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DOMINATION IN FUNCTIGRAPHS

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Abstract

Let G_1 and G_2 be disjoint copies of a graph G, and let $f : V(G_1) \to V(G_2)$ be a function. Then a *functigraph* $C(G, f) = (V, E)$ has the vertex set $V = V(G_1) \cup V(G_2)$ and the edge set $E = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2), v = f(u)\}$. A functigraph is a generalization of a *permutation graph* (also known as a *generalized prism*) in the sense of Chartrand and Harary. In this paper, we study domination in functigraphs. Let $\gamma(G)$ denote the domination number of G. It is readily seen that $\gamma(G) \leq \gamma(C(G, f)) \leq 2\gamma(G)$. We investigate for graphs generally, and for cycles in great detail, the functions which achieve the upper and lower bounds, as well as the realization of the intermediate values.

Keywords: domination, permutation graphs, generalized prisms, functigraphs.

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1. Introduction and Definitions

Throughout this paper, $G = (V(G), E(G))$ stands for a finite, undirected, simple and connected graph with order $|V(G)|$ and size $|E(G)|$. A set $D \subseteq V(G)$ is a *dominating set* of G if for every vertex $v \in V(G) \setminus D$, there exists a vertex $u \in D$ such that v and u are adjacent. The *domination number* of a graph G, denoted by $\gamma(G)$, is the minimum of the cardinalities of all dominating sets of G. For earlier discussions on domination in graphs, see [3, 4, 10, 16]. For further reading on domination, refer to [13] and [14].

For any vertex $v \in V(G)$, the *open neighborhood* of v in G, denoted by $N_G(v)$, is the set of all vertices adjacent to v in G. The *closed neighborhood* of v, denoted by $N_G[v]$, is the set $N_G(v) \cup \{v\}$. Throughout the paper, we denote by $N(v)$ (resp., $N[v]$) the open (resp., closed) neighborhood of v in $C(G, f)$. The maximum degree of G is denoted by $\Delta(G)$. For a given graph G and $S \subseteq V(G)$, we denote by $\langle S \rangle$ the subgraph induced by S. Refer to [8] for additional graph theory terminology.

Chartrand and Harary studied planar permutation graphs in [7]. Hedetniemi introduced two graphs (not necessarily identical copies) with a function relation between them; he called the resulting object a "function graph" [15]. Independently, Dörfler introduced a "mapping graph", which consists of two disjoint identical copies of a graph and additional edges between the two vertex sets specified by a function [11]. Later, an extension of permutation graphs, called *functigraph*, was rediscovered and studied in [9]. In the current paper, we study domination in functigraphs. We recall the definition of a functigraph in [9].

Definition. Let G_1 and G_2 be two disjoint copies of a graph G, and let f be a function from $V(G_1)$ to $V(G_2)$. Then a functigraph $C(G, f)$ has the vertex set $V(C(G, f)) = V(G_1) \cup V(G_2)$, and the edge set $E(C(G, f)) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2), v = f(u)\}$.

Throughout the paper, $V(G_1)$ denotes the *domain* of a function f; $V(G_2)$ denotes the *codomain* of f; *Range*(f) denotes the *range* of f. For a set $S \subseteq V(G_2)$, we denote by $f^{-1}(S)$ the set of all pre-images of the elements of S; i.e., $f^{-1}(S) = \{v \in V(G_1) \mid f(v) \in S\}$. Also, C_n denotes a cycle of length $n \geq 3$, and *id* denotes the identity function. Let $V(G_1) = \{u_1, u_2, \ldots, u_n\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_n\}$. For simplicity, we sometimes refer to each vertex of the graph G_1 (resp., G_2) by the index i (resp., i') of its label u_i (resp., v_i) for $1 \leq i, i' \leq n$. When $G = C_n$, we assume that the vertices of G_1 and G_2 are labeled cyclically. It is readily seen that $\gamma(G) \leq \gamma(C(G, f)) \leq 2\gamma(G)$. We study the domination of $C(C_n, f)$ in great detail: for $n \equiv 0 \pmod 3$, we characterize the domination number for an infinite class of functions and state conditions under which the upper bound is not achieved; for $n \equiv 1, 2 \pmod 3$, we prove that, for any function f, the domination number of $C(C_n, f)$ is strictly less than $2\gamma(C_n)$. These results extend and generalize a result by Burger, Mynhardt, and Weakley in [6].

Domination number on permutation graphs (generalized prisms) has been extensively investigated in a great many articles, among these are [1, 2, 5, 6, 12]; the present paper primarily deepens — and secondarily broadens — the current state of knowledge.

2. Domination Number of Functigraphs

First we consider the lower and upper bounds of the domination number of $C(G, f)$.

Proposition 1. *For any graph G, $\gamma(G) \leq \gamma(C(G, f)) \leq 2\gamma(G)$.*

Proof. Let D be a dominating set of G. Since a copy of D in G_1 together with a copy of D in G_2 form a dominating set of $C(G, f)$ for any function f, the upper bound follows. For the lower bound, assume there is a dominating set D of $C(G, f)$ such that $|D| < \gamma(G)$. Let $D_1 = D \cap V(G_1) \neq \emptyset$ and $D_2 = D \cap V(G_2) \neq \emptyset$, with $D_1 \cup D_2 = D$. Now, for each $x \in D_1$, x dominates exactly one vertex in G_2, namely $f(x)$. And so $D_2 \cup \{f(x) \mid x \in D_1\}$ is a dominating set of G_2 of cardinality less than or equal to $|D|$, but $|D| < \gamma(G_2)$ — a contradiction. ■

Next we consider realization results for an arbitrary graph G.

