

## EDGE MAXIMAL $C_{2k+1}$ -EDGE DISJOINT FREE GRAPHS

M.S.A. BATAINEH

*Department of Mathematics*  
*Yarmouk University*  
*Irbid-Jordan*

**e-mail:** bataineh71@hotmail.com

AND

M.M.M. JARADAT

*Yarmouk University*  
*Department of Mathematics*  
*Irbid-Jordan*  
*Department of Mathematics, Physics and Statistics*  
*Qatar University*  
*Doha-Qatar*

**e-mail:** mmjst4@yu.edu.jo; mmjst4@qu.edu.qa

### Abstract

For two positive integers  $r$  and  $s$ ,  $\mathcal{G}(n; r, s)$  denotes to the class of graphs on  $n$  vertices containing no  $r$  of  $s$ -edge disjoint cycles and  $f(n; r, s) = \max\{|\mathcal{E}(G)| : G \in \mathcal{G}(n; r, s)\}$ . In this paper, for integers  $r \geq 2$  and  $k \geq 1$ , we determine  $f(n; r, 2k+1)$  and characterize the edge maximal members in  $\mathcal{G}(n; r, 2k+1)$ .

**Keywords:** extremal graphs, edge disjoint, cycles.

**2010 Mathematics Subject Classification:** 05C38, 05C35.

### 1. INTRODUCTION

The graphs considered in this paper are finite, undirected and have no loops or multiple edges. Most of the notations that follow can be found in [5]. For a given graph  $G$ , we denote the vertex set of a graph  $G$  by  $V(G)$  and the edge set by  $E(G)$ . The cardinalities of these sets are denoted by  $\nu(G)$  and  $\mathcal{E}(G)$ , respectively. The cycle on  $n$  vertices is denoted by  $C_n$ .

Let  $G_1$  and  $G_2$  be graphs. The union of  $G_1$  and  $G_2$  is a graph with vertex set  $V(G_1) \cup V(G_2)$  and edge set  $E(G_1) \cup E(G_2)$ . Two graphs  $G_1$  and  $G_2$  are vertex disjoint if and only if  $V(G_1) \cap V(G_2) = \emptyset$ ;  $G_1$  and  $G_2$  are edge disjoint if  $E(G_1) \cap E(G_2) = \emptyset$ . If  $G_1$  and  $G_2$  are vertex disjoint, we denote their union by  $G_1 + G_2$ . The intersection  $G_1 \cap G_2$  of graphs  $G_1$  and  $G_2$  is defined similarly, but in this case we need to assume that  $V(G_1) \cap V(G_2) \neq \emptyset$ . The join  $G \vee H$  of two vertex disjoint graphs  $G$  and  $H$  is the graph obtained from  $G + H$  by joining each vertex of  $G$  to each vertex of  $H$ . For two vertex disjoint subgraphs  $H_1$  and  $H_2$  of  $G$ , we let  $E_G(H_1, H_2) = \{xy \in E(G) : x \in V(H_1), y \in V(H_2)\}$  and  $\mathcal{E}_G(H_1, H_2) = |E_G(H_1, H_2)|$ .

In this paper we consider the Turán-type extremal problem with the odd edge disjoint cycles being the forbidden subgraph. Since a bipartite graph contains no odd cycles, the non-bipartite graphs have been considered by some authors. First, we recall some notations and terminologies. For a positive integer  $n$  and a set of graphs  $\mathcal{F}$ , let  $\mathcal{G}(n; \mathcal{F})$  denote the class of non-bipartite  $\mathcal{F}$ -free graphs on  $n$  vertices, and

$$f(n; \mathcal{F}) = \max\{\mathcal{E}(G) : G \in \mathcal{G}(n; \mathcal{F})\}.$$

For simplicity, in the case when  $\mathcal{F}$  consists only of one member  $C_s$ , where  $s$  is an odd integer, we write  $\mathcal{G}(n; s) = \mathcal{G}(n; \mathcal{F})$  and  $f(n; s) = f(n; \mathcal{F})$ .

