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ON RAMSEY $(K_{1,2}, K_n)$-MINIMAL GRAPHS

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Abstract

Let F be a graph and let $\mathcal{G}$, $\mathcal{H}$ denote nonempty families of graphs. We write $F \to (\mathcal{G}, \mathcal{H})$ if in any 2-coloring of edges of F with red and blue, there is a red subgraph isomorphic to some graph from $\mathcal{G}$ or a blue subgraph isomorphic to some graph from $\mathcal{H}$. The graph F without isolated vertices is said to be a $(\mathcal{G}, \mathcal{H})$-minimal graph if $F \to (\mathcal{G}, \mathcal{H})$ and $F - e \not\to (\mathcal{G}, \mathcal{H})$ for every $e \in E(F)$.

We present a technique which allows to generate infinite family of $(\mathcal{G}, \mathcal{H})$-minimal graphs if we know some special graphs. In particular, we show how to receive infinite family of $(K_{1,2}, K_n)$-minimal graphs, for every $n \geq 3$.

Keywords: Ramsey minimal graph, edge coloring, 1-factor, complete graph.

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1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. By $deg_G(v_1)$, $d_G(v_1, v_2)$ we denote the degree of the vertex v_1 in G and the distance between two vertices v_1, v_2, respectively. If G is known we can shortly write $deg(v_1), d(v_1, v_2)$. We use the notation and terminology of [8].

Let F be a graph and let $\mathcal{G}$, $\mathcal{H}$ be nonempty families of graphs. We write $F \to (\mathcal{G}, \mathcal{H})$ if in any 2-coloring of edges of F with red and blue, there is a red subgraph isomorphic to some graph from $\mathcal{G}$ or a blue subgraph isomorphic to some graph from $\mathcal{H}$. Otherwise, if there exists a 2-coloring of edges such that neither a red subgraph isomorphic to some graph from $\mathcal{G}$ nor a blue subgraph isomorphic to some graph from $\mathcal{H}$ occur, then we write $F \not\to (\mathcal{G}, \mathcal{H})$. The graph

F without isolated vertices is said to be a $(\mathcal{G},\mathcal{H})$-minimal graph if $F \to (\mathcal{G},\mathcal{H})$ and $F-e \not\to (\mathcal{G},\mathcal{H})$ for any $e \in E(F)$. The Ramsey set $\Re(\mathcal{G},\mathcal{H})$ is defined to be the set of all $(\mathcal{G},\mathcal{H})$-minimal graphs (up to isomorphism). For the simplicity of the notation, instead of $\Re(\{G\},\{H\})$ we write $\Re(G,H)$.

Many papers study the problem of determining the family $\Re(G,H)$. One can easily observe that the set $\Re(K_{1,2},K_{1,2})$ is infinite and consists of star with three rays and all cycles of odd length. Burr *et al.* [6] proved that $\Re(K_{1,2k+1},K_{1,2l+1}) = \{K_{1,2(k+l)+1}\}$ and $\Re(K_{1,2k},K_{1,2l})$ is infinite, for every $k,l \geq 1$. Next Borowiecki *et al.* [3] characterized graphs belonging to $\Re(K_{1,2},K_{1,m})$ for $m \geq 3$.

The graphs belonging to $\Re(2K_2,K_{1,n})$ were characterized in [10]. Moreover, Łuczak [9] showed that $\Re(K_{1,2m},G)$ is finite if and only if G is a matching. It means that, for $n \geq 3$, $\Re(K_{1,2},K_n)$ has infinite number of graphs.

Borowiecki, *et al.* described in [4] the whole set $\Re(K_{1,2},K_3)$. In [1, 2] the authors presented how we can generate an infinite family of $(K_{1,2},C_4)$-minimal graphs. In this paper we describe a method which can be applied to the construction of infinitely many graphs belonging to $\Re(K_{1,2},K_n)$, for any $n \geq 3$.

2. The Main Results

First we extend, in the same way as in [2], the already given standard definitions by adding some restriction on a chosen set of vertices. This allows us to construct the infinite family $\Re(K_{1,2},\mathcal{G})$, for any given family $\mathcal{G}$ of 2-connected graphs.

