_1, a_2] \supseteq C_5$ , a contradiction. ■

### 3. PROOF OF THEOREM 1

Let  $G$  be a graph of order  $5k$  with minimum degree at least  $3k$ . Suppose, for a contradiction, that  $G \not\supseteq kC_5$ . We may assume that  $G$  is maximal, i.e.,  $G + xy \supseteq kC_5$  for each pair of non-adjacent vertices  $x$  and  $y$  of  $G$ . Thus  $G \supseteq P_5 \uplus (k-1)C_5$ . Our proof will follow from the following three lemmas.

**Lemma 3.1.** For each  $s \in \{1, 2, \dots, k\}$ ,  $G \not\supseteq sB \uplus (k-s)C_5$ .

**Proof.** On the contrary, suppose that  $G \supseteq sB \uplus (k-s)C_5$  for some  $s \in \{1, 2, \dots, k\}$ . Let  $s$  be the minimum number in  $\{1, 2, \dots, k\}$  such that  $G \supseteq sB \uplus (k-s)C_5$ . Say  $G \supseteq sB \uplus (k-s)C_5 = \{B_1, \dots, B_s, L_1, \dots, L_{k-s}\}$  with  $B_i \cong B$  for  $i \in \{1, 2, \dots, s\}$ . Let  $R$  be the set of the four vertices of  $B_1$  whose degrees in  $B_1$  are 2. By Lemma 2.2, Lemma 2.8 and the minimality of  $s$ , we see that  $e(R, B_i) \leq 12$  and  $e(R, L_j) \leq 12$  for all  $i \in \{2, 3, \dots, s\}$  and  $j \in \{1, 2, \dots, k-s\}$ . Therefore  $e(R, G) \leq 12(k-1) + 8 = 12k - 4$ . As the minimum degree of  $G$  is  $3k$ , we obtain  $12k - 4 \geq e(R, G) \geq 12k$ , a contradiction. ■

**Lemma 3.2.** There exists a sequence  $(D, L_1, L_2, \dots, L_{k-1})$  of disjoint subgraphs of  $G$  such that  $D \cong K_4^+$  and  $L_i \cong C_5$  for all  $i \in \{1, 2, \dots, k-1\}$ .

**Proof.** First, we claim that  $G \supseteq F \uplus (k-1)C_5$ . We choose a sequence  $(P, L_1, L_2, \dots, L_{k-1})$  of disjoint subgraphs of  $G$  such that  $P \cong P_5$  and  $L_i \cong C_5$  for

all  $i \in \{1, 2, \dots, k-1\}$  with  $\sum_{i=1}^{k-1} \tau(L_i)$  as large as possible. As  $G \not\supseteq kC_5$  and by Lemma 2.1(c),  $e(P, P) \leq 14$  and so  $e(P, G - V(P)) \geq 15k - 14 = 15(k-1) + 1$ . Thus  $e(P, L_i) \geq 16$  for some  $i \in \{1, 2, \dots, k-1\}$ . By Lemma 2.3,  $[P, L_i] \supseteq F \uplus C_5$  and so  $G \supseteq F \uplus (k-1)C_5$ .