Theorem 2. *For any pair of integers a, b such that $1 \leq a \leq b \leq 2a$, there is a connected graph G for which $\gamma(G) = a$ and $\gamma(C(G, f)) = b$ for some function f.*

Proof. Let the star $S_i \cong K_{1,4}$ have center c_i for $1 \leq i \leq a$. Let G be a chain of a stars; i.e., the disjoint union of a stars such that the centers are connected to form a path of length a (and no other additional edges) — see Figure 1. Label the stars in the chain of the domain G_1 by $S_1, S_2, \ldots, S_a$ and label their centers by $c_1, c_2, \ldots, c_a$, respectively. Likewise, label the stars in the chain of the codomain G_2 by $S'_1, S'_2, \ldots, S'_a$ and label their centers by $c'_1, c'_2, \ldots, c'_a$, respectively. More generally, denote by v' the vertex in G_2 corresponding to an arbitrary v in G_1.

We define $a + 1$ functions from G_1 to G_2 as follows. Let f_0 be the "identity function"; i.e., $f_0(v) = v'$. For each i from 1 to a, let f_i be the function which collapses S_1 through S_i to c'_1 through c'_i, respectively, and which acts as the "identity" on the remaining vertices: $f_i(S_j) = c'_j$ for $1 \leq j \leq i$ and $f_i(v) = v'$ for $v \notin \bigcup_{1 \leq j \leq i} V(S_j)$. (See Figure 1.) Notice $\gamma(G) = a$.

Claim. $\gamma(C(G, f_i)) = 2a - i$ for $0 \leq i \leq a$.

First, $\gamma(C(G, f_a)) = a$ because $D_a = \{c'_1, \ldots, c'_a\}$ clearly dominates $C(G, f_a)$.

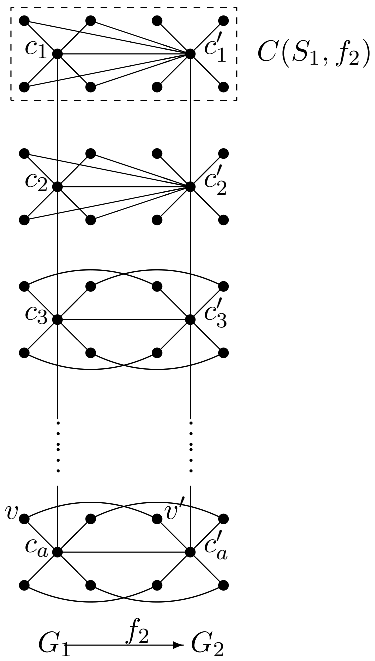


Figure 1. Realization graphs.

Second, consider $C(G, f_0)$. $D_0 = \{c_1, \ldots, c_a, c'_1, \ldots, c'_a\}$, the set of centers in G_1 or G_2, is a dominating set; so $\gamma(C(G, f_0)) \leq 2a$ as noted earlier. It suffices to show that $\gamma(C(G, f_0)) \geq 2a$. It is clear that a dominating set D consisting only of the centers must have size $2a$ — for a pendant to be dominated, its neighboring center must be in D. We need to check that the replacement of centers by some (former) pendants (of G_1 or G_2) will only result in a dominating set D' such that $|D'| > |D_0|$. It suffices to check $C(S_i, f_0)$ at each i, a subgraph of $C(G, f_0)$ – since pendant domination is a local question: the closed neighborhood of each pendant of $C(S_i, f_0)$ is contained within $C(S_i, f_0)$. It is easy to see that the unique minimum dominating set of $C(S_i, f_0)$ consists of the two centers c_i and c'_i.

Finally, the set $D_i = \{c_{i+1}, \ldots, c_a, c'_1, \ldots, c'_a\}$ is a minimum dominating set of $C(G, f_i)$: in relation to $C(G, f_0)$, the subset $\{c_1, \ldots, c_i\}$ of D_0 is not needed since the set $\{c'_1, \ldots, c'_i\}$ dominates $\bigcup_{1 \leq j \leq i} V(S_j)$ in $C(G, f_i)$. The local nature of pendant domination and the fact that $f_i|_{S_j} = f_0|_{S_j}$ for $j > i$ ensure that D_i has minimum cardinality. ∎

3. Characterization of Lower Bound

We now present a characterization for $\gamma(C(G,f)) = \gamma(G)$, in analogy with what was done for permutation-fixers in [5].

Theorem 3. *Let G_1 and G_2 be two copies of a graph G in $C(G,f)$. Then $\gamma(G) = \gamma(C(G,f))$ if, and only if, there are sets $D_1 \subseteq V(G_1)$ and $D_2 \subseteq V(G_2)$ satisfying the following conditions:*

1. *D_1 dominates $V(G_1) \setminus f^{-1}(D_2)$,*
2. *D_2 dominates $V(G_2) \setminus f(D_1)$,*
3. *$D_2 \cup f(D_1)$ is a minimum dominating set of G_2,*
4. *$|D_1| = |f(D_1)|$,*
5. *$D_2 \cap f(D_1) = \emptyset$, and*
6. *$D_1 \cap f^{-1}(D_2) = \emptyset$.*

Proof. ($\Leftarrow$) Suppose there are sets $D_1 \subseteq V(G_1)$ and $D_2 \subseteq V(G_2)$ satisfying the specified conditions. Clearly $D_1 \cup D_2$ is a dominating set of $C(G,f)$. By assumption, $D_2 \cup f(D_1)$ is a minimum dominating set of G_2. Since $|D_1| = |f(D_1)|$ and $D_2 \cap f(D_1) = \emptyset$, $\gamma(G) = \gamma(G_2) = |D_2| + |f(D_1)| = |D_2| + |D_1|$. Since $\gamma(G) \le \gamma(C(G,f)) \le |D_1| + |D_2| = \gamma(G)$, it follows that $\gamma(G) = \gamma(C(G,f))$.

