

## TREES WITH EQUAL 2-DOMINATION AND 2-INDEPENDENCE NUMBERS

MUSTAPHA CHELLALI<sup>1</sup> AND NACÉRA MEDDAH

LAMDA-RO Laboratory, Department of Mathematics  
University of Blida  
B.P. 270, Blida, Algeria

**e-mail:** m.chellali@yahoo.com  
meddahnacera@yahoo.fr

### Abstract

Let  $G = (V, E)$  be a graph. A subset  $S$  of  $V$  is a 2-dominating set if every vertex of  $V - S$  is dominated at least 2 times, and  $S$  is a 2-independent set of  $G$  if every vertex of  $S$  has at most one neighbor in  $S$ . The minimum cardinality of a 2-dominating set of  $G$  is the 2-domination number  $\gamma_2(G)$  and the maximum cardinality of a 2-independent set of  $G$  is the 2-independence number  $\beta_2(G)$ . Fink and Jacobson proved that  $\gamma_2(G) \leq \beta_2(G)$  for every graph  $G$ . In this paper we provide a constructive characterization of trees with equal 2-domination and 2-independence numbers.

**Keywords:** 2-domination number, 2-independence number, trees.

**2010 Mathematics Subject Classification:** 05C69.

### 1. INTRODUCTION

Let  $G = (V(G), E(G))$  be a simple graph with vertex set  $V(G)$  and edge set  $E(G)$ . The *open neighborhood*  $N(v)$  of a vertex  $v$  consists of the vertices adjacent to  $v$ , the *closed neighborhood* of  $v$  is defined by  $N[v] = N(v) \cup \{v\}$  and  $d_G(v) = |N(v)|$  is the *degree* of  $v$ . A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. If  $u$  is a support vertex, then  $L_u$  will denote the set of leaves attached at  $u$ . We denote by  $K_{1,t}$  a *star* of order  $t + 1$ . A tree  $T$  is a *double star* if it contains exactly two vertices that are not leaves. A double star with, respectively  $p$  and  $q$  leaves attached at each support vertex is denoted by  $S_{p,q}$ . A

---

<sup>1</sup>This research was supported by "Programmes Nationaux de Recherche: Code 8/u09/510".

graph is  $G$  called a *corona* if it is constructed from a graph of  $H$  by adding for each vertex  $v \in V(H)$ , a new vertex  $v'$  and a pendant edge  $vv'$ .

In [4], Fink and Jacobson generalized the concepts of independent and dominating sets. Let  $k$  be a positive integer, a subset  $S$  of  $V(G)$  is  *$k$ -independent* if the maximum degree of the subgraph induced by the vertices of  $S$  is less or equal to  $k - 1$ . The subset  $S$  is  *$k$ -dominating* if every vertex of  $V(G) - S$  has at least  $k$  neighbors in  $S$ . The  *$k$ -domination number*  $\gamma_k(G)$  is the minimum cardinality of a  $k$ -dominating set and the  *$k$ -independence number*  $\beta_k(G)$  is the maximum cardinality of a  $k$ -independent set. A minimum  $k$ -dominating set and a maximum  $k$ -independent set of a graph  $G$  is called a  $\gamma_k(G)$ -set and  $\beta_k(G)$ -set, respectively. Thus for  $k = 1$ , the 1-independent and 1-dominating sets are the classical independent and dominating sets. A survey on  $k$ -domination and  $k$ -independence in graphs has been given by Chellali, Favaron, Hansberg and Volkmann and can be found in [2]. Also for more details on domination and its variations see the books of Haynes, Hedetniemi, and Slater [5, 6].