Next, we claim that  $G \supseteq F_1 \uplus (k-1)C_5$ . Assume for the moment that  $G \supseteq F_2 \uplus (k-1)C_5 = \{D, L_1, L_2, \dots, L_{k-1}\}$  with  $D \cong F_2$ . Let  $R$  be the three vertices of  $D$  with degree 2 in  $D$ . Then  $e(R, G - V(D)) \geq 9k - 6 = 9(k-1) + 3$ . Thus  $e(R, L_i) \geq 10$  for some  $i \in \{1, 2, \dots, k-1\}$ . By Lemma 2.4,  $[D, L_i] \supseteq F_1 \uplus C_5$  and so  $G \supseteq F_1 \uplus (k-1)C_5$ . Hence we may assume that  $G \not\supseteq F_2 \uplus (k-1)C_5$ . Then we choose a sequence  $(D, L_1, L_2, \dots, L_{k-1})$  of disjoint subgraphs of  $G$  such that  $D \cong F$  and  $L_i \cong C_5$  for all  $i \in \{1, 2, \dots, k-1\}$  with  $\sum_{i=1}^{k-1} \tau(L_i)$  as large as possible. Then  $e(D, L_i) \geq 16$  for some  $i \in \{1, 2, \dots, k-1\}$ . By Lemma 2.5 and Lemma 3.1, we may assume that there exist two labellings  $D = x_0x_1x_2x_3x_4x_1$  and  $L_1 = a_1a_2a_3a_4a_5a_1$  such that  $e(x_0, L_1) = 0$ ,  $e(x_1x_3, L_1) = 10$ ,  $N(x_2, L_1) = N(x_4, L_1) = \{a_1, a_2, a_4\}$ ,  $\tau(L_1) = 4$  and  $a_3a_5 \notin E$ . Then  $e(x_0x_2a_3a_5, G - V(D \cup L_1)) \geq 12k - 17 = 12(k-2) + 7$ . Thus  $e(x_0x_2a_3a_5, L_i) \geq 13$  for some  $i \in \{2, 3, \dots, k-1\}$ . By Lemma 2.6, we obtain  $[D, L_1, L_i] \supseteq F_1 \uplus 2C_5$  and so  $G \supseteq F_1 \uplus (k-1)C_5$ .

Suppose that  $G \supseteq K_4^+ \uplus B \uplus (k-2)C_5 = \{D, B_1, L_1, L_2, \dots, L_{k-2}\}$  with  $D \cong K_4^+$  and  $B_1 \cong B$ . Let  $R$  be the four vertices of  $B_1$  with degree 2 in  $B_1$ . Then either  $e(R, D) \geq 13$  or  $e(R, L_i) \geq 13$  for some  $i \in \{1, 2, \dots, k-2\}$ . By Lemma 2.2, Lemma 2.7 and Lemma 3.1, we see that  $G \supseteq K_4^+ \uplus (k-1)C_5$ . Hence we may suppose that  $G \not\supseteq K_4^+ \uplus B \uplus (k-2)C_5$ .

We now choose an optimal sequence  $(D, L_1, L_2, \dots, L_{k-1})$  of disjoint subgraphs of  $G$  with  $D \cong F_1$  and  $L_i \cong C_5$  for all  $i \in \{1, 2, \dots, k-1\}$ . Then  $e(D, L_i) \geq 16$  for some  $i \in \{1, 2, \dots, k-1\}$ . Say w.l.o.g.  $e(D, L_1) \geq 16$ . By Lemma 2.9 and Lemma 3.1, we may assume that there exist two labellings  $L_1 = a_1a_2a_3a_4a_5a_1$  and  $V(D) = \{x_0, x_1, x_2, x_3, x_4\}$  with  $E(D) = \{x_0x_1, x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_2x_4\}$  such that  $e(x_0, L_1) = 0$ ,  $e(a_1a_2a_4, D - x_0) = 12$ ,  $N(a_3, L_1) = N(a_5, L_1) = \{x_2, x_4\}$ ,  $\tau(L_1) = 4$  and  $a_3a_5 \notin E$ . Let  $R = \{x_0, x_3, a_3, a_5\}$  and  $G_1 = [D, L_1]$ . Then  $e(R, G_1) \leq 16$  and so  $e(R, G - V(G_1)) \geq 12k - 16 = 12(k-2) + 8$ . This implies that  $e(R, L_i) \geq 13$  for some  $i \in \{2, 3, \dots, k-1\}$ . Say w.l.o.g.  $e(R, L_2) \geq 13$ . By Lemma 2.10, it follows that  $[G_1, L_2] \supseteq K_4^+ \uplus 2C_5$  and so  $G \supseteq K_4^+ \uplus (k-1)C_5$ . ■

Let  $\sigma = (D, L_1, \dots, L_{k-1})$  be an optimal sequence of disjoint subgraphs in  $G$  with  $D \cong K_4^+$  and  $L_i \cong C_5$  for all  $i \in \{1, 2, \dots, k-1\}$ . Say  $V(D) = \{x_0, x_1, x_2, x_3, x_4\}$  with  $N(x_0, D) = \{x_1\}$ . Let  $Q = D - x_0$  and  $T = Q - x_1$ . Then  $Q \cong K_4$  and  $T \cong C_3$ .

