# ON RAMSEY $\left(K_{1,2}, K_{n}\right)$-MINIMAL GRAPHS 

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#### Abstract

Let $F$ be a graph and let $\mathcal{G}, \mathcal{H}$ denote nonempty families of graphs. We write $F \rightarrow(\mathcal{G}, \mathcal{H})$ if in any 2 -coloring of edges of $F$ with red and blue, there is a red subgraph isomorphic to some graph from $\mathcal{G}$ or a blue subgraph isomorphic to some graph from $\mathcal{H}$. The graph $F$ without isolated vertices is said to be a $(\mathcal{G}, \mathcal{H})$-minimal graph if $F \rightarrow(\mathcal{G}, \mathcal{H})$ and $F-e \nrightarrow(\mathcal{G}, \mathcal{H})$ for every $e \in E(F)$.

We present a technique which allows to generate infinite family of $(\mathcal{G}, \mathcal{H})$ minimal graphs if we know some special graphs. In particular, we show how to receive infinite family of $\left(K_{1,2}, K_{n}\right)$-minimal graphs, for every $n \geq 3$.


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## 1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Let $G$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. By $\operatorname{deg}_{G}\left(v_{1}\right)$, $d_{G}\left(v_{1}, v_{2}\right)$ we denote the degree of the vertex $v_{1}$ in $G$ and the distance between two vertices $v_{1}, v_{2}$, respectively. If $G$ is known we can shortly write $\operatorname{deg}\left(v_{1}\right), d\left(v_{1}, v_{2}\right)$. We use the notation and terminology of [8].

Let $F$ be a graph and let $\mathcal{G}, \mathcal{H}$ be nonempty families of graphs. We write $F \rightarrow(\mathcal{G}, \mathcal{H})$ if in any 2 -coloring of edges of $F$ with red and blue, there is a red subgraph isomorphic to some graph from $\mathcal{G}$ or a blue subgraph isomorphic to some graph from $\mathcal{H}$. Otherwise, if there exists a 2 -coloring of edges such that neither a red subgraph isomorphic to some graph from $\mathcal{G}$ nor a blue subgraph isomorphic to some graph from $\mathcal{H}$ occur, then we write $F \nrightarrow(\mathcal{G}, \mathcal{H})$. The graph
$F$ without isolated vertices is said to be a $(\mathcal{G}, \mathcal{H})$-minimal graph if $F \rightarrow(\mathcal{G}, \mathcal{H})$ and $F-e \nrightarrow(\mathcal{G}, \mathcal{H})$ for any $e \in E(F)$. The Ramsey set $\Re(\mathcal{G}, \mathcal{H})$ is defined to be the set of all $(\mathcal{G}, \mathcal{H})$-minimal graphs (up to isomorphism). For the simplicity of the notation, instead of $\Re(\{G\},\{H\})$ we write $\Re(G, H)$.

Many papers study the problem of determining the family $\Re(G, H)$. One can easily observe that the set $\Re\left(K_{1,2}, K_{1,2}\right)$ is infinite and consists of star with three rays and all cycles of odd length. Burr et al. [6] proved that $\Re\left(K_{1,2 k+1}, K_{1,2 l+1}\right)=$ $\left\{K_{1,2(k+l)+1}\right\}$ and $\Re\left(K_{1,2 k}, K_{1,2 l}\right)$ is infinite, for every $k, l \geq 1$. Next Borowiecki et al. [3] characterized graphs belonging to $\Re\left(K_{1,2}, K_{1, m}\right)$ for $m \geq 3$.

The graphs belonging to $\Re\left(2 K_{2}, K_{1, n}\right)$ were characterized in [10]. Moreover, Łuczak [9] showed that $\Re\left(K_{1,2 m}, G\right)$ is finite if and only if $G$ is a matching. It means that, for $n \geq 3, \Re\left(K_{1,2}, K_{n}\right)$ has infinite number of graphs.

Borowiecki, et al. described in [4] the whole set $\Re\left(K_{1,2}, K_{3}\right)$. In [1, 2] the authors presented how we can generate an infinite family of ( $K_{1,2}, C_{4}$ )-minimal graphs. In this paper we describe a method which can be applied to the construction of infinitely many graphs belonging to $\Re\left(K_{1,2}, K_{n}\right)$, for any $n \geq 3$.

## 2. The Main Results

First we extend, in the same way as in [2], the already given standard definitions by adding some restriction on a chosen set of vertices. This allows us to construct the infinite family $\Re\left(K_{1,2}, \mathcal{G}\right)$, for any given family $\mathcal{G}$ of 2 -connected graphs.

