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ON RAMSEY $(K_{1,2}, K_n)$ -MINIMAL GRAPHS

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Abstract

Let F be a graph and let \mathcal{G} , \mathcal{H} denote nonempty families of graphs. We write $F \to (\mathcal{G}, \mathcal{H})$ if in any 2-coloring of edges of F with red and blue, there is a red subgraph isomorphic to some graph from \mathcal{G} or a blue subgraph isomorphic to some graph from \mathcal{H} . The graph F without isolated vertices is said to be a $(\mathcal{G}, \mathcal{H})$ -minimal graph if $F \to (\mathcal{G}, \mathcal{H})$ and $F - e \not\to (\mathcal{G}, \mathcal{H})$ for every $e \in E(F)$.

We present a technique which allows to generate infinite family of $(\mathcal{G}, \mathcal{H})$ minimal graphs if we know some special graphs. In particular, we show how to receive infinite family of $(K_{1,2}, K_n)$ -minimal graphs, for every $n \geq 3$.

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1. INTRODUCTION

We consider only finite undirected graphs without loops or multiple edges. Let G be a graph with the vertex set V(G) and the edge set E(G). By $deg_G(v_1)$, $d_G(v_1, v_2)$ we denote the degree of the vertex v_1 in G and the distance between two vertices v_1, v_2 , respectively. If G is known we can shortly write $deg(v_1), d(v_1, v_2)$. We use the notation and terminology of [8].

Let F be a graph and let \mathcal{G} , \mathcal{H} be nonempty families of graphs. We write $F \to (\mathcal{G}, \mathcal{H})$ if in any 2-coloring of edges of F with red and blue, there is a red subgraph isomorphic to some graph from \mathcal{G} or a blue subgraph isomorphic to some graph from \mathcal{H} . Otherwise, if there exists a 2-coloring of edges such that neither a red subgraph isomorphic to some graph from \mathcal{G} nor a blue subgraph isomorphic to some graph from \mathcal{G} nor a blue subgraph isomorphic to some graph from \mathcal{H} occur, then we write $F \not\rightarrow (\mathcal{G}, \mathcal{H})$. The graph

F without isolated vertices is said to be a $(\mathcal{G}, \mathcal{H})$ -minimal graph if $F \to (\mathcal{G}, \mathcal{H})$ and $F - e \not\to (\mathcal{G}, \mathcal{H})$ for any $e \in E(F)$. The Ramsey set $\Re(\mathcal{G}, \mathcal{H})$ is defined to be the set of all $(\mathcal{G}, \mathcal{H})$ -minimal graphs (up to isomorphism). For the simplicity of the notation, instead of $\Re(\{G\}, \{H\})$ we write $\Re(G, H)$.

Many papers study the problem of determining the family $\Re(G, H)$. One can easily observe that the set $\Re(K_{1,2}, K_{1,2})$ is infinite and consists of star with three rays and all cycles of odd length. Burr *et al.* [6] proved that $\Re(K_{1,2k+1}, K_{1,2l+1}) =$ $\{K_{1,2(k+l)+1}\}$ and $\Re(K_{1,2k}, K_{1,2l})$ is infinite, for every $k, l \geq 1$. Next Borowiecki *et al.* [3] characterized graphs belonging to $\Re(K_{1,2}, K_{1,m})$ for $m \geq 3$.

The graphs belonging to $\Re(2K_2, K_{1,n})$ were characterized in [10]. Moreover, Luczak [9] showed that $\Re(K_{1,2m}, G)$ is finite if and only if G is a matching. It means that, for $n \geq 3$, $\Re(K_{1,2}, K_n)$ has infinite number of graphs.

Borowiecki, et al. described in [4] the whole set $\Re(K_{1,2}, K_3)$. In [1, 2] the authors presented how we can generate an infinite family of $(K_{1,2}, C_4)$ -minimal graphs. In this paper we describe a method which can be applied to the construction of infinitely many graphs belonging to $\Re(K_{1,2}, K_n)$, for any $n \geq 3$.

