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THE VERTEX DETOUR HULL NUMBER OF A GRAPH [1]

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Abstract

For vertices x and y in a connected graph G, the detour distance $D(x,y)$ is the length of a longest $x-y$ path in G. An $x-y$ path of length $D(x,y)$ is an $x-y$ detour. The closed detour interval $I_D[x,y]$ consists of x,y, and all vertices lying on some $x-y$ detour of G; while for $S \subseteq V(G)$, $I_D[S] = \bigcup_{x,y\in S} I_D[x,y]$. A set S of vertices is a detour convex set if $I_D[S] = S$. The detour convex hull $[S]_D$ is the smallest detour convex set containing S. The detour hull number $dh(G)$ is the minimum cardinality among subsets S of $V(G)$ with $[S]_D = V(G)$. Let x be any vertex in a connected graph G. For a vertex y in G, denoted by $I_D[y]^x$, the set of all vertices distinct from x that lie on some $x-y$ detour of G; while for $S \subseteq V(G)$, $I_D[S]^x = \bigcup_{y\in S} I_D[y]^x$. For $x \notin S$, S is an x-detour convex set if $I_D[S]^x = S$. The x-detour convex hull of S, $[S]^x_D$ is the smallest x-detour convex set containing S. A set S is an x-detour hull set if $[S]^x_D = V(G) - \{x\}$ and the minimum cardinality of x-detour hull sets is the x-detour hull number $dh_x(G)$ of G. For $x \notin S$, S is an x-detour set of G if $I_D[S]^x = V(G) - \{x\}$ and the minimum cardinality of x-detour sets is the x-detour number $d_x(G)$ of G. Certain general properties of the x-detour hull number of a graph are studied. It is shown that for

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each pair of positive integers a, b with $2 \leq a \leq b+1$, there exist a connected graph G and a vertex x such that $dh(G) = a$ and $dh_x(G) = b$. It is proved that every two integers a and b with $1 \leq a \leq b$, are realizable as the x-detour hull number and the x-detour number respectively. Also, it is shown that for integers a, b and n with $1 \leq a \leq n - b$ and $b \geq 3$, there exist a connected graph G of order n and a vertex x such that $dh_x(G) = a$ and the detour eccentricity of x, $e_D(x) = b$. We determine bounds for $dh_x(G)$ and characterize graphs G which realize these bounds.

Keywords: detour, detour number, detour hull number, x-detour number, x-detour hull number.

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1. Introduction

By a *graph* $G = (V, E)$, we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For basic definitions and terminologies, we refer to [1, 6]. For vertices x and y in a nontrivial connected graph G, the *detour distance* $D(x, y)$ is the length of a longest $x - y$ path in G. An $x - y$ path of length $D(x, y)$ is an $x - y$ *detour*. It is known that the detour distance is a metric on the vertex set $V(G)$. The *detour eccentricity* of a vertex u is $e_D(u) = \max\{D(u, v) : v \in V(G)\}$. The *detour radius*, $rad_D(G)$ of G is the minimum detour eccentricity among the vertices of G, while the *detour diameter*, $diam_D(G)$ of G is the maximum detour eccentricity among the vertices of G. The detour distance and the detour center of a graph were studied in [2]. The *closed detour interval* $I_D[x, y]$ consists of x, y, and all vertices lying on some $x - y$ detour of G; while for $S \subseteq V(G)$, $I_D[S] = \bigcup_{x,y \in S} I_D[x, y]$; S is a *detour set* if $I_D[S] = V(G)$ and the minimum cardinality of detour sets is the *detour number* $dn(G)$ of G. Any detour set of cardinality $dn(G)$ is a *minimum detour set* or *dn-set* of G. A vertex x in G is a *detour extreme vertex* if it is an initial or terminal vertex of any detour containing x. The detour number of a graph was introduced in [3] and further studied in [4, 8]. These concepts have interesting applications in Channel Assignment Problem in radio technologies [5, 7].