Definition 1. Let F be a graph with $U \subseteq V(F)$ and let $\mathcal{G}$, $\mathcal{H}$ be families of graphs. If for any red-blue coloring of edges of F, such that all vertices in U are not incident with red edges, there exists a red copy of some graph from $\mathcal{G}$ or a blue copy of some graph from $\mathcal{H}$, then we write $F(U) \to (\mathcal{G},\mathcal{H})$. Otherwise, there exists a $(\mathcal{G},\mathcal{H})$-coloring of edges of $F(U)$ and we write $F(U) \not\to (\mathcal{G},\mathcal{H})$.

Definition 2. Let F be a graph and $U \subseteq V(F)$. Let $i \in \{1,2,\ldots,|U|\}$. We say that $F(U)_i$ is $(\mathcal{G},\mathcal{H})$-minimal if

1. $F(U_i) \to (\mathcal{G},\mathcal{H})$, for every $U_i \in \binom{U}{i}$,

2. $(F-e)(U_i) \not\to (\mathcal{G},\mathcal{H})$, for every $e \in E(F)$ and every $U_i \in \binom{U}{i}$,

3. $F(U_{i-1}) \not\to (\mathcal{G},\mathcal{H})$, for every $U_{i-1} \in \binom{U}{i-1}$.

We write $F(U)_i \in \tilde{\Re}(\mathcal{G},\mathcal{H})$ if $F(U)_i$ is $(\mathcal{G},\mathcal{H})$-minimal. If $U=\emptyset$ or $i=0$, then we assume that $F(U)_i \in \tilde{\Re}(\mathcal{G},\mathcal{H}) \Leftrightarrow F \in \Re(\mathcal{G},\mathcal{H})$.
For the simplicity of the notation we write $F(v_1,\ldots,v_p)_i$ instead of $F(\{v_1,\ldots,v_p\})_i$ and $F(v_1,\ldots,v_p)$ instead of $F(v_1,\ldots,v_p)_p$.

Remark 1. $F(r_1, r_2)_1 \in \tilde{\Re}(\mathcal{G}, \mathcal{H})$ if and only if $F(r_i) \in \tilde{\Re}(\mathcal{G}, \mathcal{H})$, for $i = 1, 2$.

Lemma 2. *Let $\mathcal{G}$ be a family of 2-connected graphs. Let M_1, M_2 be disjoint graphs and $U_i \subset V(M_i)$, $|U_i| \in \{1, 2\}$ and $r_i \in U_i$, for $i = 1, 2$, and let M be a graph obtained from disjoint graphs M_1 and M_2 by identifying the vertices r_1 and r_2. If $M_1(U_1), M_2(U_2) \in \tilde{\Re}(K_{1,2}, \mathcal{G})$, then $M(U_1 \cup U_2 \setminus \{r_1, r_2\}) \in \tilde{\Re}(K_{1,2}, \mathcal{G})$.*

Proof. Let $U = U_1 \cup U_2 \setminus \{r_1, r_2\}$. First we prove that $M(U) \to (K_{1,2}, \mathcal{G})$. If we assume that $M(U) \not\to (K_{1,2}, \mathcal{G})$, then there exists a coloring of edges of M such that there is at most one red edge e incident with r_1. It means that $M_1(U_1) \not\to (K_{1,2}, \mathcal{G})$ or $M_2(U_2) \not\to (K_{1,2}, \mathcal{G})$. Hence, we obtain a contradiction.