It is well known that every graph  $G$  satisfies  $\gamma_1(G) \leq \beta_1(G)$ . In [4], Fink and Jacobson proved that  $\gamma_2(G) \leq \beta_2(G)$  and conjectured that for every graph  $G$  and positive integer  $k$ ,  $\gamma_k(G) \leq \beta_k(G)$ . The conjecture has been proved by Favaron [3] by showing that every graph  $G$  admits a set that is both a  $k$ -independent and a  $k$ -dominating. It follows from this stronger result that if  $G$  is a graph such that  $\beta_k(G) = \gamma_k(G)$ , then  $G$  has a set that is both  $\gamma_k(G)$ -set and  $\beta_k(G)$ -set. This useful property will be used in the proof of the main result. Note that trees  $T$  with  $\gamma_1(T) = \beta_1(T)$  have been characterized in [1] by Borowiecki who proved that such trees must be either  $K_1$  or coronas.

In this paper, we give a characterization of all trees  $T$  with equal 2-domination and 2-independence numbers. We will call such trees  $(\gamma_2, \beta_2)$ -trees. Note that the difference  $\beta_2(G) - \gamma_2(G)$  can be arbitrarily large even for trees. To see this consider a tree  $T_j$  obtained from a path of order  $2j + 1$  where the vertices are labelled from 1 to  $2j + 1$  by attaching a path  $P_2$  to each of the odd numbered vertices. Then  $\beta_2(T_j) = 3j + 2$  and  $\gamma_2(T_j) = 2j + 2$ .

## 2. $(\gamma_2, \beta_2)$ -TREES

### 2.1. Observations

We give some useful observations.

**Observation 1.** *Every 2-dominating set of a graph  $G$  contains every leaf.*

**Observation 2.** *Let  $T$  be a non-trivial tree and  $w \in V(T)$ . Then  $\gamma_2(T) \leq \gamma_2(T - w) + 1$ .*

**Proof.** If  $D$  is a  $\gamma_2(T - w)$ -set, then  $D \cup \{w\}$  is a 2-dominating set of  $T$  and hence  $\gamma_2(T) \leq |D| + 1$ . ■

**Observation 3.** Let  $T$  be a non-trivial tree and  $v$  a vertex of  $T$ . Then  $\beta_2(T - v) \leq \beta_2(T) \leq \beta_2(T - v) + 1$ .

**Proof.**  $\beta_2(T - v) \leq \beta_2(T)$  follows from the fact that any 2-independent set of  $T - v$  is also a 2-independent set of  $T$ . Now if  $D$  is  $\beta_2(T)$ -set, then  $D - \{v\}$  is a 2-independent set of  $T - v$  and hence  $\beta_2(T - v) \geq |D| - 1$ . ■

**Observation 4.** Let  $T$  be a tree obtained from a nontrivial tree  $T'$  and a star  $K_{1,p}$  of center vertex  $v$  by adding an edge  $vw$  at any vertex  $w$  of  $T'$ . Then,

- (1)  $\gamma_2(T') \leq \gamma_2(T) - p$ , with equality if either  $p \geq 2$  or  $w$  is a leaf of  $T'$ .
- (2) If  $p \geq 2$ , then  $\beta_2(T) = \beta_2(T') + p$ .

**Proof.** (1) Let  $D$  be a  $\gamma_2(T)$ -set. Then by Observation 1,  $L_v \subset D$  and, without loss of generality,  $v \notin D$  (else substitute  $v$  by  $w$  in  $D$ ). Then  $D \cap V(T')$  2-dominates  $T'$  and so  $\gamma_2(T') \leq |D \cap V(T')| = \gamma_2(T) - p$ . Now if  $p \geq 2$ , then every  $\gamma_2(T')$ -set can be extended to a 2-dominating set of  $T$  by adding the  $p$  leaves of the added star, and hence  $\gamma_2(T) \leq \gamma_2(T') + p$ . Assume now that  $p = 1$  and let  $v'$  be the unique leaf adjacent to  $v$ . If  $w$  is a leaf in  $T'$ , then  $w$  belongs to every  $\gamma_2(T')$ -set  $D'$  and  $D' \cup \{v'\}$  is a 2-dominating set of  $T'$ , implying that  $\gamma_2(T) \leq \gamma_2(T') + 1$ . In both cases the equality is obtained.