Definition 1. Let $F$ be a graph with $U \subseteq V(F)$ and let $\mathcal{G}, \mathcal{H}$ be families of graphs. If for any red-blue coloring of edges of $F$, such that all vertices in $U$ are not incident with red edges, there exists a red copy of some graph from $\mathcal{G}$ or a blue copy of some graph from $\mathcal{H}$, then we write $F(U) \rightarrow(\mathcal{G}, \mathcal{H})$. Otherwise, there exists a $(\mathcal{G}, \mathcal{H})$-coloring of edges of $F(U)$ and we write $F(U) \nrightarrow(\mathcal{G}, \mathcal{H})$.

Definition 2. Let $F$ be a graph and $U \subseteq V(F)$. Let $i \in\{1,2, \ldots,|U|\}$. We say that $F(U)_{i}$ is $(\mathcal{G}, \mathcal{H})$-minimal if

1. $F\left(U_{i}\right) \rightarrow(\mathcal{G}, \mathcal{H})$, for every $U_{i} \in\binom{U}{i}$,
2. $(F-e)\left(U_{i}\right) \nrightarrow(\mathcal{G}, \mathcal{H})$, for every $e \in E(F)$ and every $U_{i} \in\binom{U}{i}$,
3. $F\left(U_{i-1}\right) \nrightarrow(\mathcal{G}, \mathcal{H})$, for every $U_{i-1} \in\binom{U}{i-1}$.

We write $F(U)_{i} \in \tilde{\Re}(\mathcal{G}, \mathcal{H})$ if $F(U)_{i}$ is $(\mathcal{G}, \mathcal{H})$-minimal. If $U=\emptyset$ or $i=0$, then we assume that $F(U)_{i} \in \mathscr{\Re}(\mathcal{G}, \mathcal{H}) \Leftrightarrow F \in \Re(\mathcal{G}, \mathcal{H})$.
For the simplicity of the notation we write $F\left(v_{1}, \ldots, v_{p}\right)_{i}$ instead of $F\left(\left\{v_{1}, \ldots, v_{p}\right\}\right)_{i}$ and $F\left(v_{1}, \ldots, v_{p}\right)$ instead of $F\left(v_{1}, \ldots, v_{p}\right)_{p}$.

Remark 1. $F\left(r_{1}, r_{2}\right)_{1} \in \tilde{\Re}(\mathcal{G}, \mathcal{H})$ if and only if $F\left(r_{i}\right) \in \tilde{\Re}(\mathcal{G}, \mathcal{H})$, for $i=1,2$.
Lemma 2. Let $\mathcal{G}$ be a family of 2-connected graphs. Let $M_{1}, M_{2}$ be disjoint graphs and $U_{i} \subset V\left(M_{i}\right),\left|U_{i}\right| \in\{1,2\}$ and $r_{i} \in U_{i}$, for $i=1,2$, and let $M$ be $a$ graph obtained from disjoint graphs $M_{1}$ and $M_{2}$ by identifying the vertices $r_{1}$ and $r_{2}$. If $M_{1}\left(U_{1}\right), M_{2}\left(U_{2}\right) \in \tilde{\Re}\left(K_{1,2}, \mathcal{G}\right)$, then $M\left(U_{1} \cup U_{2} \backslash\left\{r_{1}, r_{2}\right\}\right) \in \tilde{\Re}\left(K_{1,2}, \mathcal{G}\right)$.

Proof. Let $U=U_{1} \cup U_{2} \backslash\left\{r_{1}, r_{2}\right\}$. First we prove that $M(U) \rightarrow\left(K_{1,2}, \mathcal{G}\right)$. If we assume that $M(U) \nrightarrow\left(K_{1,2}, \mathcal{G}\right)$, then there exists a coloring of edges of $M$ such that there is at most one red edge $e$ incident with $r_{1}$. It means that $M_{1}\left(U_{1}\right) \nrightarrow\left(K_{1,2}, \mathcal{G}\right)$ or $M_{2}\left(U_{2}\right) \nrightarrow\left(K_{1,2}, \mathcal{G}\right)$. Hence, we obtain a contradiction.