2. The Main Results

First we extend, in the same way as in [2], the already given standard definitions by adding some restriction on a chosen set of vertices. This allows us to construct the infinite family $\Re(K_{1,2}, \mathcal{G})$, for any given family \mathcal{G} of 2-connected graphs.

Definition 1. Let F be a graph with $U \subseteq V(F)$ and let \mathcal{G} , \mathcal{H} be families of graphs. If for any red-blue coloring of edges of F, such that all vertices in U are not incident with red edges, there exists a red copy of some graph from \mathcal{G} or a blue copy of some graph from \mathcal{H} , then we write $F(U) \to (\mathcal{G}, \mathcal{H})$. Otherwise, there exists a $(\mathcal{G}, \mathcal{H})$ -coloring of edges of F(U) and we write $F(U) \neq (\mathcal{G}, \mathcal{H})$.

Definition 2. Let F be a graph and $U \subseteq V(F)$. Let $i \in \{1, 2, ..., |U|\}$. We say that $F(U)_i$ is $(\mathcal{G}, \mathcal{H})$ -minimal if

- 1. $F(U_i) \to (\mathcal{G}, \mathcal{H})$, for every $U_i \in {\binom{U}{i}}$,
- 2. $(F-e)(U_i) \not\rightarrow (\mathcal{G}, \mathcal{H})$, for every $e \in E(F)$ and every $U_i \in {\binom{U}{i}}$,
- 3. $F(U_{i-1}) \not\rightarrow (\mathcal{G}, \mathcal{H})$, for every $U_{i-1} \in \binom{U}{i-1}$.

We write $F(U)_i \in \mathfrak{R}(\mathcal{G}, \mathcal{H})$ if $F(U)_i$ is $(\mathcal{G}, \mathcal{H})$ -minimal. If $U = \emptyset$ or i = 0, then we assume that $F(U)_i \in \mathfrak{R}(\mathcal{G}, \mathcal{H}) \Leftrightarrow F \in \mathfrak{R}(\mathcal{G}, \mathcal{H})$. For the simplicity of the notation we write $F(v_1, \ldots, v_p)_i$ instead of $F(\{v_1, \ldots, v_p\})_i$ and $F(v_1, \ldots, v_p)$ instead of $F(v_1, \ldots, v_p)_p$. ON RAMSEY $(K_{1,2}, K_n)$ -MINIMAL GRAPHS

Remark 1. $F(r_1, r_2)_1 \in \tilde{\Re}(\mathcal{G}, \mathcal{H})$ if and only if $F(r_i) \in \tilde{\Re}(\mathcal{G}, \mathcal{H})$, for i = 1, 2.

Lemma 2. Let \mathcal{G} be a family of 2-connected graphs. Let M_1, M_2 be disjoint graphs and $U_i \subset V(M_i)$, $|U_i| \in \{1, 2\}$ and $r_i \in U_i$, for i = 1, 2, and let M be a graph obtained from disjoint graphs M_1 and M_2 by identifying the vertices r_1 and r_2 . If $M_1(U_1), M_2(U_2) \in \tilde{\Re}(K_{1,2}, \mathcal{G})$, then $M(U_1 \cup U_2 \setminus \{r_1, r_2\}) \in \tilde{\Re}(K_{1,2}, \mathcal{G})$.

Proof. Let $U = U_1 \cup U_2 \setminus \{r_1, r_2\}$. First we prove that $M(U) \to (K_{1,2}, \mathcal{G})$. If we assume that $M(U) \not\to (K_{1,2}, \mathcal{G})$, then there exists a coloring of edges of M such that there is at most one red edge e incident with r_1 . It means that $M_1(U_1) \not\to (K_{1,2}, \mathcal{G})$ or $M_2(U_2) \not\to (K_{1,2}, \mathcal{G})$. Hence, we obtain a contradiction.