Now we show that $(M - e)(U) \not\to (K_{1,2}, \mathcal{G})$. Without loss of generality we can consider only the situation when $e \in E(M_1)$. We know that $(M_1 - e)(U_1) \not\to (K_{1,2}, \mathcal{G})$ and $M_2(U_2 \setminus \{r_2\}) \not\to (K_{1,2}, \mathcal{G})$. Thus, there exist a $(K_{1,2}, \mathcal{G})$-coloring of edges of $(M_1 - e)(U_1)$, let us denote it by ϕ_1, and a $(K_{1,2}, \mathcal{G})$-coloring of edges of $M_2(U_2 \setminus \{r_2\})$, let us denote it by ϕ_2. Let ϕ be a coloring of edges of $(M - e)$ such that $\phi(f) = \phi_1(f)$ for $f \in E(M_1)$ and $\phi(f) = \phi_2(f)$ for $f \in E(M_2)$. Since $M_2(U_2 \setminus \{r_2\}) \not\to (K_{1,2}, \mathcal{G})$, it is easy to notice that the vertex r_1 is incident with exactly one red edge which belongs to $E(M_2)$. We can notice that there does not exist a blue copy of a graph $G \in \mathcal{G}$ such that $|V(G) \cap V(M_1)| > 1$ and $|V(G) \cap V(M_2)| > 1$, because $\mathcal{G}$ contains only 2-connected graphs. Hence, ϕ is a $(K_{1,2}, \mathcal{G})$-coloring of edges of $(M - e)(U)$.

Finally, we prove that $M(U_i - r_i) \not\to (K_{1,2}, \mathcal{G})$ for $i = 1, 2$. Without loss of generality we can assume that $i = 1$. We know that $M_1(U_1 - \{r_1\}) \not\to (K_{1,2}, \mathcal{G})$ and $M_2(r_2) \not\to (K_{1,2}, \mathcal{G})$. Hence, there exist $(K_{1,2}, \mathcal{G})$-colorings ϕ_1 and ϕ_2 of edges of $M_1(U_1 - \{r_1\})$ and $M_2(r_2)$, respectively. Let ϕ be a coloring of edges of M such that $\phi(f) = \phi_1(f)$, if $f \in E(M_1)$ and $\phi(f) = \phi_2(f)$, otherwise. It is easy to observe that the vertex r_1 is incident with exactly one red edge belonging to $E(M_1)$ in M. For the same reason as previously we can notice that there does not exist a blue copy of a graph $G \in \mathcal{G}$ such that $|V(G) \cap V(M_1)| > 1$ and $|V(G) \cap V(M_2)| > 1$. Hence, ϕ is a $(K_{1,2}, \mathcal{G})$-coloring of edges of $M(U_1 - \{r_1\})$. This observation finishes the proof. ∎

Lemma 3. *Let $c \geq 3$ be an integer, M_1, M_2 be disjoint graphs, $\mathcal{G}$ be a family of 2-connected graphs without induced cycles of the length greater than c. Let $r_{i,1}, r_{i,2}$ be vertices of M_i, for $i = 1, 2$, such that $d_{M_1}(r_{1,1}, r_{1,2}) + d_{M_2}(r_{2,1}, r_{2,2}) > c$, and let L be a graph obtained from graphs M_1 and M_2 by identifying the vertices $r_{1,1}$ and $r_{2,1}$, and the vertices $r_{1,2}$ and $r_{2,2}$. If $M_i(r_{i,1}, r_{i,2}) \in \tilde{\Re}(K_{1,2}, \mathcal{G})$, for $i = 1, 2$, then $L(r_{1,1}, r_{1,2})_1 \in \tilde{\Re}(K_{1,2}, \mathcal{G})$.*

Proof. First we prove that $L(r_{1,1}, r_{1,2})_1 \to (K_{1,2}, \mathcal{G})$. Conversely, suppose that $L(r_{1,1}, r_{1,2})_1 \not\to (K_{1,2}, \mathcal{G})$. Without loss of generality we can assume that $L(r_{1,1}) \not\to (K_{1,2}, \mathcal{G})$. Then there exists a $(K_{1,2}, \mathcal{G})$-coloring of edges of L such that every edge

e incident with $r_{1,1}$ in L is blue and at most one edge incident with $r_{2,2}$ is red. Without loss of generality we can assume that the red edge belongs to $E(M_1)$. Hence, we obtain a contradiction with the fact that $M_2(r_{2,1}, r_{2,2}) \to (K_{1,2}, \mathcal{G})$.