($\Rightarrow$) Let D be any minimum dominating set of $C(G,f)$. Suppose then that $\gamma(G) = \gamma(C(G,f))$ such that $D_1 = D \cap V(G_1)$ and $D_2 = D \cap V(G_2)$. So $\gamma(C(G,f)) = |D_1| + |D_2|$. Note that the only vertices in G_2 that are dominated by D_1 are the vertices in $f(D_1)$ and the only vertices in G_1 that are dominated by D_2 are the vertices in $f^{-1}(D_2)$. Since D is a dominating set of $C(G,f)$, D_2 must dominate every vertex in $V(G_2) \setminus f(D_1)$, and D_1 must dominate every vertex in $V(G_1) \setminus f^{-1}(D_2)$.

Clearly $D_2 \cup f(D_1)$ is a dominating set of G_2. Note that $|D_1| \ge |f(D_1)|$. So $\gamma(G) = \gamma(C(G,f)) = |D_1| + |D_2| \ge |D_2| + |f(D_1)| \ge \gamma(G_2) = \gamma(G)$. But then these terms must all be equal. In particular, $|D_1| = |f(D_1)|$ and $D_2 \cup f(D_1)$ is a minimum dominating set of G_2. Furthermore, $D_2 \cap f(D_1) = \emptyset$, else $D_2 \cup f(D_1)$ is a dominating set of G_2 with fewer than $\gamma(G_2)$ vertices. Finally, suppose there is a vertex $v \in D_1 \cap f^{-1}(D_2)$. So $v \in D_1$ and $v \in f^{-1}(D_2)$. But then $f(v) \in f(D_1)$ and $f(v) \in D_2$. But $f(D_1)$ and D_2 are disjoint. So, $D_1 \cap f^{-1}(D_2) = \emptyset$. ■

It is known that for cycles C_n ($n \ge 3$), $\gamma(C_n) = \lceil \frac{n}{3} \rceil$. We now apply Theorem 3 to characterize the lower bound of $\gamma(C(C_n,f))$.

Theorem 4. *For the cycle C_n ($n \ge 3$), let G_1 and G_2 be copies of C_n. Then $\gamma(C_n) = \gamma(C(C_n,f))$ if, and only if, there is a minimum dominating set $D = D_1 \cup D_2$ of $C(C_n,f)$ such that either:*

1. *$D_1 = \emptyset$ and D_2 is a minimum dominating set of G_2 and $Range(f) \subseteq D_2$, or*

2. $n \equiv 1 \pmod 3$, D_2 *is a minimum dominating set for* $\langle V(G_2) \setminus \{v\}\rangle$, $D_1 = \{w\}$, $f(w) = v$, *and* $f(V(G_1) \setminus N[w]) \subseteq D_2$.

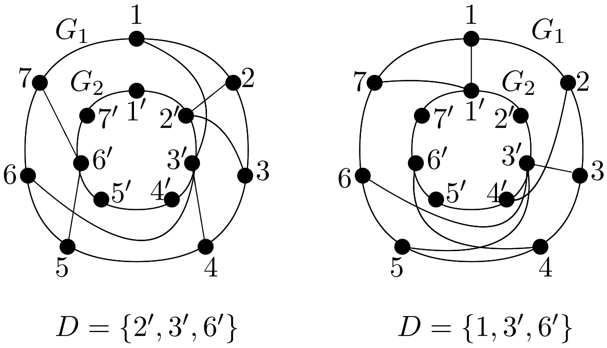


Figure 2. Examples of $\gamma(C(C_n, f)) = \gamma(C_n)$ for $n \equiv 1 \pmod 3$.

Proof. ($\Leftarrow$) Suppose that there is a minimum dominating set D of $C(C_n, f)$ satisfying the specified conditions. So $\gamma(C(C_n, f)) = |D| = |D_1| + |D_2|$. If $D_2 \subseteq V(G_2)$ is a minimum dominating set of C_n and $Range(f) \subseteq D_2$, then $D_1 = \emptyset$. So $\gamma(C_n) = |D_2| = \lceil \frac{n}{3} \rceil$. Furthermore $\gamma(C(C_n, f)) = |D| = |D_1| + |D_2| = 0 + \gamma(G_2)$.

Suppose $n \equiv 1 \pmod 3$, D_2 dominates all but one vertex v of G_2, $D_1 = \{w\}$, $f(w) = v$, and $f(V(G_1) \setminus N[w]) \subseteq D_2$. Note that, since $n \equiv 1 \pmod 3$, $n = 3k + 1$, for some positive integer k, and $\lceil \frac{n}{3} \rceil = k + 1$. By assumption, $\gamma(C(C_n, f)) = |D| = |D_1| + |D_2| = 1 + |D_2|$. Since $\gamma(C_n) = k + 1$, it remains to show that $\gamma(C(C_n, f)) = k + 1$, which is equivalent to showing that $|D_2| = k$. Since D_2 is a minimum dominating set for $\langle V(G_2) \setminus \{v\}\rangle$ and $\langle V(G_2) \setminus \{v\}\rangle$ has domination number k, $|D_2| = k$.

($\Rightarrow$) Now suppose that $\gamma(C_n) = \gamma(C(C_n, f)) = \lceil \frac{n}{3} \rceil$. Let D be a minimum dominating set satisfying the conditions of Theorem 3. There are three cases to consider: $n \equiv 0 \pmod 3$, $n \equiv 1 \pmod 3$, and $n \equiv 2 \pmod 3$. In each case, Theorem 3 implies that $D_2 \cup f(D_1)$ is a minimum dominating set of G_2 and $|D_1| = |f(D_1)|$. Since $f(D_1)$ must include all the vertices not dominated by D_2, it follows that D must contain at least $|D_2| + (n - 3|D_2|) = n - 2|D_2|$ vertices. If $n \equiv 0 \pmod 3$, then $n = 3k$ for some positive integer k and $\lceil \frac{n}{3} \rceil = k$. Note that D_2 dominates at most $3|D_2|$ vertices in G_2. There are at least $n - 3|D_2|$ vertices in G_2 which are not dominated by D_2. If $|D_2| < k$ then $\gamma(C(C_n, f)) = |D| \geq n - 2|D_2| > n - 2k = 3k - 2k = k$, contradicting the assumption that $\gamma(C(C_n, f)) = k$. So $|D_2| = k$. This implies $D_1 = \emptyset$. And this, in turn, implies that D_2 must dominate all the vertices in G_1. So $Range(f) \subseteq D_2$.