An important problem in extremal graph theory is that of determining the values of the function  $f(n; \mathcal{F})$ . Further, characterize the extremal graphs  $\mathcal{G}(n; \mathcal{F})$  where  $f(n; \mathcal{F})$  is attained. For a given  $r$ , the edge maximal graphs of  $\mathcal{G}(n; r)$  have been studied by a number of authors [1, 2, 3, 7, 8, 9, 10, 12]. In 1998, Jia [11] proved the following result:

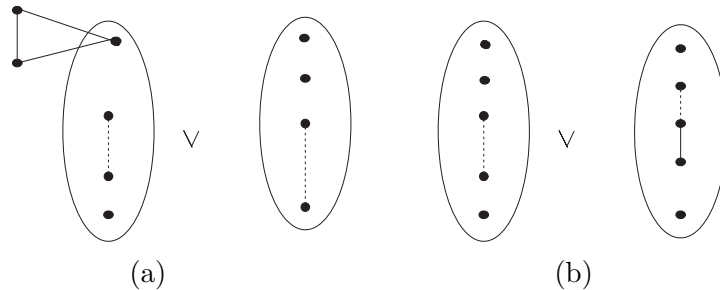


Figure 1. (a) The figure represents a member of  $\mathcal{G}^*(n)$ .

(b) The figure represents a member of  $\Omega(n, 2)$ .

**Theorem 1** (Jia). *Let  $G \in \mathcal{G}(n; 5)$ ,  $n \geq 10$ . Then  $\mathcal{E}(G) \leq \lfloor (n-2)^2/4 \rfloor + 3$ . Furthermore, equality holds if and only if  $G \in \mathcal{G}^*(n)$  where  $\mathcal{G}^*(n)$  is the class of graphs obtained by adding a triangle, two vertices of which are new, to the complete bipartite graph  $K_{\lfloor (n-2)/2 \rfloor, \lceil (n-2)/2 \rceil}$ . Figure 1(a) displays a member of  $\mathcal{G}^*(n)$ .*

Jia, also conjectured that  $f(n; 2k+1) \leq \lfloor (n-2)^2/4 \rfloor + 3$  for all  $n \geq 4k+2$ . In 2007, Bataineh, confirmed positively the conjecture. In fact, he proved the following result:

**Theorem 2** (Bataineh). *Let  $k \geq 3$  be a positive integer and  $G \in \mathcal{G}(n; 2k+1)$ . Then for large  $n$ ,  $\mathcal{E}(G) \leq \lfloor (n-2)^2/4 \rfloor + 3$ .*

*Furthermore, equality holds if and only if  $G \in \mathcal{G}^*(n)$  where  $\mathcal{G}^*(n)$  is as above.*

Let  $\mathcal{G}(n; r, s)$  denote to the class of graphs on  $n$  vertices containing no  $r$  of  $s$ -edge disjoint cycles and

$$f(n; r, s) = \max\{\mathcal{E}(G) : G \in \mathcal{G}(n; r, s)\}.$$

Note that

$$\mathcal{G}(n; 2, s) \subseteq \mathcal{G}(n; 3, s) \subseteq \cdots \subseteq \mathcal{G}(n; r, s).$$

Let  $\Omega(n, r)$  denote to the class of graphs obtained by adding  $r-1$  edges to the complete bipartite graphs  $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ . Figure 1(b) displays a member of  $\Omega(n, 2)$ .

The Turán-type extremal problem with two odd edge disjoint cycles being the forbidden subgraph, was studied by Bataineh and Jaradat [2]. In fact, they only established partial results by proving the following:

**Theorem 3** (Bataineh and Jaradat). *Let  $k = 1, 2$  and  $G \in \mathcal{G}(n; 2, 2k+1)$ . Then for large  $n$ ,*

$$\mathcal{E}(G) \leq \lfloor n^2/4 \rfloor + 1.$$

*Furthermore, equality holds if and only if  $G \in \Omega(n, 2)$ .*

In this paper, we continue the work initiated in [2] by generalizing and extending the above theorem. In fact, we determine  $f(n; r, 2k+1)$  and characterize the edge maximal members in  $\mathcal{G}(n; r, 2k+1)$ . Now, we state a number of results, which play an important role in proving our result.