Now we show that $(L-e)(r_{1,1}, r_{1,2})_1 \not\to (K_{1,2}, \mathcal{G})$. Without loss of generality we can assume that $e \in E(M_1)$. We know that $(M_1 - e)(r_{1,1}, r_{1,2}) \not\to (K_{1,2}, \mathcal{G})$ and $M_2(r_{2,1}) \not\to (K_{1,2}, \mathcal{G})$. Thus, there exist a $(K_{1,2}, \mathcal{G})$-coloring of edges of $(M_1 - e)(r_{1,1}, r_{1,2})$ and a $(K_{1,2}, \mathcal{G})$-coloring of edges of $M_2(r_{2,1})$. We denote these colorings by ϕ_1 and ϕ_2, respectively. Let ϕ be a coloring of edges $(L-e)$ such that $\phi(f) = \phi_1(f)$, if $f \in E(M_1)$ and $\phi(f) = \phi_2(f)$, otherwise. It is easy to notice that the vertex $r_{1,2}$ is incident with exactly one red edge belonging to $E(M_2)$. Since $\mathcal{G}$ contains only 2-connected graphs and $d_{M_1}(r_{1,1}, r_{1,2}) + d_{M_2}(r_{2,1}, r_{2,2}) > c$, there does not exist a blue copy of a graph $G \in \mathcal{G}$ such that $|V(G) \cap V(M_i - r_{i,1} - r_{i,2})| > 0$, for $i = 1, 2$. Hence, ϕ is a $(K_{1,2}, \mathcal{G})$-coloring of edges of $(L-e)(r_{1,1}, r_{1,2})$.

Finally, we prove that $L \not\to (K_{1,2}, \mathcal{G})$. From our assumption, it follows that $M_1(r_{1,1}) \not\to (K_{1,2}, \mathcal{G})$ and $M_2(r_{2,2}) \not\to (K_{1,2}, \mathcal{G})$. Thus once again, we can indicate two colorings ϕ_1 and ϕ_2 such that ϕ_i is a $(K_{1,2}, \mathcal{G})$-coloring of edges of $M_i(r_{i,i})$, for $i = 1, 2$. Let ϕ be a coloring of edges of L such that $\phi(f) = \phi_i(f)$ for $f \in E(M_i)$ and $i = 1, 2$. We can observe that the vertex $r_{1,1}$ is incident with exactly one red edge belonging to $E(M_2)$ and the vertex $r_{1,2}$ is incident with exactly one red edge belonging to $E(M_1)$. We can notice that there does not exist a blue copy of a graph $G \in \mathcal{G}$ such that $|V(G) \cap V(M_i - r_{i,1} - r_{i,2})| > 0$ for $i = 1, 2$, because $\mathcal{G}$ contains only 2-connected graphs and $d_{M_1}(r_{1,1}, r_{1,2}) + d_{M_2}(r_{2,1}, r_{2,2}) > c$. Therefore ϕ is a $(K_{1,2}, \mathcal{G})$-coloring edges of L. ■

Corollary 4. *Let $c \geq 3$ be an integer, M_1, M_2 be disjoint graphs, $\mathcal{G}$ be a family of 2-connected graphs without induced cycles of the length greater than c. Let $r_{i,1}, r_{i,2}$ be vertices of M_i, for $i = 1, 2$, such that $d_{M_1}(r_{1,1}, r_{1,2}) + d_{M_2}(r_{2,1}, r_{2,2}) > c$, and let B be a graph obtained from graphs M_1 and M_2 by identifying the vertices $r_{1,1}$ and $r_{2,1}$, and the vertices $r_{1,2}$ and $r_{2,2}$. If $M_i(r_{i,1}, r_{i,2}) \in \tilde{\Re}(K_{1,2}, \mathcal{G})$, for $i = 1, 2$, then $B(r_{1,1}) \in \tilde{\Re}(K_{1,2}, \mathcal{G})$.*

Proof. From Lemma 3 and Remark 1. ■

The next theorems give us a method of the construction of infinitely many graphs that belong to $\Re(K_{1,2}, \mathcal{G})$, where $\mathcal{G}$ is any given family of graphs. In this construction we use graphs with adding some restriction on a chosen set of vertices, i.e., graphs that belong to the family $\tilde{\Re}(K_{1,2}, \mathcal{G})$.