(2) Let  $S'$  be any  $\beta_2(T')$ -set. Then clearly  $S' \cup L_v$  is a 2-independent set of  $T$ , and so  $\beta_2(T) \geq \beta_2(T') + |L_v|$ . Now among all  $\beta_2(T)$ -sets, let  $S$  be one containing the maximum number of leaves. If there exists a leaf  $v' \in L_v$  such that  $v' \notin S$ , then  $v \in S$  (else  $S \cup \{v'\}$  is a 2-independent set larger than  $S$ ) but then  $\{v'\} \cup S - \{v\}$  is a 2-independent set of  $T$  containing more leaves than  $S$ , a contradiction. Hence  $L_v \subset S$  and so  $S - L_v$  is a 2-independent set of  $T'$ . It follows that  $\beta_2(T') \geq \beta_2(T) - |L_v|$  and the equality holds. ■

**Observation 5.** Let  $T$  be a tree obtained from a nontrivial tree  $T'$  and a double star  $S_{1,p}$  with support vertices  $u$  and  $v$ , where  $|L_v| = p$  by adding an edge  $vw$  at a vertex  $w$  of  $T'$ . Then,

- (1)  $\beta_2(T) = \beta_2(T') + (p + 2)$ .
- (2)  $\gamma_2(T) \leq \gamma_2(T') + (p + 2)$ , with equality if  $\beta_2(T) = \gamma_2(T)$ .

**Proof.** (1) Let  $u'$  be the unique leaf neighbor of  $u$  and let  $S$  a  $\beta_2(T)$ -set containing the maximum number of leaves. Then as seen in the proof of Observation 4,  $L_v \cup \{u'\} \subset S$ . Also  $S$  contains either  $u$  or  $v$  for otherwise  $S \cup \{u\}$  is a 2-independent set of  $T$  larger than  $S$ . Without loss of generality,  $u \in S$  and so  $S - (L_v \cup \{u, u'\})$  is a 2-independent set of  $T'$ . Hence  $\beta_2(T') \geq \beta_2(T) - (|L_v| + 2)$ .

The equality is obtained from the fact that every  $\beta_2(T')$ -set can be extended to a 2-independent set of  $T$  by adding  $L_v \cup \{u, u'\}$ .

(2) Clearly if  $D'$  is a  $\gamma_2(T')$ -set, then  $D' \cup L_v \cup \{u', v\}$  is a 2-dominating set of  $T$  and so  $\gamma_2(T) \leq \gamma_2(T') + (p + 2)$ . Now assume that  $\beta_2(T) = \gamma_2(T)$  and suppose that  $\gamma_2(T) < \gamma_2(T') + (p + 2)$ . Then by item (1) we have

$$\beta_2(T') + (p + 2) = \beta_2(T) = \gamma_2(T) < \gamma_2(T') + (p + 2),$$

implying that  $\beta_2(T') < \gamma_2(T')$ , a contradiction. Therefore if  $\beta_2(T) = \gamma_2(T)$ , then  $\gamma_2(T) = \gamma_2(T') + (p + 2)$ . ■

**Observation 6.** Let  $T$  be a tree obtained from a nontrivial tree  $T'$  and a path  $P_3 = xyz$  by adding an edge  $xw$  at a vertex  $w$  of  $T'$ . Then

- (1)  $\beta_2(T) = \beta_2(T') + 2$ .
- (2)  $\gamma_2(T) \leq \gamma_2(T') + 2$ , with equality if  $\beta_2(T) = \gamma_2(T)$ .

**Proof.** (1) If  $D'$  is a  $\beta_2(T')$ -set, then  $D' \cup \{y, z\}$  is a 2-independent set of  $T$  and so  $\beta_2(T) \geq \beta_2(T') + 2$ . Now let  $D$  be a  $\beta_2(T)$ -set. Clearly  $1 \leq |D \cap \{x, y, z\}| \leq 2$ . If  $|D \cap \{x, y, z\}| = 1$ , then, without loss of generality,  $z \in D$  but  $D \cup \{y\}$  is a larger 2-independent set of  $T$ , a contradiction. Thus  $|D \cap \{x, y, z\}| = 2$ . Also  $D \cap V(T')$  is a 2-independent set of  $T'$ , implying that  $\beta_2(T') \geq \beta_2(T) - 2$ . Hence  $\beta_2(T) = \beta_2(T') + 2$ .