Now we show that $(M-e)(U) \nrightarrow\left(K_{1,2}, \mathcal{G}\right)$. Without loss of generality we can consider only the situation when $e \in E\left(M_{1}\right)$. We know that $\left(M_{1}-e\right)\left(U_{1}\right) \nrightarrow$ $\left(K_{1,2}, \mathcal{G}\right)$ and $M_{2}\left(U_{2} \backslash\left\{r_{2}\right\}\right) \nrightarrow\left(K_{1,2}, \mathcal{G}\right)$. Thus, there exist a $\left(K_{1,2}, \mathcal{G}\right)$-coloring of edges of $\left(M_{1}-e\right)\left(U_{1}\right)$, let us denote it by $\phi_{1}$, and a $\left(K_{1,2}, \mathcal{G}\right)$-coloring of edges of $M_{2}\left(U_{2} \backslash\left\{r_{2}\right\}\right)$, let us denote it by $\phi_{2}$. Let $\phi$ be a coloring of edges of $(M-e)$ such that $\phi(f)=\phi_{1}(f)$ for $f \in E\left(M_{1}\right)$ and $\phi(f)=\phi_{2}(f)$ for $f \in E\left(M_{2}\right)$. Since $M_{2}\left(U_{2} \backslash\left\{r_{2}\right\}\right) \nrightarrow\left(K_{1,2}, \mathcal{G}\right)$, it is easy to notice that the vertex $r_{1}$ is incident with exactly one red edge which belongs to $E\left(M_{2}\right)$. We can notice that there does not exist a blue copy of a graph $G \in \mathcal{G}$ such that $\left|V(G) \cap V\left(M_{1}\right)\right|>1$ and $\left|V(G) \cap V\left(M_{2}\right)\right|>1$, because $\mathcal{G}$ contains only 2-connected graphs. Hence, $\phi$ is a $\left(K_{1,2}, \mathcal{G}\right)$-coloring of edges of $(M-e)(U)$.

Finally, we prove that $M\left(U_{i}-r_{i}\right) \nrightarrow\left(K_{1,2}, \mathcal{G}\right)$ for $i=1,2$. Without loss of generality we can assume that $i=1$. We know that $M_{1}\left(U_{1}-\left\{r_{1}\right\}\right) \nrightarrow\left(K_{1,2}, \mathcal{G}\right)$ and $M_{2}\left(r_{2}\right) \nrightarrow\left(K_{1,2}, \mathcal{G}\right)$. Hence, there exist $\left(K_{1,2}, \mathcal{G}\right)$-colorings $\phi_{1}$ and $\phi_{2}$ of edges of $M_{1}\left(U_{1}-\left\{r_{1}\right\}\right)$ and $M_{2}\left(r_{2}\right)$, respectively. Let $\phi$ be a coloring of edges of $M$ such that $\phi(f)=\phi_{1}(f)$, if $f \in E\left(M_{1}\right)$ and $\phi(f)=\phi_{2}(f)$, otherwise. It is easy to observe that the vertex $r_{1}$ is incident with exactly one red edge belonging to $E\left(M_{1}\right)$ in $M$. For the same reason as previously we can notice that there does not exist a blue copy of a graph $G \in \mathcal{G}$ such that $\left|V(G) \cap V\left(M_{1}\right)\right|>1$ and $\left|V(G) \cap V\left(M_{2}\right)\right|>1$. Hence, $\phi$ is a $\left(K_{1,2}, \mathcal{G}\right)$-coloring of edges of $M\left(U_{1}-\left\{r_{1}\right\}\right)$. This observation finishes the proof.

Lemma 3. Let $c \geq 3$ be an integer, $M_{1}, M_{2}$ be disjoint graphs, $\mathcal{G}$ be a family of 2connected graphs without induced cycles of the length greater than c. Let $r_{i, 1}, r_{i, 2}$ be vertices of $M_{i}$, for $i=1,2$, such that $d_{M_{1}}\left(r_{1,1}, r_{1,2}\right)+d_{M_{2}}\left(r_{2,1}, r_{2,2}\right)>c$, and let $L$ be a graph obtained from graphs $M_{1}$ and $M_{2}$ by identifying the vertices $r_{1,1}$ and $r_{2,1}$, and the vertices $r_{1,2}$ and $r_{2,2}$. If $M_{i}\left(r_{i, 1}, r_{i, 2}\right) \in \tilde{\Re}\left(K_{1,2}, \mathcal{G}\right)$, for $i=1,2$, then $L\left(r_{1,1}, r_{1,2}\right)_{1} \in \tilde{\Re}\left(K_{1,2}, \mathcal{G}\right)$.

Proof. First we prove that $L\left(r_{1,1}, r_{1,2}\right)_{1} \rightarrow\left(K_{1,2}, \mathcal{G}\right)$. Conversely, suppose that $L\left(r_{1,1}, r_{1,2}\right)_{1} \nrightarrow\left(K_{1,2}, \mathcal{G}\right)$. Without loss of generality we can assume that $L\left(r_{1,1}\right) \nrightarrow$ $\left(K_{1,2}, \mathcal{G}\right)$. Then there exists a $\left(K_{1,2}, \mathcal{G}\right)$-coloring of edges of $L$ such that every edge
$e$ incident with $r_{1,1}$ in $L$ is blue and at most one edge incident with $r_{2,2}$ is red. Without loss of generality we can assume that the red edge belongs to $E\left(M_{1}\right)$. Hence, we obtain a contradiction with the fact that $M_{2}\left(r_{2,1}, r_{2,2}\right) \rightarrow\left(K_{1,2}, \mathcal{G}\right)$.