In the remaining two cases, where $n \equiv 1$ or $n \equiv 2 \pmod 3$, then $n = 3k + 1$ or $n = 3k + 2$, respectively, for some positive integer k and $\gamma(C_n) = \lceil \frac{n}{3} \rceil = k + 1$.

From Theorem 3 it follows that $D_2 \cup f(D_1)$ is a minimum dominating set of G_2. Since D_2 dominates at most $3|D_2|$ vertices in G_2, D_1 must dominate at least $n - 3|D_2|$ vertices in G_2. If $|D_2| < k$, then $\gamma(C(C_n, f)) = |D| \geq n - 2|D_2| > n - 2k = (3k+1) - 2k = k+1$, contradicting the assumption that $\gamma(C(C_n, f)) = k+1$. So $|D_2| \geq k$. Since $|D| = k + 1$, $|D_2| \leq k + 1$. If $|D_2| = k + 1$, then $D_1 = \emptyset$, $f(D_1) = \emptyset$ and $D_2 \cup f(D_1) = D_2$ is a minimum dominating set of G_2. Since D is a dominating set of $C(C_n, f)$, it follows that D_2 must also dominate all the vertices in D_1 and, thus, $Range(f) \subseteq D_2$.

Let $n \equiv 1 \pmod 3$. If $|D_2| = k$, then there is at least one vertex in G_2 not dominated by D_2. If there are $c > 1$ vertices not dominated by D_2 then these vertices are a subset of $f(D_1)$ and Theorem 3 guarantees that $|D_1| = |f(D_1)| \geq c$ and, thus, $\gamma(C(C_n, f)) \geq k + c > k + 1$, contradicting our assumption. So $c = 1$. There is only one vertex $v \in V(G_2)$ which is not dominated by D_2. D_1 can only contain a single vertex w (or $|D|$ will again be too large) and $f(w) = v$. Since w dominates $N[w]$ in G_1, it follows that D_2 must dominate $V(G_1) \setminus N[w]$. So $f(V(G_1) \setminus N[w]) \subseteq D_2$.

Let $n \equiv 2 \pmod 3$. If $|D_2| = k$, then there are at least two vertices in G_2 not dominated by D_2. But then these vertices must be a subset of $f(D_1)$ and $|f(D_1)| \geq 2$. Since $|D_1| = |f(D_1)|$, $|D_1| \geq 2$. But then $k + 1 = \gamma(C(G, f)) = |D| = |D_1| + |D_2| \geq 2 + k$, which is a contradiction. So $|D_2| = k + 1$. ■

Next we consider the domination number of $C(C_3, f)$.

Lemma 5. *Let G_1 and G_2 be two copies of C_3. Then $\gamma(C(C_3, f)) = 2\gamma(C_3)$ if and only if f is not a constant function.*

Proof. ($\Leftarrow$) Suppose that f is not a constant function. Then, for each vertex $v \in V(C(C_3, f))$, $\deg(v) \leq 4$ and hence $N[v] \subsetneq V(C(C_3, f))$. Thus $\gamma(C(C_3, f)) \geq 2$. Since there exists a dominating set consisting of one vertex from each of G_1 and G_2, $\gamma(C(C_3, f)) = 2$.

($\Rightarrow$) Suppose that f is a constant function, say $f(w) = a$ for some $a \in V(G_2)$ and for all $w \in V(G_1)$. Then $N[a] = V(C(C_3, f))$, and thus $\gamma(C(C_3, f)) = 1 = \gamma(C_3)$. ■

As an immediate consequence of Theorem 4 and Lemma 5, we have the following.

Corollary 6. *There is no permutation f such that $\gamma(C(C_n, f)) = \gamma(C_n)$ for $n = 3$ or $n \geq 5$.*

Now we consider $C(G, f)$ when $G = C_n$ ($n \geq 3$) and f is the identity function.

Theorem 7. *Let G_1 and G_2 be two copies of the cycle C_n for $n \geq 3$. Then*

$$\gamma(C(C_n, id)) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \not\equiv 2 \pmod 4, \\ \frac{n}{2} + 1 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Proof. Since $C(C_n, id)$ is 3-regular, each vertex in $C(C_n, id)$ can dominate 4 vertices. We consider four cases.

Case 1. $n = 4k$. Since $|V(C(C_n, id))| = 8k$, we have $\gamma(C(C_n, id)) \geq \lceil \frac{8k}{4} \rceil = 2k$. Since $\cup_{j=0}^{k-1}\{4j+1, (4j+3)'\}$ is a dominating set of $C(C_n, id)$ with cardinality $2k$, we conclude that $\gamma(C(C_n, id)) = 2k = \lceil \frac{n}{2} \rceil$.

Case 2. $n = 4k+1$. Since $|V(C(C_n, id))| = 2(4k+1) = 8k+2$, we have $\gamma(C(C_n, id)) \geq \lceil \frac{8k+2}{4} \rceil = 2k+1$. Since $(\cup_{j=0}^{k}\{4j+1\}) \bigcup (\cup_{i=0}^{k-1}\{(4i+3)'\})$ is a dominating set of $C(C_n, id)$ with cardinality $2k+1$, we have $\gamma(C(C_n, id)) = 2k+1 = \lceil \frac{n}{2} \rceil$.