A set S of vertices of a graph G is a *detour convex set* if $I_D[S] = S$. The *detour convex hull* $[S]_D$ of S is the smallest detour convex set containing S. The detour convex hull of S can also be formed from the sequence $\{I_D^k[S], k \geq 0\}$, where $I_D^0[S] = S, I_D^1[S] = I_D[S]$ and $I_D^k = I_D[I_D^{k-1}[S]]$. From some term on, this sequence must be constant. Let p be the smallest number such that $I_D^p[S] = I_D^{p+1}[S]$. Then $I_D^p[S]$ is the *detour convex hull* $[S]_D$ of S and we call p as the *detour iteration number* $din(S)$ of S. A set S of vertices of G is a *detour hull set* if $[S]_D = V(G)$ and the minimum cardinality of detour hull sets is the *detour* *hull number* $dh(G)$ of G. The detour hull number of a graph was introduced and studied in [10].

Let x be any vertex of G. For a vertex y in G, $I_D[y]^x$ denotes the set of all vertices distinct from x that lie on some $x-y$ detour of G; while for $S \subseteq V(G)$, $I_D[S]^x = \bigcup_{y \in S} I_D[y]^x$. It is clear that $I_D[x]^x = \emptyset$. For $x \notin S$, S is an *x-detour set* if $I_D[S]^x = V(G) - \{x\}$ and the cardinality of a smallest x-detour set is the *x-detour number* $d_x(G)$ of G. Any x-detour set of cardinality $d_x(G)$ is a *minimum x-detour set* or a *d_x-set* of G. The vertex detour numbers of a graph were introduced and studied in [9]. Throughout this paper G denotes a connected graph with at least two vertices. The following theorems will be used in the sequel.

Theorem 1.1 [9]**.** *Each end vertex of G other than x (whether x is an end vertex or not) belongs to every minimum x-detour set of G.*

Theorem 1.2 [9]**.** *For any vertex x in a connected graph G of order n, $d_x(G) \leq n - e_D(x)$.*

Theorem 1.3 [10]**.** *Let G be a connected graph. Then*

(i) *Each detour extreme vertex of G belongs to every detour hull set of G.*

(ii) *No cut vertex of G belongs to any minimum detour hull set of G.*

2. THE VERTEX DETOUR HULL NUMBER OF A GRAPH

Let G be a connected graph and x a vertex in G. Let S be a set of vertices in G such that $x \notin S$. Then S is an *x-detour convex set* if $I_D[S]^x = S$. The *x-detour convex hull* of S, $[S]_D^x$ is the smallest x-detour convex set containing S. The x-detour convex hull of S can also be formed from the sequence $\{I_D^k[S]^x, k \geq 0\}$, where $I_D^0[S]^x = S$, $I_D^1[S]^x = I_D[S]^x$ and $I_D^k[S]^x = I_D[I_D^{k-1}[S]^x]^x$. From some term on, this sequence must be constant. Let p_x be the smallest number such that $I_D^{p_x}[S]^x = I_D^{p_x+1}[S]^x$. Then $I_D^{p_x}[S]^x$ is the x-detour convex hull $[S]_D^x$ of S and we call p_x as the *x-detour iteration number* $din_x(S)$ of S. The set S is an *x-detour hull set* if $[S]_D^x = V(G) - \{x\}$ and the minimum cardinality of x-detour hull sets is the *x-detour hull number* $dh_x(G)$ of G. Any x-detour hull set of cardinality $dh_x(G)$ is a *minimum x-detour hull set* or *dh_x-hull set* of G.

For the graph G in Figure 2.1, it is straightforward to compute the minimum vertex detour sets and the minimum vertex detour hull sets, and correspondingly these sets together with the minimum vertex detour hull numbers and vertex detour numbers are given in Table 2.1. Table 2.1 shows that, for a vertex x, the x-detour number and the x-detour hull number of a graph are different.