**Lemma 3.3.** *For each  $t \in \{1, 2, \dots, k-1\}$ , the following statements hold:*

- (a) *If  $e(x_0, L_t) = 5$ , then  $e(Q, L_t) \leq 5$ .*

- (b) If  $e(x_0, L_t) = 4$ , then  $e(Q, L_t) \leq 9$ .  
 (c) If  $e(x_0, L_t) = r$ , then  $e(Q, L_t) \leq 18 - 2r$  for  $r \in \{1, 3\}$  and if  $e(x_0, L_t) = 2$ , then  $e(Q, L_t) \leq 15$ .

**Proof.** For convenience, we may assume  $L_t = L_1 = a_1a_2a_3a_4a_5a_1$ . Let  $G_1 = [D, L_1]$ . As  $G_1 \not\supseteq 2C_5$ , we see that if  $x_0 \rightarrow L_1$ , then  $e(a_i, Q) \leq 1$  for all  $a_i \in V(L_1)$  and so the lemma holds. Hence we may assume that  $x_0 \not\rightarrow L_1$  and so  $e(x_0, L_1) \leq 4$ .

To prove (b), say w.l.o.g.  $e(x_0, L_1 - a_5) = 4$ . On the contrary, suppose  $e(Q, L_1) \geq 10$ . It is easy to see that  $\tau(a_5, L_1) = 0$  for otherwise  $x_0 \rightarrow L_1$  and so  $G_1 \supseteq 2C_5$ . As  $x_0 \rightarrow (L_1, a_i)$  for  $i \in \{2, 3, 5\}$ ,  $e(a_i, Q) \leq 1$  for  $i \in \{2, 3, 5\}$ . If  $e(a_5, Q) = 1$  then  $[Q + a_5] \cong K_4^+$  and  $\tau(x_0a_1a_2a_3a_4x_0) > \tau(L_1)$ , contradicting the optimality of  $\sigma$ . Hence  $e(a_5, Q) = 0$ . It follows that  $e(a_2, Q) = e(a_3, Q) = 1$  and  $e(a_1a_4, Q) = 8$ . Clearly,  $\tau(x_0a_3a_4a_5a_1x_0) \geq \tau(L_1)$  with equality only if  $a_2a_4 \in E$ . As  $[Q + a_2] \supseteq K_4^+$  and by the optimality of  $\sigma$ , we obtain  $a_2a_4 \in E$ . Thus  $[a_5, a_4, a_3, a_2, x_0] \supseteq K_4^+$  and  $[Q + a_1] \cong K_5$ . By the optimality of  $\sigma$ , we obtain  $[L_1] \cong K_5$ , a contradiction.

To prove (c), we suppose, for a contradiction, that either  $e(x_0, L_1) = r$  and  $e(Q, L_1) \geq 19 - 2r$  for some  $r \in \{1, 3\}$  or  $e(x_0, L_1) = 2$  and  $e(Q, L_1) \geq 16$ . We divide the proof into the following three cases.