Now we show that $(L-e)\left(r_{1,1}, r_{1,2}\right)_{1} \nrightarrow\left(K_{1,2}, \mathcal{G}\right)$. Without loss of generality we can assume that $e \in E\left(M_{1}\right)$. We know that $\left(M_{1}-e\right)\left(r_{1,1}, r_{1,2}\right) \nrightarrow\left(K_{1,2}, \mathcal{G}\right)$ and $M_{2}\left(r_{2,1}\right) \nrightarrow\left(K_{1,2}, \mathcal{G}\right)$. Thus, there exist a $\left(K_{1,2}, \mathcal{G}\right)$-coloring of edges of $\left(M_{1}-e\right)\left(r_{1,1}, r_{1,2}\right)$ and a $\left(K_{1,2}, \mathcal{G}\right)$-coloring of edges of $M_{2}\left(r_{2,1}\right)$. We denote these colorings by $\phi_{1}$ and $\phi_{2}$, respectively. Let $\phi$ be a coloring of edges $(L-e)$ such that $\phi(f)=\phi_{1}(f)$, if $f \in E\left(M_{1}\right)$ and $\phi(f)=\phi_{2}(f)$, otherwise. It is easy to notice that the vertex $r_{1,2}$ is incident with exactly one red edge belonging to $E\left(M_{2}\right)$. Since $\mathcal{G}$ contains only 2 -connected graphs and $d_{M_{1}}\left(r_{1,1}, r_{1,2}\right)+d_{M_{2}}\left(r_{2,1}, r_{2,2}\right)>c$, there does not exist a blue copy of a graph $G \in \mathcal{G}$ such that $\left|V(G) \cap V\left(M_{i}-r_{i, 1}-r_{i, 2}\right)\right|>$ 0 , for $i=1,2$. Hence, $\phi$ is a $\left(K_{1,2}, \mathcal{G}\right)$-coloring of edges of $(L-e)\left(r_{1,1}, r_{1,2}\right)$.

Finally, we prove that $L \nrightarrow\left(K_{1,2}, \mathcal{G}\right)$. From our assumption, it follows that $M_{1}\left(r_{1,1}\right) \nrightarrow\left(K_{1,2}, \mathcal{G}\right)$ and $M_{2}\left(r_{2,2}\right) \nrightarrow\left(K_{1,2}, \mathcal{G}\right)$. Thus once again, we can indicate two colorings $\phi_{1}$ and $\phi_{2}$ such that $\phi_{i}$ is a $\left(K_{1,2}, \mathcal{G}\right)$-coloring of edges of $M_{i}\left(r_{i, i}\right)$, for $i=1,2$. Let $\phi$ be a coloring of edges of $L$ such that $\phi(f)=\phi_{i}(f)$ for $f \in E\left(M_{i}\right)$ and $i=1,2$. We can observe that the vertex $r_{1,1}$ is incident with exactly one red edge belonging to $E\left(M_{2}\right)$ and the vertex $r_{1,2}$ is incident with exactly one red edge belonging to $E\left(M_{1}\right)$. We can notice that there does not exist a blue copy of a graph $G \in \mathcal{G}$ such that $\left|V(G) \cap V\left(M_{i}-r_{i, 1}-r_{i, 2}\right)\right|>0$ for $i=1,2$, because $\mathcal{G}$ contains only 2 -connected graphs and $d_{M_{1}}\left(r_{1,1}, r_{1,2}\right)+d_{M_{2}}\left(r_{2,1}, r_{2,2}\right)>$ $c$. Therefore $\phi$ is a $\left(K_{1,2}, \mathcal{G}\right)$-coloring edges of $L$.

Corollary 4. Let $c \geq 3$ be an integer, $M_{1}, M_{2}$ be disjoint graphs, $\mathcal{G}$ be a family of 2 -connected graphs without induced cycles of the length greater than c. Let $r_{i, 1}, r_{i, 2}$ be vertices of $M_{i}$, for $i=1,2$, such that $d_{M_{1}}\left(r_{1,1}, r_{1,2}\right)+d_{M_{2}}\left(r_{2,1}, r_{2,2}\right)>c$, and let $B$ be a graph obtained from graphs $M_{1}$ and $M_{2}$ by identifying the vertices $r_{1,1}$ and $r_{2,1}$, and the vertices $r_{1,2}$ and $r_{2,2}$. If $M_{i}\left(r_{i, 1}, r_{i, 2}\right) \in \tilde{\Re}\left(K_{1,2}, \mathcal{G}\right)$, for $i=1,2$, then $B\left(r_{1,1}\right) \in \tilde{\Re}\left(K_{1,2}, \mathcal{G}\right)$.