Now we show that $(M - e)(U) \not\rightarrow (K_{1,2}, \mathcal{G})$. Without loss of generality we can consider only the situation when $e \in E(M_1)$. We know that $(M_1 - e)(U_1) \not\rightarrow (K_{1,2}, \mathcal{G})$ and $M_2(U_2 \setminus \{r_2\}) \not\rightarrow (K_{1,2}, \mathcal{G})$. Thus, there exist a $(K_{1,2}, \mathcal{G})$ -coloring of edges of $(M_1 - e)(U_1)$, let us denote it by ϕ_1 , and a $(K_{1,2}, \mathcal{G})$ -coloring of edges of $M_2(U_2 \setminus \{r_2\})$, let us denote it by ϕ_2 . Let ϕ be a coloring of edges of (M - e)such that $\phi(f) = \phi_1(f)$ for $f \in E(M_1)$ and $\phi(f) = \phi_2(f)$ for $f \in E(M_2)$. Since $M_2(U_2 \setminus \{r_2\}) \not\rightarrow (K_{1,2}, \mathcal{G})$, it is easy to notice that the vertex r_1 is incident with exactly one red edge which belongs to $E(M_2)$. We can notice that there does not exist a blue copy of a graph $G \in \mathcal{G}$ such that $|V(G) \cap V(M_1)| > 1$ and $|V(G) \cap V(M_2)| > 1$, because \mathcal{G} contains only 2-connected graphs. Hence, ϕ is a $(K_{1,2}, \mathcal{G})$ -coloring of edges of (M - e)(U).

Finally, we prove that $M(U_i - r_i) \not\rightarrow (K_{1,2}, \mathcal{G})$ for i = 1, 2. Without loss of generality we can assume that i = 1. We know that $M_1(U_1 - \{r_1\}) \not\rightarrow (K_{1,2}, \mathcal{G})$ and $M_2(r_2) \not\rightarrow (K_{1,2}, \mathcal{G})$. Hence, there exist $(K_{1,2}, \mathcal{G})$ -colorings ϕ_1 and ϕ_2 of edges of $M_1(U_1 - \{r_1\})$ and $M_2(r_2)$, respectively. Let ϕ be a coloring of edges of Msuch that $\phi(f) = \phi_1(f)$, if $f \in E(M_1)$ and $\phi(f) = \phi_2(f)$, otherwise. It is easy to observe that the vertex r_1 is incident with exactly one red edge belonging to $E(M_1)$ in M. For the same reason as previously we can notice that there does not exist a blue copy of a graph $G \in \mathcal{G}$ such that $|V(G) \cap V(M_1)| > 1$ and $|V(G) \cap V(M_2)| > 1$. Hence, ϕ is a $(K_{1,2}, \mathcal{G})$ -coloring of edges of $M(U_1 - \{r_1\})$. This observation finishes the proof.

Lemma 3. Let $c \geq 3$ be an integer, M_1, M_2 be disjoint graphs, \mathcal{G} be a family of 2connected graphs without induced cycles of the length greater than c. Let $r_{i,1}, r_{i,2}$ be vertices of M_i , for i = 1, 2, such that $d_{M_1}(r_{1,1}, r_{1,2}) + d_{M_2}(r_{2,1}, r_{2,2}) > c$, and let L be a graph obtained from graphs M_1 and M_2 by identifying the vertices $r_{1,1}$ and $r_{2,1}$, and the vertices $r_{1,2}$ and $r_{2,2}$. If $M_i(r_{i,1}, r_{i,2}) \in \tilde{\Re}(K_{1,2}, \mathcal{G})$, for i = 1, 2, then $L(r_{1,1}, r_{1,2})_1 \in \tilde{\Re}(K_{1,2}, \mathcal{G})$.

Proof. First we prove that $L(r_{1,1}, r_{1,2})_1 \to (K_{1,2}, \mathcal{G})$. Conversely, suppose that $L(r_{1,1}, r_{1,2})_1 \not\to (K_{1,2}, \mathcal{G})$. Without loss of generality we can assume that $L(r_{1,1}) \not\to (K_{1,2}, \mathcal{G})$. Then there exists a $(K_{1,2}, \mathcal{G})$ -coloring of edges of L such that every edge

e incident with $r_{1,1}$ in *L* is blue and at most one edge incident with $r_{2,2}$ is red. Without loss of generality we can assume that the red edge belongs to $E(M_1)$. Hence, we obtain a contradiction with the fact that $M_2(r_{2,1}, r_{2,2}) \to (K_{1,2}, \mathcal{G})$.