(2) If  $S'$  is a  $\gamma_2(T')$ -set, then  $S' \cup \{z, x\}$  is a 2-dominating set of  $T$ , and so  $\gamma_2(T) \leq \gamma_2(T') + 2$ . Assume now that  $T$  satisfies  $\beta_2(T) = \gamma_2(T)$ . If  $\gamma_2(T) < \gamma_2(T') + 2$ , then by item (1) we have

$$\beta_2(T') + 2 = \beta_2(T) = \gamma_2(T) < \gamma_2(T') + 2,$$

implying that  $\beta_2(T') < \gamma_2(T')$ , a contradiction. Therefore if  $\beta_2(T) = \gamma_2(T)$ , then  $\gamma_2(T) = \gamma_2(T') + 2$ . ■

## 2.2. Main result

For the purpose of characterizing  $(\gamma_2, \beta_2)$ -trees, we define the family  $\mathcal{O}$  of all trees  $T$  that can be obtained from a sequence  $T_1, T_2, \dots, T_k$  ( $k \geq 1$ ) of trees, where  $T_1$  is a star  $K_{1,p}$  ( $p \geq 1$ ),  $T = T_k$ , and, if  $k \geq 2$ ,  $T_{i+1}$  is obtained recursively from  $T_i$  by one of the operations defined below.

- **Operation  $\mathcal{O}_1$**  : Add a star  $K_{1,p}$ ,  $p \geq 2$ , centered at a vertex  $u$  and join  $u$  by an edge to a vertex of  $T_i$ .
- **Operation  $\mathcal{O}_2$**  : Add a double star  $S_{1,p}$  with support vertices  $u$  and  $v$ , where  $|L_v| = p$  and join  $v$  by an edge to a vertex  $w$  of  $T_i$  with the condition that if  $\gamma_2(T_i - w) = \gamma_2(T_i) - 1$ , then no neighbor of  $w$  in  $T_i$  belongs to a  $\gamma_2(T_i - w)$ -set.

- **Operation  $\mathcal{O}_3$**  : Add a path  $P_2 = u'u$  and join  $u$  by an edge to a leaf  $v$  of  $T_i$  that belongs to every  $\beta_2(T_i)$ -set and satisfies in addition  $\beta_2(T_i - v) + 1 = \beta_2(T_i)$ .
- **Operation  $\mathcal{O}_4$** : Add a path  $P_3 = u'uv$  and join  $v$  by an edge to a vertex  $w$  that belongs to a  $\gamma_2(T_i)$ -set and satisfies further  $\gamma_2(T_i - w) \leq \gamma_2(T_i)$ , with the condition that if  $\gamma_2(T_i - w) = \gamma_2(T_i) - 1$ , then no neighbor of  $w$  in  $T_i$  belongs to a  $\gamma_2(T_i - w)$ -set.

We state the following lemma.

**Lemma 7.** *If  $T \in \mathcal{O}$  then,  $\gamma_2(T) = \beta_2(T)$ .*

**Proof.** Let  $T$  be a tree of  $\mathcal{O}$ . Then  $T$  is obtained from a sequence  $T_1, T_2, \dots, T_k$  ( $k \geq 1$ ) of trees, where  $T_1$  is a star  $K_{1,p}$  ( $p \geq 1$ ),  $T = T_k$ , and, if  $k \geq 2$ ,  $T_{k+1}$  is obtained recursively from  $T_k$  by one of the four operations defined above. We use an induction on the number of operations performed to construct  $T$ . Clearly the property is true if  $k = 1$ . This establishes the basis case.