**Lemma 4** (Bondy and Murty). *Let  $G$  be a graph on  $n$  vertices. If  $\mathcal{E}(G) > n^2/4$ , then  $G$  contains a cycle of length  $r$  for each  $3 \leq r \leq \lfloor (n+3)/2 \rfloor$ .*

**Theorem 5** (Brandt). *Let  $G$  be a non-bipartite graph with  $n$  vertices and more than  $\lfloor (n-1)^2/4 \rfloor + 1$  edges. Then  $G$  contains all cycles of length between 3 and the length of the longest cycle.*

In the rest of this paper,  $N_G(u)$  stands for the set of neighbors of  $u$  in the graph  $G$ . Moreover,  $G[X]$  denotes to the subgraph induced by  $X$  in  $G$ .

2. EDGE-MAXIMAL  $C_{2k+1}$ -EDGE DISJOINT FREE GRAPHS

In this section, we determine  $f(n; r, 2k+1)$  and characterize the edge maximal members in  $\mathcal{G}(n; r, 2k+1)$ . Observe that  $\Omega(n, r) \subseteq \mathcal{G}(n; r, 2k+1)$  and every graph in  $\Omega(n, r)$  contains  $\lfloor n^2/4 \rfloor + r - 1$  edges. Thus, we have established that

$$(1) \quad f(n; r, 2k+1) \geq \lfloor n^2/4 \rfloor + r - 1.$$

In the following work, we establish that equality holds. Further we characterize the edge maximal members in  $\mathcal{G}(n; r, 2k+1)$ .

**Theorem 6.** *Let  $k \geq 1, r \geq 2$  be two positive integers and  $G \in \mathcal{G}(n; r, 2k+1)$ . For large  $n$ ,*

$$\mathcal{E}(G) \leq \lfloor n^2/4 \rfloor + r - 1.$$

*Furthermore, equality holds if and only if  $G \in \Omega(n, r)$ .*

**Proof.** We prove the theorem using induction on  $r$ .

**Step 1.** We show the result for  $r = 2$ . Note that by Theorem 3, it is enough to prove the result for  $k \geq 3$ . Let  $G \in \mathcal{G}(n, 2, 2k+1)$ . If  $G$  does not have a cycle of length  $2k+1$ , then by Lemma 4,  $\mathcal{E}(G) \leq \lfloor n^2/4 \rfloor$ . Thus,  $\mathcal{E}(G) < \lfloor n^2/4 \rfloor + 1$ . So, we need to consider the case when  $G$  has cycles of length  $2k+1$ . Assume  $C = x_1x_2 \dots x_{2k+1}x_1$  be a cycle of length  $2k+1$  in  $G$ . Consider  $H = G - \{e_1 = x_1x_2, e_2 = x_2x_3, \dots, e_{2k+1} = x_{2k+1}x_1\}$ . Observe that  $H$  cannot have  $2k+1$ -cycle as otherwise  $G$  would have two edge disjoint  $2k+1$ -cycles. We now consider two cases according to  $H$ :

*Case 1.*  $H$  is not a bipartite graph. If  $k \geq 2$ , then by Theorems 1 and 2

$$\mathcal{E}(H) \leq \lfloor (n-2)^2/4 \rfloor + 3.$$

But,  $\mathcal{E}(G) = \mathcal{E}(H) + 2k + 1 \leq \lfloor \frac{(n-2)^2}{4} \rfloor + 2k + 4 \leq \lfloor \frac{n^2}{4} \rfloor - n + 2k + 5$ . Thus, for  $n \geq 2k + 5$ , we have  $\mathcal{E}(G) < \lfloor \frac{n^2}{4} \rfloor + 1$ . If  $k = 1$ , then by Theorems 5  $\mathcal{E}(H) \leq \lfloor (n-1)^2/4 \rfloor + 1$ . And so, by using the same argument as in the above, we get that for  $n \geq 7$ ,

$$\mathcal{E}(G) < \left\lfloor \frac{n^2}{4} \right\rfloor + 1.$$

*Case 2.*  $H$  is a bipartite graph. Let  $X$  and  $Y$  be the partition of  $V(H)$ . Thus,  $\mathcal{E}(H) \leq |X||Y|$ . Observe  $|X| + |Y| = n$ . The maximum of the above is when  $|X| = \lfloor \frac{n}{2} \rfloor$  and  $|Y| = \lceil \frac{n}{2} \rceil$ . Thus,  $\mathcal{E}(H) \leq \lfloor \frac{n^2}{4} \rfloor$ . Restore the edges of the cycle  $C$ . We now consider the following subcases:

(2.1). One of  $X$  and  $Y$  contains two edges of the cycle, say  $e_i$  and  $e_j$  belong to  $X$ . Let  $y_1, y_2, \dots, y_{k-1}$  be a set of vertices in  $X - \{x_i, x_{i+1}, x_j, x_{j+1}\}$ . We split this subcase into two subcases:

(2.1.1).  $i$  and  $j$  are not consecutive. Then  $|N_Y(x_i) \cap N_Y(x_{i+1}) \cap N_Y(x_j) \cap N_Y(x_{j+1}) \cap N_Y(y_1) \cap N_Y(y_2) \cap \dots \cap N_Y(y_{k-1})| \leq k+2$ , as otherwise  $G$  contains two edge disjoint  $2k+1$ -cycles. Thus,

$$\mathcal{E}_G(\{x_i, x_{i+1}, x_j, x_{j+1}, y_1, y_2, \dots, y_{k-1}\}, Y) \leq (k+2)|Y| + k+2.$$

So,

$$\begin{aligned} \mathcal{E}(G) &= \mathcal{E}_G(X - \{x_i, x_{i+1}, x_j, x_{j+1}, y_1, y_2, \dots, y_{k-1}\}, Y) \\ &\quad + \mathcal{E}_G(\{x_i, x_{i+1}, x_j, x_{j+1}, y_1, y_2, \dots, y_{k-1}\}, Y) + \mathcal{E}(G[X]) + \mathcal{E}(G[Y]) \\ &\leq (|X| - k - 3)|Y| + (k+2)|Y| + k+2 + 2k+1 \\ &\leq |X||Y| - |Y| + 3k+3 \leq (|X| - 1)|Y| + 3k+3. \end{aligned}$$

Observe that  $|X| + |Y| = n$ . The maximum of the above equation is when  $|Y| = \lceil \frac{n-1}{2} \rceil$  and  $|X| - 1 = \lfloor \frac{n-1}{2} \rfloor$ . Thus,

$$\mathcal{E}(G) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 3k+3.$$

Hence, for  $n \geq 6k+7$ , we have  $\mathcal{E}(G) < \lfloor \frac{n^2}{4} \rfloor + 1$ .

(2.1.2).  $i$  and  $j$  are consecutive, say  $j = i+1$ . Then by following, word by word, the same arguments as in (2.1.1) and by taking into the account that  $|N_Y(x_i) \cap N_Y(x_{i+1}) \cap N_Y(x_{i+2}) \cap N_Y(y_1) \cap N_Y(y_2) \cap \dots \cap N_Y(y_{k-1})| \leq k+1$  and so  $\mathcal{E}(\{x_i, x_{i+1}, x_{i+2}, y_1, y_2, \dots, y_{k-1}\}, Y) \leq (k+1)|Y| + k+1$ , we get the same inequality.

(2.2).  $\mathcal{E}(G[X]) = 1$  and  $\mathcal{E}(G[Y]) = 0$  or  $\mathcal{E}(G[X]) = 0$  and  $\mathcal{E}(G[Y]) = 1$ , say  $e_1 \in E(G[X])$ . Then  $G - e_1$  is a bipartite graph and so as in the above  $\mathcal{E}(G - e_1) \leq \lfloor \frac{n^2}{4} \rfloor$ . Thus,  $\mathcal{E}(G) = \mathcal{E}(G - e_1) + 1 \leq \lfloor \frac{n^2}{4} \rfloor + 1$ .

One can observe from the above arguments that for  $r = 2$  the only time we have equality is in case when  $G$  is obtained by adding an edge to the complete bipartite graph  $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ . This gives rise to the class  $\Omega(n, 2)$ .