Now we show that $(L-e)(r_{1,1},r_{1,2})_1 \not\rightarrow (K_{1,2},\mathcal{G})$. Without loss of generality we can assume that $e \in E(M_1)$. We know that $(M_1 - e)(r_{1,1},r_{1,2}) \not\rightarrow (K_{1,2},\mathcal{G})$ and $M_2(r_{2,1}) \not\rightarrow (K_{1,2},\mathcal{G})$. Thus, there exist a $(K_{1,2},\mathcal{G})$ -coloring of edges of $(M_1 - e)(r_{1,1}, r_{1,2})$ and a $(K_{1,2},\mathcal{G})$ -coloring of edges of $M_2(r_{2,1})$. We denote these colorings by ϕ_1 and ϕ_2 , respectively. Let ϕ be a coloring of edges (L-e) such that $\phi(f) = \phi_1(f)$, if $f \in E(M_1)$ and $\phi(f) = \phi_2(f)$, otherwise. It is easy to notice that the vertex $r_{1,2}$ is incident with exactly one red edge belonging to $E(M_2)$. Since \mathcal{G} contains only 2-connected graphs and $d_{M_1}(r_{1,1}, r_{1,2}) + d_{M_2}(r_{2,1}, r_{2,2}) > c$, there does not exist a blue copy of a graph $G \in \mathcal{G}$ such that $|V(G) \cap V(M_i - r_{i,1} - r_{i,2})| >$ 0, for i = 1, 2. Hence, ϕ is a $(K_{1,2}, \mathcal{G})$ -coloring of edges of $(L - e)(r_{1,1}, r_{1,2})$.

Finally, we prove that $L \not\rightarrow (K_{1,2}, \mathcal{G})$. From our assumption, it follows that $M_1(r_{1,1}) \not\rightarrow (K_{1,2}, \mathcal{G})$ and $M_2(r_{2,2}) \not\rightarrow (K_{1,2}, \mathcal{G})$. Thus once again, we can indicate two colorings ϕ_1 and ϕ_2 such that ϕ_i is a $(K_{1,2}, \mathcal{G})$ -coloring of edges of $M_i(r_{i,i})$, for i = 1, 2. Let ϕ be a coloring of edges of L such that $\phi(f) = \phi_i(f)$ for $f \in E(M_i)$ and i = 1, 2. We can observe that the vertex $r_{1,1}$ is incident with exactly one red edge belonging to $E(M_2)$ and the vertex $r_{1,2}$ is incident with exactly one red edge belonging to $E(M_1)$. We can notice that there does not exist a blue copy of a graph $G \in \mathcal{G}$ such that $|V(G) \cap V(M_i - r_{i,1} - r_{i,2})| > 0$ for i = 1, 2, because \mathcal{G} contains only 2-connected graphs and $d_{M_1}(r_{1,1}, r_{1,2}) + d_{M_2}(r_{2,1}, r_{2,2}) > c$. Therefore ϕ is a $(K_{1,2}, \mathcal{G})$ -coloring edges of L.

Corollary 4. Let $c \geq 3$ be an integer, M_1, M_2 be disjoint graphs, \mathcal{G} be a family of 2-connected graphs without induced cycles of the length greater than c. Let $r_{i,1}, r_{i,2}$ be vertices of M_i , for i = 1, 2, such that $d_{M_1}(r_{1,1}, r_{1,2}) + d_{M_2}(r_{2,1}, r_{2,2}) > c$, and let B be a graph obtained from graphs M_1 and M_2 by identifying the vertices $r_{1,1}$ and $r_{2,1}$, and the vertices $r_{1,2}$ and $r_{2,2}$. If $M_i(r_{i,1}, r_{i,2}) \in \tilde{\Re}(K_{1,2}, \mathcal{G})$, for i = 1, 2, then $B(r_{1,1}) \in \tilde{\Re}(K_{1,2}, \mathcal{G})$.