*Case 1.*  $e(x_0, L_1) = 3$  and  $e(Q, L_1) \geq 13$ . First, suppose that  $N(x_0, L_1) = \{a_i, a_{i+1}, a_{i+3}\}$  for some  $i \in \{1, 2, 3, 4, 5\}$ . Say w.l.o.g.  $N(x_0, L_1) = \{a_1, a_2, a_4\}$ . As  $x_0 \not\rightarrow L_1$ ,  $a_3a_5 \notin E$ . Clearly,  $x_0 \rightarrow (L_1, a_3)$  and  $x_0 \rightarrow (L_1, a_5)$ . Thus  $e(a_3, Q) \leq 1$  and  $e(a_5, Q) \leq 1$ . It follows that  $e(a_1a_2a_4, Q) \geq 11$ ,  $e(x_1, a_1a_4) \geq 1$  and  $e(x_1, a_2a_4) \geq 1$ . Thus  $[x_0, x_1, a_1, a_5, a_4] \supseteq C_5$  and  $[x_0, x_1, a_2, a_3, a_4] \supseteq C_5$ . As  $e(a_i, T) \geq 2$  for  $i \in \{1, 2\}$ , it is easy to see that  $e(a_3a_5, T) = 0$ , i.e.,  $N(a_3a_5, Q) \subseteq \{x_1\}$ , for otherwise  $G_1 \supseteq 2C_5$ .

Let  $R = \{x_0, x_3, a_3, a_5\}$ . Then  $e(R, G_1) \leq 18$  and so  $e(R, G - V(G_1)) \geq 12k - 18 = 12(k - 2) + 6$ . Then  $e(R, L_i) \geq 13$  for some  $i \in \{2, 3, \dots, k - 1\}$ . Say w.l.o.g.  $e(R, L_2) \geq 13$ . Let  $G_2 = [G_1, L_2]$ . Then  $G_2 \not\supseteq 3C_5$ . Since  $e(Q, L_1) \geq 13$  and  $N(a_3a_5, Q) \subseteq \{x_1\}$ , it is easy to check that if  $u \rightarrow (L_2; R - \{u\})$  for some  $u \in R$ , then  $G_2 \supseteq 3C_5$ . Hence  $u \not\rightarrow (L_2; R - \{u\})$  for all  $u \in R$ . By Lemma 2.1(d), there exist two labellings  $L_2 = b_1b_2b_3b_4b_5b_1$  and  $R = \{y_1, y_2, y_3, y_4\}$  such that  $e(y_1y_2, L_2 - b_5) = 8$ ,  $e(y_3, b_1b_5b_4) = 3$  and  $e(y_4, b_1b_4) = 2$ . If  $\{y_1, y_2\} = \{x_0, x_3\}$ , let  $\{s, t\} = \{1, 2\}$  with  $a_s \in I(x_0x_3, L_1)$  and then we see that  $[x_0, a_s, x_3, b_2, b_3] \supseteq C_5$ ,  $[a_3, a_5, b_1, b_5, b_4] \supseteq C_5$  and  $[Q - x_3 + a_4 + a_t] \supseteq C_5$ , a contradiction. If  $\{y_1, y_2\} = \{x_0, a_i\}$  for some  $i \in \{3, 5\}$ , we may assume w.l.o.g. that  $\{y_1, y_2\} = \{x_0, a_5\}$  and then we see that  $[x_0, a_1, a_5, b_2, b_3] \supseteq C_5$ ,  $[a_3, x_3, b_1, b_5, b_4] \supseteq C_5$  and  $[a_2, a_4, x_1, x_2, x_4] \supseteq C_5$ , a contradiction. If  $\{y_1, y_2\} = \{x_3, a_i\}$  for some  $i \in \{3, 5\}$ , we may assume w.l.o.g. that  $\{y_1, y_2\} = \{x_3, a_5\}$  and let  $\{s, t\} = \{1, 4\}$  be such that  $x_3a_s \in E$ . Then we see that  $\{x_3, a_s, a_5, b_2, b_3\} \supseteq$

$C_5$ ,  $[x_0, a_3, b_1, b_5, b_4] \supseteq C_5$  and  $[x_1, x_2, x_4, a_2, a_t] \supseteq C_5$ , a contradiction. Hence  $\{y_1, y_2\} = \{a_3, a_5\}$ . Then  $[a_3, a_4, a_5, b_2, b_3] \supseteq C_5$ ,  $[x_0, x_3, b_1, b_5, b_4] \supseteq C_5$  and  $[x_1, x_2, x_4, a_1, a_2] \supseteq C_5$ , a contradiction.