Case 3. $n = 4k+2$. Notice that $(\cup_{j=0}^{k}\{4j+1\}) \bigcup (\cup_{i=0}^{k-1}\{(4i+3)'\}) \bigcup \{(4k+2)'\}$ is a dominating set of $C(C_n, id)$ with cardinality $2k+2 = \frac{n}{2}+1$; thus $\gamma(C(C_n, id)) \leq 2k+2$. Since $|V(C(C_n, id))| = 2(4k+2) = 8k+4$, $\gamma(C(C_n, id)) \geq \lceil \frac{8k+4}{4} \rceil = 2k+1$; indeed, $\gamma(C(C_n, id)) = 2k+1$ only if every vertex is dominated by exactly one vertex of a dominating set; i.e., no double domination is allowed. However, we show that there must exist a doubly-dominated vertex for any dominating set by the following *descent* argument: Let the graph A_0 be $P_{4k+3} \times K_2$ where the bottom row is labeled $1, 2, \ldots, 4k+2, 1$ and the top row is labeled $1', 2', \ldots, (4k+2)', 1'$; note that $C(C_n, id)$ is obtained by identifying the two end-edges each with end-vertices labeled 1 and $1'$. Without loss of generality, choose $1'$ to be in a dominating set D. For each vertex to be singly dominated, we delete vertices $1'(s), 1(s), 2'$, and $(4k+2)'$, as well as their incident edges, to obtain a derived graph A_1. In A_1, vertices 2 and $4k+2$ are end-vertices and neither may belong to D as each only dominates two vertices in A_1. This forces support vertices 3 and $4k+1$ in A_1 to be in D. Deleting vertices $2, 3, 3', 4, 4k+2, 4k+1, (4k+1)'$, and $4k$ and incident edges results in the second derived graph A_2. After k iterations, A_k is the extension of $P_3 \times P_2$ by two leaves at both ends of either the top or the bottom row (see Figure 3); A_k, which has eight vertices, clearly requires three vertices to be dominated.

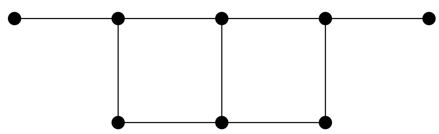


Figure 3. A_k in the $n = 4k+2$ case.

Thus, we conclude that $\gamma(C(C_n, id)) = 2k+2 = \frac{n}{2}+1$.

Case 4. $n = 4k+3$: Since $|V(C(C_n, id))| = 2(4k+3) = 8k+6$, we have $\gamma(C(C_n, id)) \geq \lceil \frac{8k+6}{4} \rceil = 2k+2$. Since $\cup_{j=0}^{k}\{4j+1, (4j+3)'\}$ is a dominating set of $C(C_n, id)$ with cardinality $2k+2$, we conclude that $\gamma(C(C_n, id) = 2k+2 = \lceil \frac{n}{2} \rceil$. ■

As a consequence of Theorem 7, we have the following result.

Corollary 8. (1) *$\gamma(C(C_n, id)) = \gamma(C_n)$ if and only if $n = 4$.*
(2) *$\gamma(C(C_n, id)) = 2\gamma(C_n)$ if and only if $n = 3$ or $n = 6$.*

By Corollary 6 and Theorem 7, we have the following result.

Proposition 9. *For a permutation f, $\gamma(C(C_n, f)) = \gamma(C_n)$ if and only if $C(C_n, f) \cong C(C_4, id)$.*

Proof. ($\Leftarrow$) If $C(C_4, f) \cong C(C_4, id)$, then $\gamma(C_4) = 2 = \gamma(C(C_4, id))$ by Theorem 7.

($\Rightarrow$) Let $\gamma(C(C_n, f)) = \gamma(C_n)$ for $n \geq 3$. By Corollary 6, $n = 4$. If f is a permutation, then $C(C_4, f)$ is isomorphic to the graph (A) or (B) in Figure 4 (refer to [7, 9] for details).

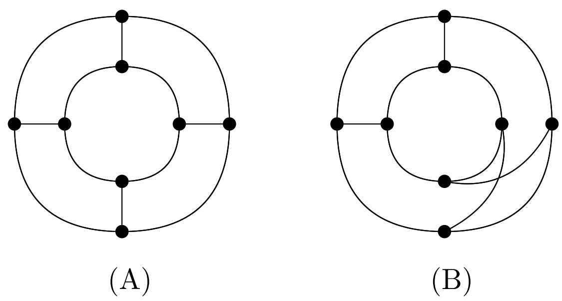


Figure 4. Two non-isomorphic graphs of $C(C_4, f)$ for a permutation f.

If $C(C_4, f) \cong C(C_4, id)$, then we are done. If $C(C_4, f)$ is as in (B) of Figure 4, we claim that $\gamma(C(C_4, f)) \geq 3$.

Since $|V(C(C_4, f))| = 8$ and $C(C_4, f)$ is 3-regular, $D = \{w_1, w_2\}$ dominates $C(C_4, f)$ only if no vertex in $C(C_4, f)$ is dominated by both w_1 and w_2. It suffices to consider two cases, using the fact that $C(C_4, f) \cong C(C_4, f^{-1})$.

(i) $D = \{w_1, w_2\} \subseteq V(G_1)$,

(ii) $w_1 \in V(G_1)$ and $w_2 \in V(G_2)$.