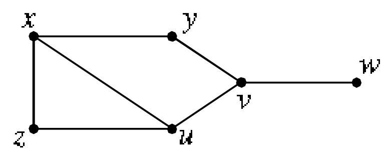

Figure 2.1

Vertex	Minimum vertex detour sets	Minimum vertex detour hull sets	Vertex detour number	Vertex detour hull number
x	$\{y,w\},\{z,w\},\{u,w\}$	$\{w\}$	2	1
y	$\{w\}$	$\{w\}$	1	1
z	$\{w\}$	$\{w\}$	1	1
u	$\{w\}$	$\{w\}$	1	1
v	$\{y,w\},\{z,w\},\{u,w\}$	$\{x,w\},\{y,w\},\{z,w\},\{u,w\}$	2	2
w	$\{y\},\{z\},\{u\}$	$\{x\},\{y\},\{z\},\{u\}$	1	1

Table 2.1

It is clear that every minimum x-detour hull set of a connected graph G of order n contains at least one vertex and at most $n-1$ vertices. Also, since every x-detour set is a x-detour hull set, we have the following proposition.

Proposition 2.1. *Let G be a connected graph of order n. Then* $1 \leq dh_x(G) \leq d_x(G) \leq n-1$ *for every vertex x in G.*

A graph G is said to be *hypohamiltonian* if G is not Hamiltonian but every graph formed by removing a single vertex from G is Hamiltonian.

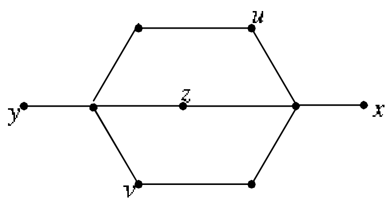

Figure 2.2

Theorem 2.2. *If a graph G is Hamiltonian or hypohamiltonian of order n, then* $dh_x(G) = 1$ *for every vertex x in G.*

Proof. This follows from the fact that $e_D(u) = n-1$ for each vertex u in G. ■

The converse of Theorem 2.2 is not true. For the graph $H = G - \{x, y\}$, where G is the graph given in Figure 2.2, it is easily seen that $dh_x(H) = 1$ for each vertex x in H. However, H is neither Hamiltonian nor hypohamiltonian.

Theorem 2.3. *Let x be any vertex of a connected graph G. Then $dh_x(G) = 1$ if and only if there exists a vertex $y \neq x$ such that $V(G) - x$ is the only x-detour convex set containing y.*

Proof. Suppose that $dh_x(G) = 1$. Let $S = \{y\}$ be a minimum x-detour hull set of G. Then $[S]_D^x = V(G) - \{x\}$ is the smallest x-detour convex set containing y and so the result follows. The converse is obvious. ∎

Proposition 2.4. *Let S be a minimum x-detour hull set of G and let $y \in S$. If z is a vertex distinct from y such that $z \in I_D[y]^x$, then $z \notin S$.*

Proof. Assume to the contrary, that $z \in S$. Since $z \in I_D[y]^x$, we have $I_D[S]^x \subseteq I_D^x[S - \{z\}]$. This gives $[S]_D^x \subseteq [S - \{z\}]_D^x$. Also, since S is an x-detour hull set, we have $[S]_D^x = V(G) - \{x\}$. It follows that $[S - \{z\}]_D^x = V(G) - \{x\}$ and hence $S - \{z\}$ is an x-detour hull set of G, which is a contradiction to S being a minimum x-detour hull set of G. Hence the result follows. ∎

Definition. Let x be a vertex in a connected graph G. A vertex $z \neq x$ is an *x-detour extreme vertex* if $z \notin I_D[y]^x$ for any vertex y in G with $y \neq z$.

Example 2.5. Each end vertex of a graph G other than the vertex x (whether x is an end vertex or not) is an x-detour extreme vertex of G. Moreover, each detour extreme vertex other than x (whether x is detour extreme or not) is an x-detour extreme vertex of G. For the graph G in Figure 2.2, it is clear that $z \notin I_D[y]^x$ for all $y \neq z$ and so z is an x-detour extreme vertex of G. It is to be noted that $z \in I_D[u, v]$ and so z is not a detour extreme vertex of G.