Proof. From Lemma 3 and Remark 1.
The next theorems give us a method of the construction of infinitely many graphs that belong to $\Re\left(K_{1,2}, \mathcal{G}\right)$, where $\mathcal{G}$ is any given family of graphs. In this construction we use graphs with adding some restriction on a chosen set of vertices, i.e., graphs that belong to the family $\tilde{\Re}\left(K_{1,2}, \mathcal{G}\right)$.

Theorem 5. Let $c \geq 3$ be an integer, $L, M$ be disjoint graphs, $\mathcal{G}$ be a family of 2-connected graphs without induced cycles of the length greater than c. Let $\left\{r_{1,1}, r_{1,2}\right\} \subset V(L)$ and $\left\{r_{2,1}, r_{2,2}\right\} \subset V(M)$ such that $d_{L}\left(r_{1,1}, r_{1,2}\right)+d_{M}\left(r_{2,1}, r_{2,2}\right)>c$, and let $F$ be a graph obtained from graphs $L$ and $M$ by identifying the vertices $r_{1,1}$
and $r_{2,1}$ and the vertices $r_{1,2}$, and $r_{2,2}$. If $L\left(r_{1,1}, r_{1,2}\right), M\left(r_{2,1}, r_{2,2}\right) \in \tilde{\mathscr{R}}\left(K_{1,2}, \mathcal{G}\right)$, then $F \in \Re\left(K_{1,2}, \mathcal{G}\right)$.
Proof. We start with proving that $F \rightarrow\left(K_{1,2}, \mathcal{G}\right)$. Suppose, on the contrary, that there exists a $\left(K_{1,2}, \mathcal{G}\right)$-coloring of edges of $F$. From the fact that $L\left(r_{1,1}, r_{1,2}\right)_{1} \rightarrow$ $\left(K_{1,2}, \mathcal{G}\right)$ it follows that in this coloring one edge incident with $r_{1,1}$ and one edge incident with $r_{1,2}$ in $L$ is red. Hence, every edge incident with $r_{2,1}$ and $r_{2,2}$ in $M$ is blue. We obtain a contradiction with the assumption that $M\left(r_{2,1}, r_{2,2}\right) \rightarrow$ $\left(K_{1,2}, \mathcal{G}\right)$.

It remains to prove that $F-e \nrightarrow\left(K_{1,2}, \mathcal{G}\right)$, for every $e \in E(F)$.
Case 1. Let $e \in E(L)$. We know that $(L-e)\left(r_{1, i}\right) \nrightarrow\left(K_{1,2}, \mathcal{G}\right)$ and $M\left(r_{2,3-i}\right) \nrightarrow\left(K_{1,2}, \mathcal{G}\right)$, for $i=1,2$. Without loss of generality we can assume that $i=1$. Thus, there exists a $\left(K_{1,2}, \mathcal{G}\right)$-coloring of edges of $(L-e)\left(r_{1,1}\right)$ and a $\left(K_{1,2}, \mathcal{G}\right)$-coloring of edges of $M\left(r_{2,2}\right)$. Let us denote these colorings by $\phi_{1}$ and $\phi_{2}$, respectively. Let $\phi$ be a coloring of edges of $(F-e)$ such that $\phi(f)=\phi_{1}(f)$, if $f \in E(L)$ and $\phi(f)=\phi_{2}(f)$, otherwise. Let us notice that the vertices $r_{1,1}$ and $r_{1,2}$ must be incident with at most one red edge in the graph $F-e$. We also know that there does not exist a blue copy of a graph $G \in \mathcal{G}$ such that $\left|V(G) \cap V\left(L-r_{1,1}-r_{1,2}\right)\right|>0$ and $\left|V(G) \cap V\left(M-r_{2,1}-r_{2,2}\right)\right|>0$. This observation follows from the fact that $\mathcal{G}$ contains only 2 -connected graphs and $d_{L}\left(r_{1,1}, r_{1,2}\right)+d_{M}\left(r_{2,1}, r_{2,2}\right)>c$. Hence, $\phi$ is a $\left(K_{1,2}, \mathcal{G}\right)$-coloring of edges of $F-e$.

Case 2. Let $e \in E(M)$. From the fact that $L \nrightarrow\left(K_{1,2}, \mathcal{G}\right)$ and ( $M-$ $e)\left(r_{2,1}, r_{2,2}\right) \nrightarrow\left(K_{1,2}, \mathcal{G}\right)$ it follows that there exist a $\left(K_{1,2}, \mathcal{G}\right)$-coloring $\phi_{1}$ of edges of $L$ and a $\left(K_{1,2}, \mathcal{G}\right)$-coloring $\phi_{2}$ of edges of $(M-e)\left(r_{2,1}, r_{2,2}\right)$. Let $\phi$ be a coloring of edges of $(F-e)$ such that $\phi(f)=\phi_{1}(f)$, if $f \in E(L)$ and $\phi(f)=\phi_{2}(f)$, otherwise. Since $L\left(r_{1, i}\right) \nrightarrow\left(K_{1,2}, \mathcal{G}\right)$, for $i=1,2$ and $L \nrightarrow\left(K_{1,2}, \mathcal{G}\right)$, the vertices $r_{1,1}$ and $r_{1,2}$ are incident with exactly one red edge in $F-e$. For the same reason as in Case 1 we know that there does not exist a blue copy of a graph $G \in \mathcal{G}$ such that $\left|V(G) \cap V\left(L-r_{1,1}-r_{1,2}\right)\right|>0$ and $\left|V(G) \cap V\left(M-r_{2,1}-r_{2,2}\right)\right|>0$. Hence, we can conclude that $\phi$ is a $\left(K_{1,2}, \mathcal{G}\right)$-coloring of edges of $F-e$.