Assume now that  $k \geq 2$  and that the result holds for all trees  $T \in \mathcal{O}$  that can be constructed from a sequence of length at most  $k - 1$ , and let  $T' = T_{k-1}$ . By the inductive hypothesis,  $T'$  is a  $(\gamma_2, \beta_2)$ -tree. Let  $T$  be a tree obtained from  $T'$  by using one of the operations  $\mathcal{O}_1$ ,  $\mathcal{O}_2$ ,  $\mathcal{O}_3$  and  $\mathcal{O}_4$ . We examine each of the following cases. Note that we will use in the proof the same notation as used for the construction.

*Case 1.*  $T$  is obtained from  $T'$  by using operation  $\mathcal{O}_1$ . By Observation 4,  $\gamma_2(T) = \gamma_2(T') + p$  and  $\beta_2(T) = \beta_2(T') + p$ . Since  $T'$  is a  $(\gamma_2, \beta_2)$ -tree it follows that  $\gamma_2(T) = \beta_2(T)$ .

*Case 2.*  $T$  is obtained from  $T'$  by using operation  $\mathcal{O}_2$ . By Observation 5,  $\beta_2(T) = \beta_2(T') + (p + 2)$  and  $\gamma_2(T) \leq \gamma_2(T') + (p + 2)$ . Now assume that  $\gamma_2(T) < \gamma_2(T') + (p + 2)$  and let  $D$  be a  $\gamma_2(T)$ -set. Then, without loss of generality,  $D$  contains  $L_v \cup \{v\}$  and the unique leaf neighbor of  $u$ . If  $w \in D$ , then  $D \cap V(T')$  is a 2-dominating set of  $T'$  with cardinality  $\gamma_2(T) - (p + 2) < \gamma_2(T')$ , which is impossible. Hence  $w \notin D$  and so  $D' = D \cap V(T')$  is a 2-dominating set of  $T' - w$ . Note that since  $w \notin D$  and  $v \in D$ ,  $D'$  contains a neighbor of  $w$  in  $T'$ . Hence  $\gamma_2(T' - w) \leq |D'| = \gamma_2(T) - (p + 2) < \gamma_2(T')$ . It follows from Observation 2 that  $\gamma_2(T' - w) = \gamma_2(T') - 1$  and  $D'$  is a  $\gamma_2(T' - w)$ -set containing a neighbor of  $w$ , a contradiction with the construction. Therefore  $\gamma_2(T) = \gamma_2(T') + (p + 2)$ . Now using the fact that  $\gamma_2(T') = \beta_2(T')$  we obtain  $\gamma_2(T) = \beta_2(T)$ , that is  $T$  is a  $(\gamma_2, \beta_2)$ -tree.

*Case 3.*  $T$  is obtained from  $T'$  by using operation  $\mathcal{O}_3$ . By Observation 4,  $\gamma_2(T') = \gamma_2(T) - 1$ . Also  $\beta_2(T) \geq \beta_2(T') + 1$  since every  $\beta_2(T')$ -set can be extended to a 2-independent set of  $T$  by adding  $u'$ . Now assume that  $\beta_2(T) > \beta_2(T') + 1$  and let  $S$  be a  $\beta_2(T)$ -set. Since  $\beta_2(T') \geq |S \cap V(T')|$ , it follows that

$u, u' \in S$ . Hence  $v \notin S$  and  $S \cap V(T')$  is a 2-independent set of  $T' - v$ . Thus  $\beta_2(T' - v) \geq |S \cap V(T')| = \beta_2(T) - 2$ . Also from the construction  $v$  satisfies  $\beta_2(T' - v) + 1 = \beta_2(T')$ . Therefore

$$\beta_2(T') - 1 = \beta_2(T' - v) \geq \beta_2(T) - 2 > (\beta_2(T') + 1) - 2,$$

a contradiction. Consequently  $\beta_2(T) = \beta_2(T') + 1$ . Since  $\gamma_2(T') = \beta_2(T')$  we obtain  $\gamma_2(T) = \beta_2(T)$ .