Theorem 5. *Let $c \geq 3$ be an integer, L, M be disjoint graphs, $\mathcal{G}$ be a family of 2-connected graphs without induced cycles of the length greater than c. Let $\{r_{1,1}, r_{1,2}\} \subset V(L)$ and $\{r_{2,1}, r_{2,2}\} \subset V(M)$ such that $d_L(r_{1,1}, r_{1,2}) + d_M(r_{2,1}, r_{2,2}) > c$, and let F be a graph obtained from graphs L and M by identifying the vertices $r_{1,1}$*

and $r_{2,1}$ and the vertices $r_{1,2}$, and $r_{2,2}$. If $L(r_{1,1}, r_{1,2}), M(r_{2,1}, r_{2,2}) \in \tilde{\Re}(K_{1,2}, \mathcal{G})$, then $F \in \Re(K_{1,2}, \mathcal{G})$.

Proof. We start with proving that $F \to (K_{1,2}, \mathcal{G})$. Suppose, on the contrary, that there exists a $(K_{1,2}, \mathcal{G})$-coloring of edges of F. From the fact that $L(r_{1,1}, r_{1,2})_1 \to (K_{1,2}, \mathcal{G})$ it follows that in this coloring one edge incident with $r_{1,1}$ and one edge incident with $r_{1,2}$ in L is red. Hence, every edge incident with $r_{2,1}$ and $r_{2,2}$ in M is blue. We obtain a contradiction with the assumption that $M(r_{2,1}, r_{2,2}) \to (K_{1,2}, \mathcal{G})$.

It remains to prove that $F - e \not\to (K_{1,2}, \mathcal{G})$, for every $e \in E(F)$.

Case 1. Let $e \in E(L)$. We know that $(L - e)(r_{1,i}) \not\to (K_{1,2}, \mathcal{G})$ and $M(r_{2,3-i}) \not\to (K_{1,2}, \mathcal{G})$, for $i = 1, 2$. Without loss of generality we can assume that $i = 1$. Thus, there exists a $(K_{1,2}, \mathcal{G})$-coloring of edges of $(L - e)(r_{1,1})$ and a $(K_{1,2}, \mathcal{G})$-coloring of edges of $M(r_{2,2})$. Let us denote these colorings by ϕ_1 and ϕ_2, respectively. Let ϕ be a coloring of edges of $(F - e)$ such that $\phi(f) = \phi_1(f)$, if $f \in E(L)$ and $\phi(f) = \phi_2(f)$, otherwise. Let us notice that the vertices $r_{1,1}$ and $r_{1,2}$ must be incident with at most one red edge in the graph $F - e$. We also know that there does not exist a blue copy of a graph $G \in \mathcal{G}$ such that $|V(G) \cap V(L - r_{1,1} - r_{1,2})| > 0$ and $|V(G) \cap V(M - r_{2,1} - r_{2,2})| > 0$. This observation follows from the fact that $\mathcal{G}$ contains only 2-connected graphs and $d_L(r_{1,1}, r_{1,2}) + d_M(r_{2,1}, r_{2,2}) > c$. Hence, ϕ is a $(K_{1,2}, \mathcal{G})$-coloring of edges of $F - e$.

Case 2. Let $e \in E(M)$. From the fact that $L \not\to (K_{1,2}, \mathcal{G})$ and $(M - e)(r_{2,1}, r_{2,2}) \not\to (K_{1,2}, \mathcal{G})$ it follows that there exist a $(K_{1,2}, \mathcal{G})$-coloring ϕ_1 of edges of L and a $(K_{1,2}, \mathcal{G})$-coloring ϕ_2 of edges of $(M - e)(r_{2,1}, r_{2,2})$. Let ϕ be a coloring of edges of $(F - e)$ such that $\phi(f) = \phi_1(f)$, if $f \in E(L)$ and $\phi(f) = \phi_2(f)$, otherwise. Since $L(r_{1,i}) \not\to (K_{1,2}, \mathcal{G})$, for $i = 1, 2$ and $L \not\to (K_{1,2}, \mathcal{G})$, the vertices $r_{1,1}$ and $r_{1,2}$ are incident with exactly one red edge in $F - e$. For the same reason as in Case 1 we know that there does not exist a blue copy of a graph $G \in \mathcal{G}$ such that $|V(G) \cap V(L - r_{1,1} - r_{1,2})| > 0$ and $|V(G) \cap V(M - r_{2,1} - r_{2,2})| > 0$. Hence, we can conclude that ϕ is a $(K_{1,2}, \mathcal{G})$-coloring of edges of $F - e$. ■