Proof. From Lemma 3 and Remark 1.

The next theorems give us a method of the construction of infinitely many graphs that belong to $\Re(K_{1,2}, \mathcal{G})$, where \mathcal{G} is any given family of graphs. In this construction we use graphs with adding some restriction on a chosen set of vertices, i.e., graphs that belong to the family $\Re(K_{1,2}, \mathcal{G})$.

Theorem 5. Let $c \geq 3$ be an integer, L, M be disjoint graphs, \mathcal{G} be a family of 2-connected graphs without induced cycles of the length greater than c. Let $\{r_{1,1}, r_{1,2}\} \subset V(L)$ and $\{r_{2,1}, r_{2,2}\} \subset V(M)$ such that $d_L(r_{1,1}, r_{1,2}) + d_M(r_{2,1}, r_{2,2}) > c$, and let F be a graph obtained from graphs L and M by identifying the vertices $r_{1,1}$

and $r_{2,1}$ and the vertices $r_{1,2}$, and $r_{2,2}$. If $L(r_{1,1}, r_{1,2}), M(r_{2,1}, r_{2,2}) \in \mathfrak{H}(K_{1,2}, \mathcal{G})$, then $F \in \mathfrak{H}(K_{1,2}, \mathcal{G})$.

Proof. We start with proving that $F \to (K_{1,2}, \mathcal{G})$. Suppose, on the contrary, that there exists a $(K_{1,2}, \mathcal{G})$ -coloring of edges of F. From the fact that $L(r_{1,1}, r_{1,2})_1 \to (K_{1,2}, \mathcal{G})$ it follows that in this coloring one edge incident with $r_{1,1}$ and one edge incident with $r_{1,2}$ in L is red. Hence, every edge incident with $r_{2,1}$ and $r_{2,2}$ in M is blue. We obtain a contradiction with the assumption that $M(r_{2,1}, r_{2,2}) \to (K_{1,2}, \mathcal{G})$.

It remains to prove that $F - e \not\rightarrow (K_{1,2}, \mathcal{G})$, for every $e \in E(F)$.

Case 1. Let $e \in E(L)$. We know that $(L - e)(r_{1,i}) \not\rightarrow (K_{1,2}, \mathcal{G})$ and $M(r_{2,3-i}) \not\rightarrow (K_{1,2}, \mathcal{G})$, for i = 1, 2. Without loss of generality we can assume that i = 1. Thus, there exists a $(K_{1,2}, \mathcal{G})$ -coloring of edges of $(L - e)(r_{1,1})$ and a $(K_{1,2}, \mathcal{G})$ -coloring of edges of $M(r_{2,2})$. Let us denote these colorings by ϕ_1 and ϕ_2 , respectively. Let ϕ be a coloring of edges of (F - e) such that $\phi(f) = \phi_1(f)$, if $f \in E(L)$ and $\phi(f) = \phi_2(f)$, otherwise. Let us notice that the vertices $r_{1,1}$ and $r_{1,2}$ must be incident with at most one red edge in the graph F - e. We also know that there does not exist a blue copy of a graph $G \in \mathcal{G}$ such that $|V(G) \cap V(L - r_{1,1} - r_{1,2})| > 0$ and $|V(G) \cap V(M - r_{2,1} - r_{2,2})| > 0$. This observation follows from the fact that \mathcal{G} contains only 2-connected graphs and $d_L(r_{1,1}, r_{1,2}) + d_M(r_{2,1}, r_{2,2}) > c$. Hence, ϕ is a $(K_{1,2}, \mathcal{G})$ -coloring of edges of F - e.