*Case 4.*  $T$  is obtained from  $T'$  by using operation  $\mathcal{O}_4$ . By Observation 6,  $\beta_2(T) = \beta_2(T') + 2$  and  $\gamma_2(T) \leq \gamma_2(T') + 2$ . Assume that  $\gamma_2(T) < \gamma_2(T') + 2$  and let  $D$  be a  $\gamma_2(T)$ -set. Clearly  $u' \in D$  and  $|D \cap \{u', u, v\}| = 2$ . If  $u \in D$ , then  $v \notin D$  and so  $w \in D$ . Hence  $D \cap V(T')$  is a 2-dominating set of  $T'$  having cardinality  $|D| - 2 < \gamma_2(T')$ , a contradiction. Therefore  $u \notin D$  and so  $v \in D$ . If  $w \in D$ , then using the same argument than used above leads to a contradiction. Thus  $w \notin D$  and hence  $D \cap V(T')$  is a 2-dominating set of  $T' - w$ . It follows that  $\gamma_2(T' - w) \leq |D| - 2 < \gamma_2(T')$  and by Observation 2 we obtain  $\gamma_2(T' - w) = \gamma_2(T') - 1$ . Therefore  $D \cap V(T')$  is a  $\gamma_2(T' - w)$ -set. Note that  $w$  is 2-dominated in  $T$  by  $v$  and some vertex, say  $w' \in V(T')$ . But then  $w'$  belongs to a  $\gamma_2(T' - w)$ -set, a contradiction with the construction. Consequently,  $\gamma_2(T) = \gamma_2(T') + 2$  implying that  $\gamma_2(T) = \beta_2(T)$ , that is,  $T$  is a  $(\gamma_2, \beta_2)$ -tree. ■

We now are ready to state our main result.

**Theorem 8.** *Let  $T$  be a tree of order  $n$ . Then  $\gamma_2(T) = \beta_2(T)$  if and only if  $T = K_1$  or  $T \in \mathcal{O}$ .*

**Proof.** If  $T = K_1$ , then  $\gamma_2(T) = \beta_2(T)$ . If  $T \in \mathcal{O}$ , then by Lemma 7,  $\gamma_2(T) = \beta_2(T)$ . Let us prove the necessity. Obviously,  $\gamma_2(K_1) = \beta_2(K_1)$ , so assume  $n \geq 2$ . We use an induction on the order  $n$  of  $T$ . If  $n = 2$ , then  $T = K_{1,1}$  that belongs to  $\mathcal{O}$ . Assume that every  $(\gamma_2, \beta_2)$ -tree  $T'$  of order  $2 \leq n' < n$  is in  $\mathcal{O}$ . Let  $T$  be  $(\gamma_2, \beta_2)$ -tree of order  $n$ . If  $T$  is a star, then  $T \in \mathcal{O}$ . If  $T$  is a double star, then  $T$  is obtained from  $T_1$  by using Operation  $\mathcal{O}_1$  if  $n \geq 5$ , and  $T$  is obtained from  $T_1 = K_{1,1}$  by using Operation  $\mathcal{O}_3$  if  $n = 4$ . Therefore both stars and double stars are in  $\mathcal{O}$ . Thus we may assume that  $T$  has diameter at least four.

We now root  $T$  at a leaf  $r$  of a longest path. Among all vertices at distance  $\text{diam}(T) - 1$  from  $r$  on a longest path starting at  $r$ , let  $u$  be one of maximum degree. Since  $\text{diam}(T) \geq 4$ , let  $v, w$  be the parents of  $u$  and  $v$ , respectively. Also let  $D$  be a set that is both  $\beta_2(T)$ -set and  $\gamma_2(T)$ -set. Recall that such a set exists as mentioned in the introduction (see [3]). Denote by  $T_x$  the subtree induced by a vertex  $x$  and its descendants in the rooted tree  $T$ . We examine the following cases.