Corollary 6. *Let B_1, B_2 be disjoint graphs, $\mathcal{G}$ be a family of 2-connected graphs. Let r_1, r_2 be vertices of B_1 and B_2, respectively, and let F be a graph obtained from graphs B_1 and B_2 by identifying the vertices r_1 and r_2. If $B_1(r_1), B_2(r_2) \in \tilde{\Re}(K_{1,2}, \mathcal{G})$, then $F \in \Re(K_{1,2}, \mathcal{G})$.*

Proof. From Lemma 2. ■

Theorem 7. *Let $c \geq 3$ be an integer, L be a graph, $\mathcal{G}$ be a family of 2-connected graphs without induced cycles of the length greater than c. Let r_1, r_2 be vertices of L such that $d_L(r_1, r_2) > c$, and let F be a graph obtained from the graph L by identifying the vertices r_1 and r_2. If $L(r_{1,1}, r_{1,2})_1 \in \tilde{\Re}(K_{1,2}, \mathcal{G})$, then $F \in \Re(K_{1,2}, \mathcal{G})$.*

Proof. First we show that $F \to (K_{1,2}, \mathcal{G})$. Suppose, on the contrary, that there exists a $(K_{1,2}, \mathcal{G})$-coloring of edges of F such that there is at most one red edge incident with r_1. Since $d_L(r_1, r_2) > c$, it follows that $r_1 r_2 \notin E(L)$, so $L(r_1) \not\to (K_{1,2}, \mathcal{G})$ or $L(r_2) \not\to (K_{1,2}, \mathcal{G})$, what leads us to a contradiction.

To finish the proof we show that $F - e \not\to (K_{1,2}, \mathcal{G})$, for $e \in E(F)$. We know that $(L-e)(r_1) \not\to (K_{1,2}, \mathcal{G})$. Hence, there exists a $(K_{1,2}, \mathcal{G})$-coloring of edges of $(L-e)(r_1)$. It is easy to notice that the vertex r_2 is incident with at most one red edge in the graph $F - e$. Since $\mathcal{G}$ contains only 2-connected graphs and $d_L(r_1, r_2) > c$, there does not exist a blue copy of a graph $G \in \mathcal{G}$. Hence, ϕ is a $(K_{1,2}, \mathcal{G})$-coloring of edges of $F - e$. ■

3. The Families $\tilde{\Re}(K_{1,2}, K_n)$ and $\Re(K_{1,2}, K_n)$

On the basis of results of Borowiecki *et al.* [4] we can observe the following facts:

Observation 1.

(i) $K_3(r_1, r_2) \in \tilde{\Re}(K_{1,2}, K_3)$.

(ii) Let r be a vertex of degree 3 of $K_4 - e$. Then $(K_4 - e)(r) \in \tilde{\Re}(K_{1,2}, K_3)$.

(iii) Let $TC_n = K_3$-cycle, which we obtain from $n \geq 4$ copies of K_3 by identifying the second vertex of the i-th copy of K_3 with the first vertex of the $((i \bmod n)+1)$-th copy of K_3, for $i = 1, 2, \ldots, n$. Then $TC_n(r) \in \tilde{\Re}(K_{1,2}, K_3)$, where $r \in V(TC_n)$.

(iv) Let r_1, r_2 be vertices of degree 3 of $K_4 - e$. Then $(K_4 - e)(r_1, r_2)_1 \in \tilde{\Re}(K_{1,2}, K_3)$.

(v) Graphs $L_i(r_1, r_2)_1$, for $i = 1, \ldots, 6$, in Figure 1 belong to $\tilde{\Re}(K_{1,2}, K_3)$.