Case 2. Let $e \in E(M)$. From the fact that $L \not\rightarrow (K_{1,2}, \mathcal{G})$ and $(M - e)(r_{2,1}, r_{2,2}) \not\rightarrow (K_{1,2}, \mathcal{G})$ it follows that there exist a $(K_{1,2}, \mathcal{G})$ -coloring ϕ_1 of edges of L and a $(K_{1,2}, \mathcal{G})$ -coloring ϕ_2 of edges of $(M - e)(r_{2,1}, r_{2,2})$. Let ϕ be a coloring of edges of (F - e) such that $\phi(f) = \phi_1(f)$, if $f \in E(L)$ and $\phi(f) = \phi_2(f)$, otherwise. Since $L(r_{1,i}) \not\rightarrow (K_{1,2}, \mathcal{G})$, for i = 1, 2 and $L \not\rightarrow (K_{1,2}, \mathcal{G})$, the vertices $r_{1,1}$ and $r_{1,2}$ are incident with exactly one red edge in F - e. For the same reason as in Case 1 we know that there does not exist a blue copy of a graph $G \in \mathcal{G}$ such that $|V(G) \cap V(L - r_{1,1} - r_{1,2})| > 0$ and $|V(G) \cap V(M - r_{2,1} - r_{2,2})| > 0$. Hence, we can conclude that ϕ is a $(K_{1,2}, \mathcal{G})$ -coloring of edges of F - e.

Corollary 6. Let B_1, B_2 be disjoint graphs, \mathcal{G} be a family of 2-connected graphs. Let r_1, r_2 be vertices of B_1 and B_2 , respectively, and let F be a graph obtained from graphs B_1 and B_2 by identifying the vertices r_1 and r_2 . If $B_1(r_1), B_2(r_2) \in \widetilde{\Re}(K_{1,2}, \mathcal{G})$, then $F \in \Re(K_{1,2}, \mathcal{G})$.

Proof. From Lemma 2.

Theorem 7. Let $c \geq 3$ be an integer, L be a graph, \mathcal{G} be a family of 2-connected graphs without induced cycles of the length greater than c. Let r_1, r_2 be vertices of L such that $d_L(r_1, r_2) > c$, and let F be a graph obtained from the graph Lby identifying the vertices r_1 and r_2 . If $L(r_{1,1}, r_{1,2})_1 \in \tilde{\Re}(K_{1,2}, \mathcal{G})$, then $F \in$ $\Re(K_{1,2}, \mathcal{G})$.

Proof. First we show that $F \to (K_{1,2}, \mathcal{G})$. Suppose, on the contrary, that there exists a $(K_{1,2}, \mathcal{G})$ -coloring of edges of F such that there is at most one red edge incident with r_1 . Since $d_L(r_1, r_2) > c$, it follows that $r_1r_2 \notin E(L)$, so $L(r_1) \nleftrightarrow (K_{1,2}, \mathcal{G})$ or $L(r_2) \nleftrightarrow (K_{1,2}, \mathcal{G})$, what leads us to a contradiction.

To finish the proof we show that $F - e \nleftrightarrow (K_{1,2}, \mathcal{G})$, for $e \in E(F)$. We know that $(L - e)(r_1) \nleftrightarrow (K_{1,2}, \mathcal{G})$. Hence, there exists a $(K_{1,2}, \mathcal{G})$ -coloring of edges of $(L - e)(r_1)$. It is easy to notice that the vertex r_2 is incident with at most one red edge in the graph F - e. Since \mathcal{G} contains only 2-connected graphs and $d_L(r_1, r_2) > c$, there does not exist a blue copy of a graph $G \in \mathcal{G}$. Hence, ϕ is a $(K_{1,2}, \mathcal{G})$ -coloring of edges of F - e.

3. The Families $\tilde{\Re}(K_{1,2}, K_n)$ and $\Re(K_{1,2}, K_n)$

On the basis of results of Borowiecki *et al.* [4] we can observe the following facts:

Observation 1.