Corollary 6. Let $B_{1}, B_{2}$ be disjoint graphs, $\mathcal{G}$ be a family of 2-connected graphs. Let $r_{1}, r_{2}$ be vertices of $B_{1}$ and $B_{2}$, respectively, and let $F$ be a graph obtained from graphs $B_{1}$ and $B_{2}$ by identifying the vertices $r_{1}$ and $r_{2}$. If $B_{1}\left(r_{1}\right), B_{2}\left(r_{2}\right) \in$ $\tilde{\Re}\left(K_{1,2}, \mathcal{G}\right)$, then $F \in \Re\left(K_{1,2}, \mathcal{G}\right)$.

Proof. From Lemma 2.
Theorem 7. Let $c \geq 3$ be an integer, $L$ be a graph, $\mathcal{G}$ be a family of 2 -connected graphs without induced cycles of the length greater than $c$. Let $r_{1}, r_{2}$ be vertices of $L$ such that $d_{L}\left(r_{1}, r_{2}\right)>c$, and let $F$ be a graph obtained from the graph $L$ by identifying the vertices $r_{1}$ and $r_{2}$. If $L\left(r_{1,1}, r_{1,2}\right)_{1} \in \tilde{\Re}\left(K_{1,2}, \mathcal{G}\right)$, then $F \in$ $\Re\left(K_{1,2}, \mathcal{G}\right)$.

Proof. First we show that $F \rightarrow\left(K_{1,2}, \mathcal{G}\right)$. Suppose, on the contrary, that there exists a ( $K_{1,2}, \mathcal{G}$ )-coloring of edges of $F$ such that there is at most one red edge incident with $r_{1}$. Since $d_{L}\left(r_{1}, r_{2}\right)>c$, it follows that $r_{1} r_{2} \notin E(L)$, so $L\left(r_{1}\right) \nrightarrow$ $\left(K_{1,2}, \mathcal{G}\right)$ or $L\left(r_{2}\right) \nrightarrow\left(K_{1,2}, \mathcal{G}\right)$, what leads us to a contradiction.

To finish the proof we show that $F-e \nrightarrow\left(K_{1,2}, \mathcal{G}\right)$, for $e \in E(F)$. We know that $(L-e)\left(r_{1}\right) \nrightarrow\left(K_{1,2}, \mathcal{G}\right)$. Hence, there exists a $\left(K_{1,2}, \mathcal{G}\right)$-coloring of edges of $(L-e)\left(r_{1}\right)$. It is easy to notice that the vertex $r_{2}$ is incident with at most one red edge in the graph $F-e$. Since $\mathcal{G}$ contains only 2 -connected graphs and $d_{L}\left(r_{1}, r_{2}\right)>c$, there does not exist a blue copy of a graph $G \in \mathcal{G}$. Hence, $\phi$ is a ( $K_{1,2}, \mathcal{G}$ )-coloring of edges of $F-e$.

## 3. The Families $\tilde{\Re}\left(K_{1,2}, K_{n}\right)$ and $\Re\left(K_{1,2}, K_{n}\right)$

On the basis of results of Borowiecki et al. [4] we can observe the following facts:

## Observation 1.

(i) $K_{3}\left(r_{1}, r_{2}\right) \in \tilde{\Re}\left(K_{1,2}, K_{3}\right)$.
(ii) Let $r$ be a vertex of degree 3 of $K_{4}-e$. Then $\left(K_{4}-e\right)(r) \in \tilde{\mathscr{R}}\left(K_{1,2}, K_{3}\right)$.
(iii) Let $T C_{n}=K_{3}$-cycle, which we obtain from $n \geq 4$ copies of $K_{3}$ by identifying the second vertex of the $i$-th copy of $K_{3}$ with the first vertex of the ( $i \bmod$ $n)+1)$-th copy of $K_{3}$, for $i=1,2, \ldots, n$. Then $T C_{n}(r) \in \tilde{\Re}\left(K_{1,2}, K_{3}\right)$, where $r \in V\left(T C_{n}\right)$.
(iv) Let $r_{1}, r_{2}$ be vertices of degree 3 of $K_{4}-e$. Then $\left(K_{4}-e\right)\left(r_{1}, r_{2}\right)_{1} \in$ $\tilde{\Re}\left(K_{1,2}, K_{3}\right)$.
(v) Graphs $L_{i}\left(r_{1}, r_{2}\right)_{1}$, for $i=1, \ldots, 6$, in Figure 1 belong to $\tilde{\Re}\left(K_{1,2}, K_{3}\right)$.