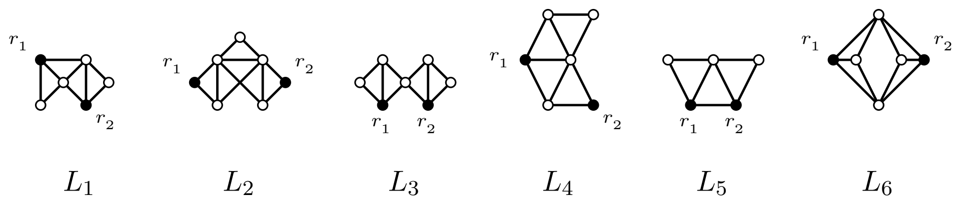

Figure 1. All presented graphs $L_i(r_1, r_2)_1$ belong to $\tilde{\Re}(K_{1,2}, K_3)$.

In the next three theorems we indicate some special graphs. These graphs together with our previous results allow us to construct infinitely many $(K_{1,2}, K_n)$-minimal graphs, i.e. graphs that belong to the Ramsey set $\Re(K_{1,2}, K_n)$ for every $n \geq 3$.

Theorem 8. *Let $n \geq 3$. Let $M = K_{2n-3} - (n-3)K_2$ and r_1, r_2 be vertices of degree $2n-4$ of M. Then $M(r_1, r_2) \in \tilde{\Re}(K_{1,2}, K_n)$.*

Proof. Note that for $n = 3$ the graph $M(r_1, r_2) = K_3(r_1, r_2) \in \tilde{\Re}(K_{1,2}, K_n)$ from Observation 1(i). Hence, we can consider only $n \geq 4$.

In the first step of the proof we show that $M(r_1, r_2) \to (K_{1,2}, K_n)$. Provided that the vertices r_1 and r_2 are not incident with red edges, we consider every red-blue coloring ϕ of edges of M, such that there is no red copy of the graph $K_{1,2}$. Let $E_1 = E(\overline{M})$ and $E_2 = \{e \in E(M) : \phi(e) = red\}$. We can notice that the graph $H = (V(M) \setminus \{r_1, r_2\}, E_1 \cup E_2)$ is bipartite and $\Delta(H) \leq 2$. Hence, we can divide the set $V(H)$ into V_1 and V_2 such that $H[V_1]$ and $H[V_2]$ are edgeless. Without loss of generality we can assume that $|V_1| > |V_2|$. This implies that $|V_1| \geq n-2$. One can see that the subgraph of M induced by $V_1 \cup \{r_1, r_2\}$ contains only blue edges and is isomorphic to K_n.

Now we show that $(M-e)(r_1, r_2) \not\to (K_{1,2}, K_n)$. Let $E(\overline{M}) = \{v_{i,1}v_{i,2} : i = 1, 2, \ldots, n-3\}$ and $v \in V(M) \setminus \{r_1, r_2\}$, where $deg(v) = 2n-4$. Without loss of generality we can consider only the case when $e \in \{v_{1,1}r_1, v_{1,1}v, v_{1,1}v_{2,1}\}$. If $n \geq 5$, then for any choice of e we color red edges $vv_{1,2}$, $v_{i,1}v_{i+1,2}$, for $i = 1, 2, \ldots, n-4$. If $n = 4$, then we color red edges $vv_{1,2}$ and $v_{1,1}r_1$. We color the remaining edges blue. These colorings of $(M-e)(r_1, r_2)$ contain neither a red copy of $K_{1,2}$ nor a blue copy of K_n.

To finish the proof we show that $M(r_1) \not\to (K_{1,2}, K_n)$. Let us consider the following coloring of edges of M. If $n \geq 5$, then we color red edges $r_2v_{n-3,1}, vv_{1,2}$, $v_{i,1}v_{i+1,2}$, for $i = 1, 2, \ldots, n-4$. If $n = 4$, then we color red edges $r_2v_{1,1}$ and $vv_{1,2}$. The remaining edges we color blue. One can see that this coloring of M contains neither a red copy of $K_{1,2}$ nor a blue copy of K_n. Similarly, we can prove that $M(r_2) \not\to (K_{1,2}, K_n)$. ∎

Theorem 9. *Let $n \geq 3$. Let $B = K_{2n-2} - (n-2)K_2$ and r be a vertex of degree $2n-3$ of B. Then $B(r) \in \tilde{\Re}(K_{1,2}, K_n)$.*

Proof. Notice that for $n = 3$ the graph $B(r) = (K_4 - e)(r) \in \tilde{\Re}(K_{1,2}, K_3)$ from Observation 1(ii). Hence, we can consider only $n \geq 4$.