- (i) $K_3(r_1, r_2) \in \Re(K_{1,2}, K_3).$
- (ii) Let r be a vertex of degree 3 of $K_4 e$. Then $(K_4 e)(r) \in \Re(K_{1,2}, K_3)$.
- (iii) Let $TC_n = K_3$ -cycle, which we obtain from $n \ge 4$ copies of K_3 by identifying the second vertex of the *i*-th copy of K_3 with the first vertex of the ((*i* mod n)+1)-th copy of K_3 , for i = 1, 2, ..., n. Then $TC_n(r) \in \mathfrak{R}(K_{1,2}, K_3)$, where $r \in V(TC_n)$.
- (iv) Let r_1, r_2 be vertices of degree 3 of $K_4 e$. Then $(K_4 e)(r_1, r_2)_1 \in \Re(K_{1,2}, K_3)$.
- (v) Graphs $L_i(r_1, r_2)_1$, for i = 1, ..., 6, in Figure 1 belong to $\Re(K_{1,2}, K_3)$.



Figure 1. All presented graphs $L_i(r_1, r_2)_1$ belong to $\Re(K_{1,2}, K_3)$.

In the next three theorems we indicate some special graphs. These graphs together with our previous results allow us to construct infinitely many $(K_{1,2}, K_n)$ minimal graphs, i.e. graphs that belong to the Ramsey set $\Re(K_{1,2}, K_n)$ for every $n \geq 3$.

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Theorem 8. Let $n \ge 3$. Let $M = K_{2n-3} - (n-3)K_2$ and r_1, r_2 be vertices of degree 2n - 4 of M. Then $M(r_1, r_2) \in \mathfrak{R}(K_{1,2}, K_n)$.

Proof. Note that for n = 3 the graph $M(r_1, r_2) = K_3(r_1, r_2) \in \Re(K_{1,2}, K_n)$ from Observation 1(i). Hence, we can consider only $n \ge 4$.

In the first step of the proof we show that $M(r_1, r_2) \to (K_{1,2}, K_n)$. Provided that the vertices r_1 and r_2 are not incident with red edges, we consider every red-blue coloring ϕ of edges of M, such that there is no red copy of the graph $K_{1,2}$. Let $E_1 = E(\overline{M})$ and $E_2 = \{e \in E(M) : \phi(e) = red\}$. We can notice that the graph $H = (V(M) \setminus \{r_1, r_2\}, E_1 \cup E_2)$ is bipartite and $\Delta(H) \leq 2$. Hence, we can divide the set V(H) into V_1 and V_2 such that $H[V_1]$ and $H[V_2]$ are edgeless. Without loss of generality we can assume that $|V_1| > |V_2|$. This implies that $|V_1| \geq n-2$. One can see that the subgraph of M induced by $V_1 \cup \{r_1, r_2\}$ contains only blue edges and is isomorphic to K_n .

Now we show that $(M - e)(r_1, r_2) \not\rightarrow (K_{1,2}, K_n)$. Let $E(\overline{M}) = \{v_{i,1}v_{i,2} : i = 1, 2, \ldots, n-3\}$ and $v \in V(M) \setminus \{r_1, r_2\}$, where deg(v) = 2n - 4. Without loss of generality we can consider only the case when $e \in \{v_{1,1}r_1, v_{1,1}v, v_{1,1}v_{2,1}\}$. If $n \ge 5$, then for any choice of e we color red edges $vv_{1,2}$, $v_{i,1}v_{i+1,2}$, for $i = 1, 2, \ldots, n-4$. If n = 4, then we color red edges $vv_{1,2}$ and $v_{1,1}r_1$. We color the remaining edges blue. These colorings of $(M - e)(r_1, r_2)$ contain neither a red copy of $K_{1,2}$ nor a blue copy of K_n .

To finish the proof we show that $M(r_1) \not\rightarrow (K_{1,2}, K_n)$. Let us consider the following coloring of edges of M. If $n \geq 5$, then we color red edges $r_2v_{n-3,1}, vv_{1,2}, v_{i,1}v_{i+1,2}$, for $i = 1, 2, \ldots, n-4$. If n = 4, then we color red edges $r_2v_{1,1}$ and $vv_{1,2}$. The remaining edges we color blue. One can see that this coloring of M contains neither a red copy of $K_{1,2}$ nor a blue copy of K_n . Similarly, we can prove that $M(r_2) \not\rightarrow (K_{1,2}, K_n)$.