Proposition 2.6. *Let G be a connected graph. Then a vertex z in G is detour extreme if and only if z is an x-detour extreme vertex for each vertex $x \neq z$.*

Proof. Suppose that z is a detour extreme vertex of G. Then z is either an initial vertex or a terminal vertex of any detour that contains z. Let $x \neq z$. Then for any $y \neq z$, we have $z \notin I_D[y]^x$. Thus z is an x-detour extreme vertex of G for each $x \neq z$.

Conversely, suppose that z is an x-detour extreme vertex for each $x \neq z$. Then $z \notin I_D[y]^x$ for any $y \neq z$. That is, $z \notin I_D[x, y]$ for any $x \neq z$ and $y \neq z$. This implies that z is a detour extreme vertex of G. ∎

Theorem 2.7. *Let x be a vertex of a connected graph G. Let S be any x-detour hull set of G. Then*

(i) *Each x-detour extreme vertex of G belongs to S.*

(ii) *If v is a cut vertex of G and C is a component of $G-v$ such that $x \notin V(C)$, then $S \cap V(C) \neq \emptyset$.*

(iii) *No cut-vertex of G belongs to any minimum x-detour hull set of G.*

Proof. (i) Let y be an x-detour extreme vertex of G. Then $y \neq x$. Suppose that $y \notin S$. Then $y \in I_D^k[S]^x$ for some $k \geq 1$. Let l be the smallest positive integer such that $l \leq k$ and $y \in I_D^l[S]^x$. Then $l \geq 1$ and $y \notin I_D^{l-1}[S]^x$. Hence $y \in I_D[z]^x$ for some $z \in I_D^{l-1}[S]^x$. This implies that $y \neq z$, which is a contradiction to y being an x-detour extreme vertex of G. Thus y belongs to every x-detour hull set of G.

(ii) Suppose that $S \cap V(C) = \emptyset$. It is clear that for each $y \in V(G) - V(C)$, $I_D[y]^x \subseteq V(G) - V(C)$. Since $S \cap V(C) = \emptyset$, it follows that $I_D^k[S]^x \subseteq V(G) - (V(C) \cup \{x\})$ for all $k \geq 0$ and so $[S]_D^x \neq V(G) - \{x\}$, which in turn implies that S is not an x-detour hull set of G, a contradiction. Thus $V(C) \cap S \neq \emptyset$.

(iii) Let S be any minimum x-detour hull set of G. Let v be a cut vertex of G and $C_1, C_2, \ldots, C_k (k \geq 2)$ the components of $G - v$. If $x = v$, then by definition, $x \notin S$. Assume that $x \in V(C_1)$. By (ii), we have $S \cap V(C_2) \neq \emptyset$. Let $y \in S \cap V(C_2)$. Then $v \in I_D[y]^x$ and it follows from Proposition 2.4 that $v \notin S$. ■

Corollary 2.8. *Let T be a tree with k end vertices. Then $dh_x(T) = k - 1$ or $dh_x(T) = k$ according to whether x is an end vertex or not. In fact, if W is the set of all end vertices of T, then $W - \{x\}$ is the unique minimum x-detour hull set of G.*

Proof. This follows from Theorem 2.7(i) and (iii). ■

Theorem 2.9. *For any vertex x in a connected graph G, $dh(G) \leq dh_x(G) + 1$.*