Also, we only need to consider w_1 and w_2 such that $w_1 w_2 \notin E(C(C_4, f))$. By symmetry, there is only one specific case to check in case (i). In case (ii), by fixing a vertex in $V(G_1)$, we see that there are three cases to check. In each case, for any $D = \{w_1, w_2\}$, $N[w_1] \cap N[w_2] \neq \emptyset$. Thus $\gamma(C(C_4, f)) > 2$. ∎

4. Upper Bound of $\gamma(C(C_n, f))$

In this section we investigate domination number of functigraphs for cycles: We show that $\gamma(C(C_n, f)) < 2\gamma(C_n)$ for $n \equiv 1, 2 \pmod 3$. For $n \equiv 0 \pmod 3$, we characterize the domination number for an infinite class of functions and state conditions under which the upper bound is not achieved. Our result in this section generalizes a result of Burger, Mynhardt, and Weakley in [6] which states that no cycle other than C_3 and C_6 is a *universal doubler* (i.e., only for $n = 3, 6$, $\gamma(C(C_n, f)) = 2\gamma(C_n)$ for any permutation f).

4.1. A characterization of $\gamma(C(C_{3k+1}, f))$

Proposition 10. *For any function f, $\gamma(C(C_{3k+1}, f)) < 2\gamma(C_{3k+1})$ for $k \in \mathbb{Z}^+$.*

Proof. Without loss of generality, we may assume that $u_1v_1 \in E(C(C_n, f))$. Since $D = \{v_1\} \cup \{u_{3j}, v_{3j} \mid 1 \le j \le k\}$ is a dominating set of $C(C_{3k+1}, f)$ with $|D| = 2k + 1$ for any function f, $\gamma(C(C_{3k+1}, f)) < 2\gamma(C_{3k+1})$ for $k \in \mathbb{Z}^+$. ∎

4.2. A characterization of $\gamma(C(C_{3k+2}, f))$

We begin with the following example showing $\gamma(C(C_5, f)) < 2\gamma(C_5)$ for any function f.

Example 11. For any function f, $\gamma(C(C_5, f)) < 2\gamma(C_5)$.

Proof. Let $G = C_5$, $V(G_1) = \{1, 2, 3, 4, 5\}$, and $V(G_2) = \{1', 2', 3', 4', 5'\}$. If $|Range(f)| \le 2$, we can choose a dominating set consisting of all vertices in the range and, if necessary, an additional vertex. If $|Range(f)| = 3$, then we can choose the range as a dominating set.

So, let $|Range(f)| \ge 4$. Then f is bijective on at least three vertices in the domain and their image. By the pigeonhole principle, there exist two adjacent vertices, say 1 and 2, on which f is bijective. Let $f(1) = 1'$. Then, by relabeling if necessary, $f(2) = 2'$ or $f(2) = 3'$. Suppose $f(2) = 3'$. Then $D = \{1', 3', 4\}$ forms a dominating set, and we are done. Suppose then $f(2) = 2'$. We consider two cases.

Case 1. $|Range(f)| = 4$. By symmetry, $5' \notin Range(f)$ is the same as $3' \notin Range(f)$. So, consider two distinct cases, $5' \notin Range(f)$ and $4' \notin Range(f)$. If $5' \notin Range(f)$, then $D = \{1, 3', 4'\}$ forms a dominating set. If $4' \notin Range(f)$, then $D = \{1, 3', 5'\}$ forms a dominating set. In either case, we have $\gamma(C(C_5, f)) < 2\gamma(C_5)$.

Case 2. f is a bijection (permutation). Recall $f(1) = 1'$ and $f(2) = 2'$; there are thus 3!=6 permutations to consider. Using the standard cycle notation, the permutations are $(3,4)$, $(3,5)$, $(4,5)$, $(3,4,5)$, $(3,5,4)$, and identity. However, they induce only four non-isomorphic graphs, since $(3,4)$ and $(4,5)$ induce isomorphic graphs and $(3,4,5)$ and $(3,5,4)$ induce isomorphic graphs. If f is either $(3,4)$ or $(3,4,5)$, then $D = \{2, 3', 5'\}$ is a dominating set. If f is $(3,5)$, then $D = \{1', 3, 3'\}$ is a dominating set. When f is the identify function, $D = \{1', 3, 5'\}$ is a dominating set. It is thus verified that $\gamma(C(C_5, f)) < 2\gamma(C_5)$. ■

Remark 12. Example 11 has the following implication. Given $C(C_{3k+2}, f)$ for $k \in \mathbb{Z}^+$, suppose there exist five consecutive vertices being mapped by f into five consecutive vertices. Then $\gamma(C(C_{3k+2}, f)) < 2\gamma(C_{3k+2}) = 2k+2$, and here is a proof. Relabeling if necessary, we may assume that $\{u_1, u_2, u_3, u_4, u_5\}$ are mapped into $\{v_1, v_2, v_3, v_4, v_5\}$; let $S = \{u_i, v_i \mid 1 \le i \le 5\}$. Then $\langle S \rangle$ in $C(C_{3k+2}, f)$ and the additional edge set $\{u_1u_5, v_1v_5\}$ form a graph isomorphic to a $C(C_5, f)$, which has a dominating set S_0 with $|S_0| \le 3$. In $C(C_{3k+2}, f)$, if S is dominated by S_0, then $D = S_0 \cup \{u_{3j+1} \mid 2 \le j \le k\} \cup \{v_{3j+1} \mid 2 \le j \le k\}$ forms a dominating set for $C(C_{3k+2}, f)$ with at most $2k+1$ vertices. If u_1 is not dominated by S_0 in $C(C_{3k+2}, f)$, then it is dominated solely by u_5 of S_0 in $C(C_5, f)$. But then u_6 is dominated by u_5 in $C(C_{3k+2}, f)$ and we can replace $\{u_{3j+1} \mid 2 \le j \le k\}$ with $\{u_{3j+2} \mid 2 \le j \le k\}$ to form D. Similarly, if u_5 is not dominated by S_0 in $C(C_{3k+2}, f)$, then it is dominated solely by u_1 of S_0 in $C(C_5, f)$. Then u_{3k+2} is dominated by u_1 in $C(C_{3k+2}, f)$ and we can replace $\{u_{3j+1} \mid 2 \le j \le k\}$ with $\{u_{3j} \mid 2 \le j \le k\}$ to form D. The cases where v_1 or v_5 is not dominated by S_0 in $C(C_{3k+2}, f)$ can be likewise handled. Thus, if five consecutive vertices are mapped by f into five consecutive vertices, then $\gamma(C(C_{3k+2}, f)) \le 2k+1 < 2k+2 = 2\gamma(C_{3k+2})$.