$L_{1}$

$L_{2}$

$L_{3}$

$L_{4}$

$L_{5}$

$L_{6}$

Figure 1. All presented graphs $L_{i}\left(r_{1}, r_{2}\right)_{1}$ belong to $\tilde{\mathscr{R}}\left(K_{1,2}, K_{3}\right)$.
In the next three theorems we indicate some special graphs. These graphs together with our previous results allow us to construct infinitely many ( $K_{1,2}, K_{n}$ )minimal graphs, i.e. graphs that belong to the Ramsey set $\Re\left(K_{1,2}, K_{n}\right)$ for every $n \geq 3$.

Theorem 8. Let $n \geq 3$. Let $M=K_{2 n-3}-(n-3) K_{2}$ and $r_{1}, r_{2}$ be vertices of degree $2 n-4$ of $M$. Then $M\left(r_{1}, r_{2}\right) \in \Re\left(K_{1,2}, K_{n}\right)$.
Proof. Note that for $n=3$ the graph $M\left(r_{1}, r_{2}\right)=K_{3}\left(r_{1}, r_{2}\right) \in \tilde{\Re}\left(K_{1,2}, K_{n}\right)$ from Observation 1(i). Hence, we can consider only $n \geq 4$.

In the first step of the proof we show that $M\left(r_{1}, r_{2}\right) \rightarrow\left(K_{1,2}, K_{n}\right)$. Provided that the vertices $r_{1}$ and $r_{2}$ are not incident with red edges, we consider every red-blue coloring $\phi$ of edges of $M$, such that there is no red copy of the graph $K_{1,2}$. Let $E_{1}=E(\bar{M})$ and $E_{2}=\{e \in E(M): \phi(e)=r e d\}$. We can notice that the graph $H=\left(V(M) \backslash\left\{r_{1}, r_{2}\right\}, E_{1} \cup E_{2}\right)$ is bipartite and $\Delta(H) \leq 2$. Hence, we can divide the set $V(H)$ into $V_{1}$ and $V_{2}$ such that $H\left[V_{1}\right]$ and $H\left[V_{2}\right]$ are edgeless. Without loss of generality we can assume that $\left|V_{1}\right|>\left|V_{2}\right|$. This implies that $\left|V_{1}\right| \geq n-2$. One can see that the subgraph of $M$ induced by $V_{1} \cup\left\{r_{1}, r_{2}\right\}$ contains only blue edges and is isomorphic to $K_{n}$.

Now we show that $(M-e)\left(r_{1}, r_{2}\right) \nrightarrow\left(K_{1,2}, K_{n}\right)$. Let $E(\bar{M})=\left\{v_{i, 1} v_{i, 2}: i=\right.$ $1,2, \ldots, n-3\}$ and $v \in V(M) \backslash\left\{r_{1}, r_{2}\right\}$, where $\operatorname{deg}(v)=2 n-4$. Without loss of generality we can consider only the case when $e \in\left\{v_{1,1} r_{1}, v_{1,1} v, v_{1,1} v_{2,1}\right\}$. If $n \geq 5$, then for any choice of $e$ we color red edges $v v_{1,2}, v_{i, 1} v_{i+1,2}$, for $i=1,2, \ldots, n-4$. If $n=4$, then we color red edges $v v_{1,2}$ and $v_{1,1} r_{1}$. We color the remaining edges blue. These colorings of $(M-e)\left(r_{1}, r_{2}\right)$ contain neither a red copy of $K_{1,2}$ nor a blue copy of $K_{n}$.

To finish the proof we show that $M\left(r_{1}\right) \nrightarrow\left(K_{1,2}, K_{n}\right)$. Let us consider the following coloring of edges of $M$. If $n \geq 5$, then we color red edges $r_{2} v_{n-3,1}, v v_{1,2}$, $v_{i, 1} v_{i+1,2}$, for $i=1,2, \ldots, n-4$. If $n=4$, then we color red edges $r_{2} v_{1,1}$ and $v v_{1,2}$. The remaining edges we color blue. One can see that this coloring of $M$ contains neither a red copy of $K_{1,2}$ nor a blue copy of $K_{n}$. Similarly, we can prove that $M\left(r_{2}\right) \nrightarrow\left(K_{1,2}, K_{n}\right)$.