Theorem 9. Let $n \ge 3$. Let $B = K_{2n-2} - (n-2)K_2$ and r be a vertex of degree 2n-3 of B. Then $B(r) \in \tilde{\Re}(K_{1,2}, K_n)$.

Proof. Notice that for n = 3 the graph $B(r) = (K_4 - e)(r) \in \Re(K_{1,2}, K_3)$ from Observation 1(ii). Hence, we can consider only $n \ge 4$.

First we prove that $B(r) \to (K_{1,2}, K_n)$. Consider a red-blue coloring ϕ of edges of B. Suppose that in this coloring there is no red copy of $K_{1,2}$. Let $E_1 = E(\overline{B})$ and $E_2 = \{e \in E(B) : \phi(e) = red\}$. If we consider the graph $H = (V(B) \setminus \{r\}, E_1 \cup E_2)$, then we can notice that H is bipartite and $\Delta(H) \leq 2$. Therefore we can divide the set V(H) into V_1 and V_2 such that $H[V_1]$ and $H[V_2]$ are edgeless. Without loss of generality we can assume that $|V_1| > |V_2|$. Hence $|V_1| \geq n - 1$. Now, we can notice that the subgraph of B induced by $V_1 \cup \{r\}$ contains only blue edges and is isomorphic to K_n .

Let $E(\overline{B}) = \{v_{i,1}v_{i,2} : i = 1, 2, ..., n-2\}$ and $v \in V(B) \setminus \{r\}$, where deg(v) = 2n-3. In the next step of the proof we show that $(B-e)(r) \not\rightarrow (K_{1,2}, K_n)$. Without

loss of generality we can consider only the case when $e \in \{v_{1,1}r, v_{1,1}v, v_{1,1}v_{2,1}\}$. Regardless of the choice of e we color red edges $vv_{1,2}, v_{i,1}v_{i+1,2}$, for $i = 1, 2, \ldots, n-3$. The remaining uncolored edges we color blue. Clearly, such a coloring of (B-e)(r) contains neither a red copy of $K_{1,2}$ nor a blue copy of K_n .

Finally, we show that $B \not\rightarrow (K_{1,2}, K_n)$. One can see that a coloring of B such that edges $rv_{n-2,1}, vv_{1,2}, v_{i,1}v_{i+1,2}$, for $i = 1, 2, \ldots, n-3$, are red and the other edges are blue contains neither a red copy of $K_{1,2}$ nor a blue copy K_n . This observation finishes the proof.

Theorem 10. Let $n \ge 3$. Let $L = K_{2n-2} - (n-2)K_2$ and r_1, r_2 be vertices of degree 2n-3 of L. Then $L(r_1, r_2)_1 \in \tilde{\Re}(K_{1,2}, K_n)$.

Proof. From Remark 1 and Theorem 9.

In the next theorem we indicate one more graph belonging to $\Re(K_{1,2}, K_n)$, for every $n \geq 3$. Moreover, from [7] this graph is minimal with respect to the number of vertices.

Theorem 11. Let $F = K_{2n-1} - (n-1)K_2$, $n \ge 3$. Then $F \in \Re(K_{1,2}, K_n)$.

Proof. From Theorem 9 we have $B(r) = (K_{2n} - (n-1)K_2)(r) \to (K_{1,2}, K_{n+1})$, where deg(r) = 2n - 1. It easy to see that B - r = F and $F \to (K_{1,2}, K_n)$.

Let $E(F) = \{v_{i,1}v_{i,2} : i = 1, 2, ..., n-1\}$ and $v \in V(B) \setminus \{r\}$, where deg(v) = 2n-2. We show that $(F-e) \not\rightarrow (K_{1,2}, K_n)$. Without loss of generality we can consider only the case when $e \in \{v_{1,1}v, v_{1,1}v_{2,1}\}$. Regardless of the choice of e we color red edges $vv_{1,2}, v_{i,1}v_{i+1,2}$, for i = 1, 2, ..., n-2. We color the remaining uncolored edges blue. Clearly, such a coloring of F contains neither a red copy of $K_{1,2}$ nor a blue copy of K_n .

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