Proof. Let $x \in V(G)$. First, we show that for any set $S \subseteq V(G) - \{x\}$, $I_D^k[S]^x \subseteq I_D^k[S \cup \{x\}]$ for all $k \geq 0$. We use induction on k. If $k = 0$, then the result is obvious. Let $y \in I_D[S]^x$. If $y \in S$, then $y \in I_D[S \cup \{x\}]$. If $y \notin S$, then $y \in I_D[z]^x$ for some $z \in S$. Since $I_D[z]^x \subseteq I_D[z]^x \cup \{x\} = I_D[x, z]$, we see that $y \in I_D[S \cup \{x\}]$. Hence $I_D[S]^x \subseteq I_D[S \cup \{x\}]$. Now, assume that $I_D^l[S]^x \subseteq I_D^l[S \cup \{x\}]$ for some integer $l \geq 1$. Let $y \in I_D^{l+1}[S]^x$. If $y \in I_D^l[S]^x$, then by induction hypothesis, $y \in I_D^l[S \cup \{x\}] \subseteq I_D^{l+1}[S \cup \{x\}]$. If $y \notin I_D^l[S]^x$, then $y \in I_D[z]^x$ for some $z \in I_D^l[S]^x$. Hence by induction hypothesis, $z \in I_D^l[S \cup \{x\}]$. Since $I_D[z]^x \subseteq I_D[z]^x \cup \{x\} = I_D[x, z]$ and $x, z \in I_D^l[S \cup \{x\}]$, we have $y \in I_D^{l+1}[S \cup \{x\}]$. Thus by induction, $I_D^k[S]^x \subseteq I_D^k[S \cup \{x\}]$ for all $k \geq 0$. Now, let S be a minimum x-detour hull set of G. Then there exists an integer $k \geq 0$ such that $I_D^k[S]^x = V(G) - \{x\}$. It follows from the above claim that $I_D^k[S \cup \{x\}] = V(G)$. Hence $S \cup \{x\}$ is a detour hull set of G. Thus $dh(G) \leq |S \cup \{x\}| = |S| + 1 = dh_x(G) + 1$. ■

In view of Theorem 2.9, we have the following realization result.

Theorem 2.10. *For each pair of integers a, b with $2 \leq a \leq b+1$, there exists a connected graph G such that $dh(G) = a$ and $dh_x(G) = b$ for some vertex x in G.*

Proof. Let $C_6 : v_1, v_2, v_3, v_4, v_5, v_6, v_1$ be a cycle of order 6. Let H be the graph obtained from C_6 by adding the new vertices $u_1, u_2, \ldots, u_a$, joining u_a to v_4, and joining $u_1, u_2, \ldots, u_{a-1}$ to v_1. Let $\overline{K}_{b-a+1}$ be the totally disconnected graph on $b-a+1$ vertices with the vertex set $V(\overline{K}_{b-a+1}) = \{w_1, w_2, \ldots, w_{b-a+1}\}$ such that H and $\overline{K}_{b-a+1}$ are vertex disjoint. Let G be the graph in Figure 2.3 obtained from H and $\overline{K}_{b-a+1}$ by joining w_i for each i, $1 \leq i \leq b-a+1$ to both v_1 and v_4.

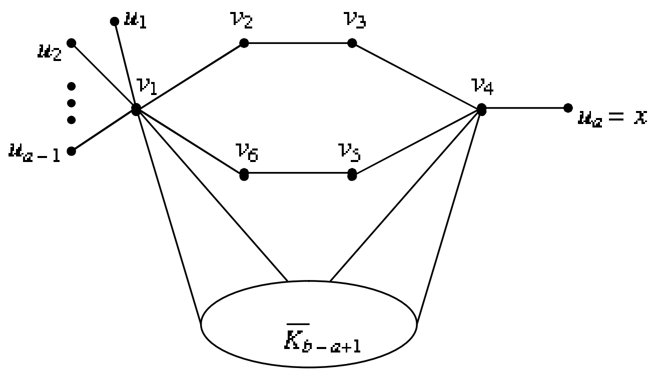

Figure 2.3

Let $S' = \{u_1, u_2, \ldots, u_a\}$ be the set of end vertices of G. Then $v_i \in I_D[S']$ for each i, $1 \leq i \leq 6$. Since $D(v_2, v_5) = 6$, it is clear that $w_j \in I_D[v_2, v_5]$ for each j, $1 \leq j \leq b-a+1$. Hence $I_D^2[S'] = V(G)$. Thus, by Theorem 1.3, S' is a minimum detour hull set of G so that $dh(G) = |S'| = a$.