**Step 2.** Assume that the result is true for  $r-1$ . We now show the result is true for  $r \geq 3$ . To accomplish that we use similar arguments to those in Step 1. Let  $G \in \mathcal{G}(n; r, 2k+1)$ . If  $G$  contains no  $r-1$  edge disjoint cycles of length  $2k+1$ , then by the inductive step  $\mathcal{E}(G) \leq \lfloor n^2/4 \rfloor + r-2$ . Thus,  $\mathcal{E}(G) < \lfloor n^2/4 \rfloor + r-1$ . So, we need to consider the case when  $G$  has  $r-1$  edge disjoint cycles of length  $2k+1$ . Assume that  $\{C^i = x_{i1}, x_{i2}, \dots, x_{i2k+1}, x_{i1}\}_{i=1}^{r-1}$  be the set of cycles of length  $2k+1$ . Consider  $H = G - \cup_{i=1}^{r-1} E(C^i)$ . Observe that  $H$  cannot have  $2k+1$ -cycles as otherwise  $G$  would have  $r$  of edges disjoint  $2k+1$ -cycles. As in Step 1, we consider two cases:

*Case I.*  $H$  is not a bipartite graph. If  $k \geq 2$ , then by Theorems 1 and 2  $\mathcal{E}(H) \leq \lfloor (n-2)^2/4 \rfloor + 3$ . Thus,  $\mathcal{E}(G) = \mathcal{E}(H) + (r-1)(2k+1) \leq \lfloor \frac{n^2}{4} \rfloor + (r-1) - n + 4 + 2k(r-1)$ . Hence, for  $n > 4 + 2k(r-1)$ , we have  $\mathcal{E}(G) < \lfloor \frac{n^2}{4} \rfloor + r - 1$ . If  $k = 1$ , then by Theorems 5  $\mathcal{E}(H) \leq \lfloor (n-1)^2/4 \rfloor + 1$ .

By using the same argument as in the above, we get that for  $n \geq 4(r-1) + 1$ ,

$$\mathcal{E}(G) < \left\lfloor \frac{n^2}{4} \right\rfloor + 1.$$

*Case II.*  $H$  is a bipartite graph. Let  $X$  and  $Y$  be the partition of  $V(H)$ . Thus,  $\mathcal{E}(H) \leq |X||Y|$ . Observe  $|X| + |Y| = n$ . The maximum of the above is when  $|X| = \lfloor \frac{n}{2} \rfloor$  and  $|Y| = \lceil \frac{n}{2} \rceil$ . Thus,  $\mathcal{E}(H) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$ . Now, we consider the following two subcases:

(II.I). There is  $1 \leq m \leq r-1$  such that  $C^m$  contains at least two edges, say  $e_i = x_{mi}x_{m(i+1)}$  and  $e_j = x_{mj}x_{m(j+1)}$ , joining vertices of one of  $X$  and  $Y$ , say  $X$ . Let  $y_1, y_2, \dots, y_{k-1}$  be a set of vertices in  $X - \{x_{mi}, x_{m(i+1)}, x_{mj}, x_{m(j+1)}\}$ . To this end we have two subcases:

(II.I.I).  $i$  and  $j$  are not consecutive. Then  $|N_Y(x_{mi}) \cap N_Y(x_{m(i+1)}) \cap N_Y(x_{mj}) \cap N_Y(x_{m(j+1)}) \cap N_Y(y_1) \cap N_Y(y_2) \cap \dots \cap N_Y(y_{k-1})| \leq k+2$ , as otherwise  $H \cup \{e_i, e_j\}$  contains two edges disjoint  $2k+1$ -cycles and so  $G$  contains  $r$  edge disjoint cycles of length  $2k+1$ . Thus, as in (2.1.1) of Step 1,

$$\mathcal{E}_H(\{x_{mi}, x_{m(i+1)}, x_{mj}, x_{m(j+1)}, y_1, y_2, \dots, y_{k-1}\}, Y) \leq (k+2)|Y| + k + 2.$$

And so,

$$\begin{aligned} \mathcal{E}(G) &= \mathcal{E}(H) + |\cup_{i=1}^{r-1} E(C^i)| \\ &= \mathcal{E}_H(X - \{x_{mi}, x_{m(i+1)}, x_{mj}, x_{m(j+1)}, y_1, y_2, \dots, y_{k-1}\}, Y) \\ &\quad + \mathcal{E}_H(\{x_{mi}, x_{m(i+1)}, x_{mj}, x_{m(j+1)}, y_1, y_2, \dots, y_{k-1}\}, Y) + |\cup_{i=1}^{r-1} E(C^i)| \\ &\leq (|X| - k - 3)|Y| + (k+2)|Y| + k + 2 + (r-1)(2k+1) \\ &= (|X| - 1)|Y| + k + 2 + (r-1)(2k+1). \end{aligned}$$