*Case 1.*  $\deg_T(u) \geq 3$ , that is  $u$  is adjacent to at least two leaves. Let  $T' = T - T_u$ . By Observation 4,  $\gamma_2(T) = \gamma_2(T') + |L_u|$  and  $\beta_2(T) = \beta_2(T') + |L_u|$ .

Hence  $\gamma_2(T') = \beta_2(T')$ . By induction on  $T'$ ,  $T' \in \mathcal{O}$  and so  $T \in \mathcal{O}$  because it is obtained from  $T'$  by using operation  $\mathcal{O}_1$ .

*Case 2.*  $\deg_T(u) = 2$ . Let  $u'$  be the unique leaf neighbor of  $u$ . By our choice of  $u$ , every child of  $v$  has degree at most two. First we claim that every child of  $v$  besides  $u$  (if any) is a leaf. Suppose to the contrary that a child  $b$  of  $v$  is a support vertex with  $L_b = \{b'\}$ . Then  $u', b' \in D$ . If  $v \in D$ , then  $u, b \notin D$  (since  $D$  is a  $\beta_2(T)$ -set) but  $\{u, b\} \cup D - \{v\}$  would be a 2-independent set of  $T$  larger than  $D$ , a contradiction. Hence  $v \notin D$  and so  $u, b \in D$  but  $\{v\} \cup D - \{u, b\}$  would be a 2-dominating set of  $T$  smaller than  $D$ , a contradiction too. Thus every child of  $v$  besides  $u$  is a leaf. We consider two subcases.

*Subcase 2.1.*  $\deg_T(v) \geq 3$ . Hence  $v$  is a support vertex and  $T_v$  is a double star  $S_{1, |L_v|}$ . Let  $T' = T - T_v$ . Clearly  $T'$  is nontrivial. By Observation 5,  $\gamma_2(T) = \gamma_2(T') + |L_v| + 2$  and  $\beta_2(T) = \beta_2(T') + |L_v| + 2$ . It follows that  $\gamma_2(T') = \beta_2(T')$  and by induction on  $T'$ ,  $T' \in \mathcal{O}$ . Assume now that  $T' - w$  admits a  $\gamma_2(T' - w)$ -set  $D''$  such that  $|D''| = \gamma_2(T') - 1$  and  $D''$  contains at least one vertex adjacent to  $w$  in  $T'$ . Then  $D'' \cup L_v \cup \{u', v\}$  is a 2-dominating set of  $T'$ , and so

$$\begin{aligned} \gamma_2(T) &\leq |D'' \cup L_v \cup \{u', v\}| = \gamma_2(T' - w) + |L_v| + 2 \\ &= \gamma_2(T') - 1 + |L_v| + 2 < \gamma_2(T') + |L_v| + 2, \end{aligned}$$

a contradiction. Hence such a case cannot occur and so  $T$  can be obtained from  $T'$  by using operation  $\mathcal{O}_2$ . Therefore  $T \in \mathcal{O}$ .

*Subcase 2.2.*  $\deg_T(v) = 2$ . Clearly  $u' \in D$ . Three possibilities can occur ( $u \notin D$  and  $v, w \in D$ ), ( $u, w \notin D$  and  $v \in D$ ) and ( $u, w \in D$  and  $v \notin D$ ). Observe that if the first situation occurs, then  $\{u\} \cup D - \{v\}$  is both  $\beta_2(T)$ -set and  $\gamma_2(T)$ -set too. Hence we have to consider only the last two situations.