Now, take $x = u_a$. Then $D(x, u_i) = 5$ for each i, $1 \leq i \leq a-1$ and so $w_j \notin I_D[u_i]^x$ for any j, $1 \leq j \leq b-a+1$. Similarly, $D(x, v_i) = 6$ for $i = 3, 5$ and any $x - v_i$ path that contains w_j for j. $1 \leq j \leq b-a+1$ has length 5 and so $w_j \notin I_D[v_3]^x$ and $w_j \notin I_D[v_5]^x$ for any j, $1 \leq j \leq b-a+1$. Also, $D(x, v_i) = 5$ for $i = 2, 6$ and any x-v_i path that contains w_j for j, $1 \leq j \leq b-a+1$ has length 4 and so $w_j \notin I_D[v_2]^x$ and $w_j \notin I_D[v_6]^x$. Also, $D(x, w_i) = 5$ and $w_j \notin I_D[w_i]^x$ for $i \neq j$ and it is clear that $w_j \notin I_D[v_1]^x$ and $w_j \notin I_D[v_4]^x$ for any j, $1 \leq j \leq b-a+1$. Hence it follows that $w_1, w_2, \ldots, w_{b-a+1}$ are x-detour extreme vertices of G. Since $u_1, u_2, \ldots, u_{a-1}$ are also x-detour extreme vertices of G and the set $S = \{u_1, u_2, \ldots, u_{a-1}, w_1, w_2, \ldots, w_{b-a+1}\}$ is an x-detour hull set of G, it follows from Theorem 2.7(i) that S is a minimum x-detour hull set of G. Thus $dh_x(G) = |S| = a-1+b-a+1 = b$. This completes the proof. ■

In view of Proposition 2.1, we have the following realization result.

Theorem 2.11. *For each pair a, b of integers with $1 \leq a \leq b$, there exists a connected graph G and a vertex x in G such that $dh_x(G) = a$ and $d_x(G) = b$.*

Proof. If $a = b$, then let $G = K_{1,a+1}$. Let x be an end vertex of G. Then G has the desired properties. So, assume that $a < b$. For each $i = 1, 2, \ldots, b-a$, let $C_{6,i} : v_{1,i}, v_{2,i}, v_{3,i}, v_{4,i}, v_{5,i}, v_{6,i}, v_{1,i}$ be vertex disjoint cycles of order 6. Let H be the graph obtained from the cycles $C_{6,i}$, $1 \leq i \leq b-a$ by joining the vertices $v_{2,i}$ and $v_{6,i+1}$ for $i = 1, 2, \ldots, b-a-1$. Let G be the graph in Figure 2.4 obtained from H by adding $a+1$ new vertices $x, u_1, u_2, \ldots, u_a$ and joining u_i for $1 \leq i \leq a$ to $v_{6,1}$ and x to $v_{2,(b-a)}$.

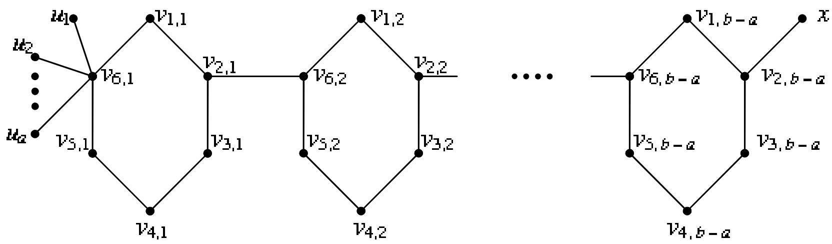

Figure 2.4

Let $S = \{u_1, u_2, \ldots, u_a, x\}$ be the set of end vertices of G. We have $D(u_i, x) = 5(b-a)+1$ and $I_D[S]^x = V(G) - \{x, v_{1,1}, v_{1,2}, \ldots, v_{1,b-a}\}$. Since $v_{1,i} \in I_D[v_{3,i}]^x$ for $i = 1, 2, \ldots, b-a$, it follows that $I_D^2[S]^x = V(G) - \{x\}$ and it follows from Theorem 2.7(i) that S is a minimum x-detour hull set of G. Thus $dh_x(G) = a$.

Next, we prove that $d_x(G) = b$. For $i = 1, 2, \ldots, b-a$, it is clear that $v_{1,i} \notin I_D[y]^x$ for any $y \notin V(C_{6,i})$. Let $T_i = \{v_{1,i}, v_{3,i}, v_{5,i}\}$. Then it is straight forward to verify that every x-detour set contains at least one vertex from each T_i and by Theorem 1.1, $d_x(G) \geq a + b - a = b$. Since $T = S \cup \{v_{1,1}, v_{1,2}, \ldots, v_{1,b-a}\}$ is an x-detour set of G, we have $d_x(G) = b$. ■

The following theorem is an immediate consequence of Theorem 1.2 and Proposition 2.1.