Next, suppose that  $N(x_0, L_1) = \{a_i, a_{i+1}, a_{i+2}\}$  for some  $i \in \{1, 2, 3, 4, 5\}$ . Say w.l.o.g.  $N(x_0, L_1) = \{a_1, a_2, a_3\}$ . Then  $e(a_2, Q) \leq 1$  as  $G_1 \not\supseteq 2C_5$  and so  $e(Q, L_1 - a_2) \geq 12$ . First, assume  $e(x_1, a_4a_5) \geq 1$ . Say w.l.o.g.  $x_1a_5 \in E$ . Then  $[x_0, x_1, a_5, a_1, a_2] \supseteq C_5$ . Then  $e(a_3a_4, T) \leq 3$  as  $G_1 \not\supseteq 2C_5$ . If we also have  $x_1a_4 \in E$ , then similarly,  $e(a_1a_5, T) \leq 3$  and so  $e(Q, L_1 - a_2) \leq 11$ , a contradiction. Hence  $x_1a_4 \notin E$ . As  $e(Q, L_1) \geq 13$ , it follows that  $e(a_1a_5, Q) = 8$ ,  $e(a_3a_4, T) = 3$ ,  $x_1a_3 \in E$  and  $e(a_2, Q) = 1$ . Clearly,  $[T + a_4 + a_5] \not\supseteq C_5$  as  $G_1 \not\supseteq 2C_5$ . This implies that  $e(a_4, T) = 0$  and so  $e(a_3, Q) = 4$ . Obviously,  $G_1 \supseteq 2C_5$ , a contradiction. Hence  $e(x_1, a_4a_5) = 0$ . Next, assume  $e(x_1, a_1a_3) \geq 1$ . Then  $[x_0, x_1, a_1, a_2, a_3] \supseteq C_5$  and so  $e(a_4a_5, T) \leq 3$ . It follows that  $e(Q, L_1 - a_2) \leq 12$ , a contradiction. Hence  $e(x_1, L_1 - a_2) = 0$ . Thus  $e(T, L_1 - a_2) = 12$ . Obviously,  $G_1 \supseteq 2C_5$ , a contradiction.

*Case 2.*  $e(x_0, L_1) = 2$  and  $e(Q, L_1) \geq 16$ . First, suppose that  $N(x_0, L_1) = \{a_i, a_{i+2}\}$  for some  $i \in \{1, 2, 3, 4, 5\}$ . Say,  $N(x_0, L_1) = \{a_1, a_3\}$ . Then  $e(a_2, Q) \leq 1$  and  $e(Q, L_1 - a_2) \geq 15$ . Thus  $e(x_1, a_1a_3) \geq 1$ . Then  $[x_0, x_1, a_1, a_2, a_3] \supseteq C_5$  and so  $e(a_4a_5, T) \leq 3$ . Thus  $e(Q, L_1 - a_2) \leq 13$ , a contradiction. Therefore we may assume w.l.o.g. that  $N(x_0, L_1) = \{a_1, a_2\}$ . First, assume  $x_1a_4 \in E$ . Then  $[x_0, x_1, a_4, a_5, a_1] \supseteq C_5$  and  $[x_0, x_1, a_4, a_3, a_2] \supseteq C_5$ . As  $G_1 \not\supseteq 2C_5$ ,  $e(a_2a_3, T) \leq 3$  and  $e(a_1a_5, T) \leq 3$ . Thus  $e(Q, L_1) \leq 14$ , a contradiction. Hence  $x_1a_4 \notin E$ . Next, assume  $e(x_1, a_3a_5) \geq 1$ . Say w.l.o.g.  $x_1a_5 \in E$ . Then  $[x_0, x_1, a_5, a_1, a_2] \supseteq C_5$  and so  $e(a_3a_4, T) \leq 3$ . As  $e(Q, L_1) \geq 16$ , it follows that  $e(a_5a_1a_2, Q) = 12$ ,  $e(a_3a_4, T) = 3$  and  $x_1a_3 \in E$ . Thus  $e(x_3, a_2a_5) = 2$  and so  $G_1 \supseteq 2C_5$ , a contradiction. Hence  $e(x_1, a_3a_4a_5) = 0$ . Thus  $e(T, L_1) \geq 14$ . This implies that  $e(x_i, a_2a_5) = 2$  and  $a_1x_j \in E$  for some  $\{i, j\} \subseteq \{2, 3, 4\}$  with  $i \neq j$ . Consequently,  $H \supseteq 2C_5$ , a contradiction.