Assume that  $u, w \notin D$  and  $v \in D$  and let  $T' = T - \{u, u'\}$ . By Observation 4,  $\gamma_2(T') = \gamma_2(T) - 1$ . Also it is clear that  $\beta_2(T) \geq \beta_2(T') + 1$ . If  $\beta_2(T) > \beta_2(T') + 1$ , then  $\gamma_2(T') + 1 = \gamma_2(T) = \beta_2(T) > \beta_2(T') + 1$ , implying that  $\gamma_2(T') > \beta_2(T')$ , a contradiction. Hence  $\beta_2(T) = \beta_2(T') + 1$  and so  $\gamma_2(T') = \beta_2(T')$ . By induction on  $T'$ ,  $T' \in \mathcal{O}$ . Note that  $v$  belongs to every  $\beta_2(T')$ -set, for otherwise if  $S'$  is a  $\beta_2(T')$ -set such that  $v \notin S'$ , then  $S' \cup \{u, u'\}$  would be a 2-independent set of  $T$  larger than  $D$ , a contradiction. On the other hand, by Observation 3,  $\beta_2(T' - v) \leq \beta_2(T') \leq \beta_2(T' - v) + 1$ . Clearly if  $\beta_2(T' - v) = \beta_2(T')$ , then every  $\beta_2(T' - v)$ -set is also a  $\beta_2(T')$ -set but does not contain  $v$ , a contradiction with the fact that  $v$  belongs to every  $\beta_2(T')$ -set. Therefore  $v$  satisfies  $\beta_2(T') = \beta_2(T' - v) + 1$ . It follows that  $T \in \mathcal{O}$  because it is obtained from  $T'$  by using Operation  $\mathcal{O}_3$ .

Finally assume that  $u, w \in D$  and  $v \notin D$ . Let  $T' = T - \{v, u, u'\}$ . Then by Observation 6,  $\beta_2(T) = \beta_2(T') + 2$  and  $\gamma_2(T) = \gamma_2(T') + 2$ . Note that  $D \cap V(T')$  is a  $\gamma_2(T')$ -set that contains  $w$ . Also by Observation 2,  $\gamma_2(T' - w) \geq \gamma_2(T') - 1$ .

Assume that  $\gamma_2(T' - w) > \gamma_2(T')$ . Then using the fact that  $\beta_2(T) \geq \beta_2(T' - w) + 2$ , it follows that

$$\beta_2(T) \geq \beta_2(T' - w) + 2 \geq \gamma_2(T' - w) + 2 > \gamma_2(T') + 2 = \gamma_2(T),$$

and so  $\beta_2(T) > \gamma_2(T)$ , a contradiction. Therefore  $\gamma_2(T') \geq \gamma_2(T' - w) \geq \gamma_2(T') - 1$ . Now we note that if  $\gamma_2(T' - w) = \gamma_2(T') - 1$ , then no neighbor of  $w$  in  $T'$  belongs to a  $\gamma_2(T' - w)$ -set, for otherwise such a set can be extended to 2-dominating set of  $T$  by adding  $u', v$  which leads to  $\beta_2(T) > \gamma_2(T)$ . Under these conditions it is clear that  $T$  is obtained from  $T'$  by using Operation  $\mathcal{O}_4$  and since  $T' \in \mathcal{O}$  it follows immediately that  $T \in \mathcal{O}$ . ■

#### REFERENCES

- [1] M. Borowiecki, *On a minimaximal kernel of trees*, Discuss. Math. **1** (1975) 3–6.
- [2] M. Chellali, O. Favaron, A. Hansberg and L. Volkmann, *k-domination and k-independence in graphs: A Survey*, Graphs and Combinatorics, **28** (2012) 1–55.  
doi:10.1007/s00373-011-1040-3
- [3] O. Favaron, *On a conjecture of Fink and Jacobson concerning k-domination and k-dependence*, J. Combinat. Theory (B) **39** (1985) 101–102.  
doi:10.1016/0095-8956(85)90040-1
- [4] J.F. Fink and M.S. Jacobson, *n-domination in graphs*, in: Graph Theory with Applications to Algorithms and Computer Science., Y. Alavi and A.J. Schwenk (Ed(s)), (Wiley, New York, 1985) 283–300.
- [5] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs* (Marcel Dekker, New York, 1998).
- [6] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Domination in Graphs: Advanced Topics* (Marcel Dekker, New York 1998).

Received 14 September 2010

Revised 10 May 2011

Accepted 11 May 2011