Remark 13. Unlike $C(C_5, f)$, it is easily checked that $\gamma(C(P_5, f)) = 2\gamma(P_5)$ for the function f given in Figure 5, where P_5 is the path on five vertices.

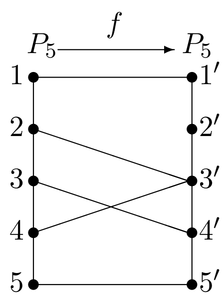


Figure 5. An example where $\gamma(C(P_5, f)) = 2\gamma(P_5)$.

Now we consider the domination number of $C(C_{3k+2}, f)$ for a non-permutation function f, where $k \in \mathbb{Z}^+$.

Theorem 14. *Let $f : V(C_{3k+2}) \to V(C_{3k+2})$ be a function which is not a permutation. Then $\gamma(C(C_{3k+2}, f)) < 2\gamma(C_{3k+2}) = 2k + 2$.*

Proof. Suppose f is a function from C_{3k+2} to C_{3k+2} and f is not a permutation. There must be a vertex v_1 in G_2 such that $\deg(v_1) \geq 4$ in $C(C_{3k+2}, f)$. Define the sets $V_1 = \{v_{3i+1} \mid 0 \leq i \leq k\}$, $V_2 = \{v_{3i+2} \mid 0 \leq i \leq k\}$, and $V_3 = \{v_{3i} \mid 1 \leq i \leq k\} \cup \{v_1\}$. Notice that each of these three sets is a minimum dominating set of G_2 of cardinality $k + 1$. Also, notice that $|f^{-1}(V_1)| + |f^{-1}(V_2)| + |f^{-1}(V_3)|$ counts every vertex in the pre-image of $V(G_2) \setminus \{v_1\}$ once and every vertex in the pre-image of $\{v_1\}$ twice, so $|f^{-1}(V_1)| + |f^{-1}(V_2)| + |f^{-1}(V_3)| \geq 3k + 4$. By the Pigeonhole Principle, $|f^{-1}(V_i)| \geq \lceil \frac{3k+4}{3} \rceil = k + 2$ for some i. Set $D_2 = V_i$ for this i and notice that D_2 is a dominating set of G_2 with cardinality $k + 1$ and $|f^{-1}(D_2)| \geq k + 2$.

Without loss of generality, we may assume that u_1 is in $f^{-1}(D_2)$. If there exists $0 \leq i \leq k$ such that u_{3i+2} is also in the pre-image of D_2, then $D_1 = \{u_{3j} \mid 1 \leq j \leq i\} \cup \{u_{3j+1} \mid i+1 \leq j \leq k\}$ dominates the remaining vertices of G_1. Otherwise, there are at least $k+1$ vertices in $f^{-1}(D_2) \cap \{u_{3j}, u_{3j+1} \mid 1 \leq j \leq k\}$. By the Pigeonhole Principle, there exist two vertices u_{3j_0} and u_{3j_0+1} in $f^{-1}(D_2)$ which are adjacent in G_1. Then $D_1 = \{u_1\} \cup \{u_{3j+1} \mid 1 \leq j \leq j_0 - 1\} \cup \{u_{3j'} \mid j_0 + 1 \leq j' \leq k\}$ dominates the remaining vertices of G_1. In either case, $D_1 \cup D_2$ is a dominating set of $C(C_{3k+2}, f)$ with $2k + 1$ vertices. ■

For $G_i \subseteq C(G, f)$ $(i = 1, 2)$, the distance between x and y in $\langle V(G_i) \rangle$ is denoted by $d_{G_i}(x, y)$.

Theorem 15. *Let $f : V(C_{3k+2}) \to V(C_{3k+2})$ be a function, where $k \in \mathbb{Z}^+$. For the cycle C_{3k+2}, if there exist two vertices x and y in G_1 such that $d_{G_1}(x, y) \equiv 1 \pmod 3$ and $d_{G_2}(f(x), f(y)) \not\equiv 1 \pmod 3$, then $\gamma(C(C_{3k+2}, f)) < 2\gamma(C_{3k+2})$.*

Proof. Let $x = 1$ and $y = 3a + 2$ for a nonnegative integer a. By relabeling, if necessary, we may assume that $f(x) = 1'$. Note that $D_1 = (\cup_{i=1}^{a}\{3i\}) \cup (\cup_{i=a+1}^{k}\{3i + 1\})$ dominates vertices in $V(G_1) \setminus \{x, y\}$. If $f(x) = 1' = f(y)$, let D_2 be any minimum dominating set of G_2 containing $1'$. Then $D = D_1 \cup D_2$ is a dominating set of $C(C_{3k+2}, f)$ with $|D| \leq 2k + 1$. Thus, we assume that $f(x) \neq f(y)$. Since $d_{G_2}(f(x), f(y)) \not\equiv 1 \pmod 3$, $f(y) = (3\ell)'$ or $f(y) = (3\ell + 1)'$ for some ℓ $(1 \leq \ell \leq k)$. First, consider when $\ell > 1$. If $f(y) = (3\ell)'$, let $D_2 = (\cup_{i=1}^{\ell-1}\{(3i + 1)'\}) \cup (\cup_{i=\ell+1}^{k}\{(3i)'\}) \cup \{1', (3\ell)'\}$; and if $f(y) = (3\ell + 1)'$, let $D_2 = (\cup_{i=1}^{\ell-1}\{(3i + 1)'\}) \cup (\cup_{i=\ell+1}^{k}\{(3i + 1)'\}) \cup \{1', (3\ell + 1)'\}$. Second, consider when $\ell = 1$. If $f(y) = (3\ell)'$, let $D_2 = (\cup_{i=1}^{k}\{(3i)'\}) \cup \{1'\}$; if $f(y) = (3\ell + 1)'$, let $D_2 = (\cup_{i=1}^{k}\{(3i + 1)'\}) \cup \{1'\}$. Notice that D_2 dominates $V(G_2) \cup \{x, y\}$ in each case. Thus $D = D_1 \cup D_2$ is a dominating set of $C(C_{3k+2}, f)$ with $|D| = |D_1| + |D_2| = k + k + 1 = 2k + 1 < 2\gamma(C_{3k+2}) = 2k + 2$. ■