Moreover, the maximum of the above inequality is when  $|Y| = \lceil \frac{n-1}{2} \rceil$  and  $|X| - 1 = \lfloor \frac{n-1}{2} \rfloor$ . Thus,

$$\mathcal{E}(G) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + k + 2 + (r-1)(2k+1).$$

Hence, for  $n \geq 6k(r-1) + 7$ , we have  $\mathcal{E}(G) < \lfloor \frac{n^2}{4} \rfloor + (r-1)$ .

(II.I.II).  $i$  and  $j$  are consecutive, say  $j = i + 1$ . Then by following word by word the same arguments as in (2.1.2) of Step 1 and (II.I.I) of Step 2, we get the same inequality

$$\mathcal{E}(G) < \left\lfloor \frac{n^2}{4} \right\rfloor + (r - 1).$$

(II.II). Each  $1 \leq m \leq r - 1$ ,  $C^m$  has exactly one edge belonging to one of  $X$  and  $Y$ . Let  $e$  be the edge of  $C^1$  that belongs to one of  $X$  and  $Y$ . Then  $G - e \in \mathcal{G}(n; r - 1, 2k + 1)$  and so by inductive step,  $\mathcal{E}(G) = \mathcal{E}(G - e) + 1 \leq \left\lfloor \frac{n^2}{4} \right\rfloor + r - 2 + 1 = \left\lfloor \frac{n^2}{4} \right\rfloor + r - 1$ .

We now characterize the extremal graphs. Throughout the proof, we notice that the only time we have equality is in case when  $G$  obtained by adding  $r - 1$  edges to the complete bipartite graph  $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ . This gives rise to the class  $\Omega(n, r)$ . This completes the proof of the theorem. ■

From Theorem 6 and the inequality (1), we get that for  $k \geq 1$ ,  $r \geq 2$  and large  $n$ ,  $f(n; r, 2k + 1) = \left\lfloor \frac{n^2}{4} \right\rfloor + r - 1$ .

#### REFERENCES

- [1] M.S. Bataineh, Some Extremal Problems in Graph Theory, Ph.D Thesis, Curtin University of Technology (Australia, 2007).
- [2] M.S. Bataineh and M.M.M. Jaradat, *Edge maximal  $C_3$  and  $C_5$ -edge disjoint free graphs*, International J. Math. Combin. **1** (2011) 82–87.
- [3] J. Bondy, *Large cycle in graphs*, Discrete Math. **1** (1971) 121–132.  
doi:10.1016/0012-365X(71)90019-7
- [4] J. Bondy, *Pancyclic graphs*, J. Combin. Theory (B) **11** (1971) 80–84.  
doi:10.1016/0095-8956(71)90016-5
- [5] J. Bondy and U. Murty, Graph Theory with Applications (The MacMillan Press, London, 1976).
- [6] S. Brandt, *A sufficient condition for all short cycles*, Discrete Appl. Math. **79** (1997) 63–66.  
doi:10.1016/S0166-218X(97)00032-2
- [7] L. Caccetta, *A problem in extremal graph theory*, Ars Combin. **2** (1976) 33–56.
- [8] L. Caccetta and R. Jia, Edge maximal non-bipartite Hamiltonian graphs without cycles of length 5, Technical Report.14/97. School of Mathematics and Statistics, Curtin University of Technology (Australia, 1997).
- [9] L. Caccetta and R. Jia, *Edge maximal non-bipartite graphs without odd cycles of prescribed length*, Graphs and Combin. **18** (2002) 75–92.  
doi:10.1007/s003730200004

- [10] Z. Füredi, *On the number of edges of quadrilateral-free graphs*, J. Combin. Theory (B) **68** (1996) 1–6.  
doi:10.1006/jctb.1996.0052
- [11] R. Jia, *Some Extremal Problems in Graph Theory*, Ph.D Thesis, Curtin University of Technology (Australia, 1998).
- [12] P. Turán, *On a problem in graph theory*, Mat. Fiz. Lapok **48** (1941) 436–452.

Received 27 August 2010

Revised 15 March 2011

Accepted 12 May 2011