Theorem 2.12. *For any vertex x in a connected graph G of order n, $dh_x(G) \leq n - e_D(x)$.*

In view of Theorem 2.12, we have the following realization result.

Theorem 2.13. *For integers a, b and n with $1 \leq a \leq n-b$ and $b \geq 3$, there exists a connected graph G of order n and a vertex x in G such that $dh_x(G) = a$ and $e_D(x) = b$.*

Proof. Let $P_b : x = u_0, u_1, \ldots, u_b$ be a path of length b. Let H be the graph obtained from P_b by adding $a-1$ new vertices $v_1, v_2, \ldots, v_{a-1}$ and joining v_i for i, $1 \leq i \leq a-1$ to u_{b-1}. Let G be the graph in Figure 2.5 obtained from H by adding $n-a-b$ new vertices $w_1, w_2, \ldots, w_{n-a-b}$ and joining w_i for i, $1 \leq i \leq n-a-b$ to both u_0 and u_2. Then G has order n and $e_D(x) = b$. Also, it is clear that the set $S = \{v_1, v_2, \ldots, v_{a-1}, u_b\}$ of end vertices is an x-detour hull set of G and so by Theorem 2.7(i), S is the unique minimum x-detour hull set of G. Hence $dh_x(G) = |S| = a$. ∎

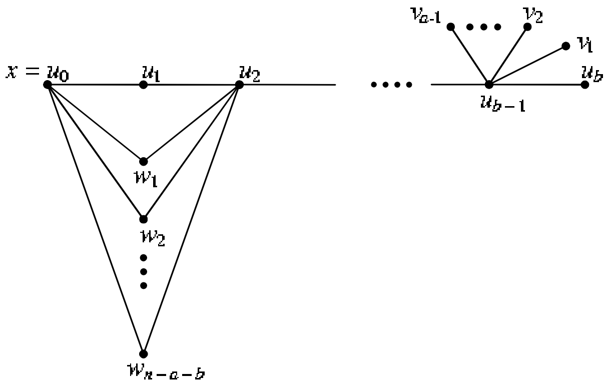

Figure 2.5

Theorem 2.14. *Let G be a connected graph of order $n \geq 2$. Then $dh_x(G) = n-1$ for every vertex x in G if and only if $G = K_2$.*

Proof. Suppose that $G = K_2$. Then $dh_x(G) = 1 = n-1$. The converse follows from Theorem 2.12. ∎

Theorem 2.15. *Let G be a connected graph of order $n \geq 3$. Then $dh_x(G) = n-2$ for every vertex x in G if and only if $G = K_3$.*

Proof. Suppose that $G = K_3$. Then by Theorem 2.2, $dh_x(G) = 1 = n-2$ for every vertex x in G. Conversely, suppose that $dh_x(G) = n-2$ for every vertex x in G. Then by Theorem 2.12, $e_D(x) \leq 2$ for every vertex x in G. Now, if $e_D(x) = 1$ for every vertex x in G, then $G = K_2$, and so by Theorem 2.15, $dh_x(G) = n-1$, which is a contradiction. Thus $e_D(x) = 2$ for every vertex x in G; or the vertex set can be partitioned into V_1 and V_2 such that $e_D(x) = 1$ for $x \in V_1$ and $e_D(x) = 2$ for $x \in V_2$. Thus either $rad_D(G) = diam_D(G) = 2$ or we have $rad_D(G) = 1$ and $diam_D(G) = 2$. This implies that either $G = K_3$ or $G = K_{1,n-1}$. If $G = K_{1,n-1}$, then by Corollary 2.8, $dh_x(G) = n-1$ for the cut vertex x and $dh_y(G) = n-2$ for any end vertex y in G, which is a contradiction to the hypothesis. Hence $G = K_3$. ∎

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