Theorem 9. Let $n \geq 3$. Let $B=K_{2 n-2}-(n-2) K_{2}$ and $r$ be a vertex of degree $2 n-3$ of $B$. Then $B(r) \in \tilde{\Re}\left(K_{1,2}, K_{n}\right)$.
Proof. Notice that for $n=3$ the graph $B(r)=\left(K_{4}-e\right)(r) \in \tilde{\mathscr{R}}\left(K_{1,2}, K_{3}\right)$ from Observation 1(ii). Hence, we can consider only $n \geq 4$.

First we prove that $B(r) \rightarrow\left(K_{1,2}, K_{n}\right)$. Consider a red-blue coloring $\phi$ of edges of $B$. Suppose that in this coloring there is no red copy of $K_{1,2}$. Let $E_{1}=E(\bar{B})$ and $E_{2}=\{e \in E(B): \phi(e)=r e d\}$. If we consider the graph $H=\left(V(B) \backslash\{r\}, E_{1} \cup E_{2}\right)$, then we can notice that $H$ is bipartite and $\Delta(H) \leq 2$. Therefore we can divide the set $V(H)$ into $V_{1}$ and $V_{2}$ such that $H\left[V_{1}\right]$ and $H\left[V_{2}\right]$ are edgeless. Without loss of generality we can assume that $\left|V_{1}\right|>\left|V_{2}\right|$. Hence $\left|V_{1}\right| \geq n-1$. Now, we can notice that the subgraph of $B$ induced by $V_{1} \cup\{r\}$ contains only blue edges and is isomorphic to $K_{n}$.
Let $E(\bar{B})=\left\{v_{i, 1} v_{i, 2}: i=1,2, \ldots, n-2\right\}$ and $v \in V(B) \backslash\{r\}$, where $\operatorname{deg}(v)=2 n-$ 3. In the next step of the proof we show that $(B-e)(r) \nrightarrow\left(K_{1,2}, K_{n}\right)$. Without
loss of generality we can consider only the case when $e \in\left\{v_{1,1} r, v_{1,1} v, v_{1,1} v_{2,1}\right\}$. Regardless of the choice of $e$ we color red edges $v v_{1,2}, v_{i, 1} v_{i+1,2}$, for $i=1,2, \ldots, n-$ 3. The remaining uncolored edges we color blue. Clearly, such a coloring of $(B-e)(r)$ contains neither a red copy of $K_{1,2}$ nor a blue copy of $K_{n}$.

Finally, we show that $B \nrightarrow\left(K_{1,2}, K_{n}\right)$. One can see that a coloring of $B$ such that edges $r v_{n-2,1}, v v_{1,2}, v_{i, 1} v_{i+1,2}$, for $i=1,2, \ldots, n-3$, are red and the other edges are blue contains neither a red copy of $K_{1,2}$ nor a blue copy $K_{n}$. This observation finishes the proof.

Theorem 10. Let $n \geq 3$. Let $L=K_{2 n-2}-(n-2) K_{2}$ and $r_{1}, r_{2}$ be vertices of degree $2 n-3$ of $L$. Then $L\left(r_{1}, r_{2}\right)_{1} \in \tilde{\Re}\left(K_{1,2}, K_{n}\right)$.

Proof. From Remark 1 and Theorem 9.
In the next theorem we indicate one more graph belonging to $\Re\left(K_{1,2}, K_{n}\right)$, for every $n \geq 3$. Moreover, from [7] this graph is minimal with respect to the number of vertices.

Theorem 11. Let $F=K_{2 n-1}-(n-1) K_{2}, \quad n \geq 3$. Then $F \in \Re\left(K_{1,2}, K_{n}\right)$.
Proof. From Theorem 9 we have $B(r)=\left(K_{2 n}-(n-1) K_{2}\right)(r) \rightarrow\left(K_{1,2}, K_{n+1}\right)$, where $\operatorname{deg}(r)=2 n-1$. It easy to see that $B-r=F$ and $F \rightarrow\left(K_{1,2}, K_{n}\right)$.

Let $E(\bar{F})=\left\{v_{i, 1} v_{i, 2}: i=1,2, \ldots, n-1\right\}$ and $v \in V(B) \backslash\{r\}$, where $\operatorname{deg}(v)=$ $2 n-2$. We show that $(F-e) \nrightarrow\left(K_{1,2}, K_{n}\right)$. Without loss of generality we can consider only the case when $e \in\left\{v_{1,1} v, v_{1,1} v_{2,1}\right\}$. Regardless of the choice of $e$ we color red edges $v v_{1,2}, v_{i, 1} v_{i+1,2}$, for $i=1,2, \ldots, n-2$. We color the remaining uncolored edges blue. Clearly, such a coloring of $F$ contains neither a red copy of $K_{1,2}$ nor a blue copy of $K_{n}$.

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