First we prove that $B(r) \to (K_{1,2}, K_n)$. Consider a red-blue coloring ϕ of edges of B. Suppose that in this coloring there is no red copy of $K_{1,2}$. Let $E_1 = E(\overline{B})$ and $E_2 = \{e \in E(B) : \phi(e) = red\}$. If we consider the graph $H = (V(B) \setminus \{r\}, E_1 \cup E_2)$, then we can notice that H is bipartite and $\Delta(H) \leq 2$. Therefore we can divide the set $V(H)$ into V_1 and V_2 such that $H[V_1]$ and $H[V_2]$ are edgeless. Without loss of generality we can assume that $|V_1| > |V_2|$. Hence $|V_1| \geq n-1$. Now, we can notice that the subgraph of B induced by $V_1 \cup \{r\}$ contains only blue edges and is isomorphic to K_n.
Let $E(\overline{B}) = \{v_{i,1}v_{i,2} : i = 1, 2, \ldots, n-2\}$ and $v \in V(B) \setminus \{r\}$, where $deg(v) = 2n-3$. In the next step of the proof we show that $(B-e)(r) \not\to (K_{1,2}, K_n)$. Without

loss of generality we can consider only the case when $e \in \{v_{1,1}r, v_{1,1}v, v_{1,1}v_{2,1}\}$. Regardless of the choice of e we color red edges $vv_{1,2}, v_{i,1}v_{i+1,2}$, for $i = 1, 2, \ldots, n-3$. The remaining uncolored edges we color blue. Clearly, such a coloring of $(B - e)(r)$ contains neither a red copy of $K_{1,2}$ nor a blue copy of K_n.

Finally, we show that $B \not\to (K_{1,2}, K_n)$. One can see that a coloring of B such that edges $rv_{n-2,1}, vv_{1,2}, v_{i,1}v_{i+1,2}$, for $i = 1, 2, \ldots, n-3$, are red and the other edges are blue contains neither a red copy of $K_{1,2}$ nor a blue copy K_n. This observation finishes the proof. ∎

Theorem 10. *Let $n \geq 3$. Let $L = K_{2n-2} - (n-2)K_2$ and r_1, r_2 be vertices of degree $2n-3$ of L. Then $L(r_1, r_2)_1 \in \tilde{\Re}(K_{1,2}, K_n)$.*

Proof. From Remark 1 and Theorem 9. ∎

In the next theorem we indicate one more graph belonging to $\Re(K_{1,2}, K_n)$, for every $n \geq 3$. Moreover, from [7] this graph is minimal with respect to the number of vertices.

Theorem 11. *Let $F = K_{2n-1} - (n-1)K_2$, $n \geq 3$. Then $F \in \Re(K_{1,2}, K_n)$.*

Proof. From Theorem 9 we have $B(r) = (K_{2n} - (n-1)K_2)(r) \to (K_{1,2}, K_{n+1})$, where $deg(r) = 2n - 1$. It easy to see that $B - r = F$ and $F \to (K_{1,2}, K_n)$.

Let $E(\overline{F}) = \{v_{i,1}v_{i,2} : i = 1, 2, \ldots, n-1\}$ and $v \in V(B) \setminus \{r\}$, where $deg(v) = 2n - 2$. We show that $(F - e) \not\to (K_{1,2}, K_n)$. Without loss of generality we can consider only the case when $e \in \{v_{1,1}v, v_{1,1}v_{2,1}\}$. Regardless of the choice of e we color red edges $vv_{1,2}, v_{i,1}v_{i+1,2}$, for $i = 1, 2, \ldots, n-2$. We color the remaining uncolored edges blue. Clearly, such a coloring of F contains neither a red copy of $K_{1,2}$ nor a blue copy of K_n. ∎

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