*Case 3.*  $e(x_0, L_1) = 1$  and  $e(Q, L_1) \geq 17$ . Say w.l.o.g.  $x_0a_1 \in E$ . Suppose  $e(x_1, a_3a_4) \geq 1$ . Say  $x_1a_3 \in E$ . Then  $[x_1, x_0, a_1, a_2, a_3] \supseteq C_5$  and so  $e(a_4a_5, T) \leq 3$  as  $G_1 \not\supseteq 2C_5$ . As  $e(Q, L_1) \geq 17$ , it follows that  $e(a_1a_2a_3, Q) = 12$ ,  $e(a_4a_5, T) = 3$  and  $e(x_1, a_4a_5) = 2$ . Then  $[x_0, x_1, a_4, a_5, a_1] \supseteq C_5$  and  $[T, a_2, a_3] \supseteq C_5$ , a contradiction. Hence  $e(x_1, a_3a_4) = 0$ . As  $e(Q, L_1) \geq 17$ ,  $e(T, L_1) \geq 14$ . This implies that  $e(x_i, a_2a_5) = 2$  and  $a_1x_j \in E$  for some  $\{i, j\} \subseteq \{2, 3, 4\}$  with  $i \neq j$ . Consequently,  $H \supseteq 2C_5$ , a contradiction. ■

We are now in the position to complete the proof of Theorem 1. Let  $\mathcal{A}_r = \{L_t | e(x_0, L_t) = r, 1 \leq t \leq k-1\}$  for each  $0 \leq r \leq 5$ . Set  $p_r = |\mathcal{A}_r|$  for each  $0 \leq r \leq 5$ . Clearly,  $p_0 + p_1 + p_2 + p_3 + p_4 + p_5 = k-1$ . By Lemma 3.3, we obtain

$$\begin{aligned}
e(x_0, G) &= e(x_0, D) + \sum_{r=0}^5 \sum_{L_t \in \mathcal{A}_r} e(x_0, L_t) \\
(2) \qquad &= 1 + p_1 + 2p_2 + 3p_3 + 4p_4 + 5p_5;
\end{aligned}$$

$$\begin{aligned}
e(D, G) &= e(D, D) + \sum_{r=0}^5 \sum_{L_t \in \mathcal{A}_r} e(D, L_t) \\
(3) \qquad &\leq 14 + 20p_0 + 17p_1 + 17p_2 + 15p_3 + 13p_4 + 10p_5.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
e(x_0, G) + e(D, G) &\leq 15 + 20p_0 + 18p_1 + 19p_2 + 18p_3 + 17p_4 + 15p_5 \\
(4) \qquad &= 18k + 2p_0 + p_2 - p_4 - 3p_5 - 3.
\end{aligned}$$

As  $3 \sum_{r=0}^5 p_r = 3k - 3$  and  $e(x_0, G) \geq 3k$ , we obtain, by using (2), the following

$$\begin{aligned}
&1 + p_1 + 2p_2 + 3p_3 + 4p_4 + 5p_5 \\
(5) \qquad &\geq 3 + 3p_0 + 3p_1 + 3p_2 + 3p_3 + 3p_4 + 3p_5.
\end{aligned}$$

This implies that  $3p_0 + 2p_1 + p_2 - p_4 - 2p_5 + 2 \leq 0$ . Thus  $2p_0 + p_2 - p_4 - 3p_5 \leq -2$ . Together with (4), we obtain  $e(x_0, G) + e(D, G) \leq 18k - 5$ . But by the degree condition on  $G$ , we have  $e(x_0, G) + e(D, G) \geq 3k + 15k = 18k$ , a contradiction. This proves Theorem 1.

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