Next we consider $C(C_{3k+2}, f)$ for a permutation f.

Lemma 16. *Let f be a monotone increasing function from $S = \{1, 2, \ldots, n\}$ to $\mathbb{Z}$ such that $f(1) = 1$. If $|j - i| \equiv 1 \pmod 3$ implies $|f(j) - f(i)| \equiv 1 \pmod 3$ for any $i, j \in S$, then $f(i) \equiv i \pmod 3$.*

Proof. The monotonicity of f — and the rest of the hypotheses — provides that $f(i+1) - f(i) \equiv 1 \pmod 3$, for each $1 \le i < n$; apply it inductively to reach the conclusion. ■

Theorem 17. *Let $G = C_{3k+2}$ for a positive integer k, and let $f : V(G_1) \to V(G_2)$ be a permutation, where the vertices in both the domain and codomain are labeled 1 through $3k+2$. Assume*

(1) $$d_{G_2}(f(x), f(y)) \equiv 1 \pmod 3 \text{ whenever } d_{G_1}(x, y) \equiv 1 \pmod 3.$$

If $f(1) = 1$, then $C(C_{3k+2}, f) \cong C_{3k+2} \times K_2$.

Proof. Denote by $F(n)$ the sequence of inequalities $f(1) < f(2) < \cdots < f(n-1) < f(n)$. By cyclically relabeling (equivalent to going to an isomorphic graph) if necessary, we may assume $F(3)$; now the graph $C(C_{3k+2}, f)$, along with the labeling of all its vertices, is fixed. Without loss of generality, let $f(1) = 1$, $f(2) = 3y_0+2$, and $f(3) = 3z_0+3$ for $0 \le y_0 \le z_0 < k$. Notice $|x-y| \equiv 1 \pmod 3$ if and only if $d_G(x, y) \equiv 1 \pmod 3$ for $G = C_{3k+2}$; we will use $|\cdot|$ in distance considerations. We will prove that f is monotone increasing on vertices in G_1 (and hence f is the identity function) in two steps: Step I is the extension to $F(5)$ from $F(3)$. Step II is the extension to $F(3(m+1)+2)$ from $F(3m+2)$ if $1 \le m \le k-1$.

Step I. Suppose for the sake of contradiction that $F(5)$ is false. We first prove $F(4)$ and then $F(5)$.

Suppose $f(4) < f(3)$. This means, by condition (1), that $f(4) \equiv 2 \pmod 3$. If $f(5) < f(4)$, then condition (1) implies $f(5) \equiv 1 \pmod 3$. If $f(5) > f(4)$, then condition (1) implies $f(5) \equiv 0 \pmod 3$. Now notice $|1-5| \equiv 1 \pmod 3$. If $f(5) < f(4)$, then $|f(1) - f(5)| = f(5) - f(1) \equiv 0 \pmod 3$; if $f(5) > f(4)$, then $|f(1) - f(5)| = f(5) - f(1) \equiv 2 \pmod 3$. In either case, condition (1) is violated. Thus $f(3) < f(4)$, and $f(4) \equiv 1 \pmod 3$.

Suppose $f(5) < f(4)$. This means, by condition (1), that $f(5) \equiv 0 \pmod 3$. Then $|f(1) - f(5)| = f(5) - f(1) \equiv 2 \pmod 3$, which contradicts condition (1) since, again, $|1-5| \equiv 1 \pmod 3$. Thus we have $f(4) < f(5)$, and $f(5) \equiv 2 \pmod 3$.

Step II. Suppose $F(3m+2)$ for $1 \le m \le k-1$; we will show $F(3(m+1)+2)$. Observe that

(2) $$f(3m+5) - f(1) \equiv 1 \pmod 3 \text{ implies } f(3m+5) \equiv 2 \pmod 3.$$

First, assume $f(3m+3) < f(3m+2)$. This means, by condition (1) and Lemma 16, that $f(3m+3) \equiv 1 \pmod 3$. Assuming $f(3m+4) > f(3m+3)$, then $f(3m+4) \equiv 2 \pmod 3$; which in turn implies that $f(3m+5) \equiv 0$ or $1 \pmod 3$, either way a contradiction to (2). Assuming $f(3m+4) < f(3m+3)$, then $f(3m+4) \equiv 0 \pmod 3$; however, comparing with $f(3)$, $f(3m+4) \equiv 1$ or $2 \pmod 3$, either way a contradiction again. We have thus shown that $f(3m+3) > f(3m+2)$, which means $f(3m+3) \equiv 0 \pmod 3$.

Second, assume $f(3m+4) < f(3m+3)$. This means, by condition (1) and Lemma 16, that $f(3m+4) \equiv 2 \pmod 3$. Assuming $f(3m+5) > f(3m+4)$, we have $f(3m+5) \equiv 0 \pmod 3$. Assuming $f(3m+5) < f(3m+4)$, we have $f(3m+5) \equiv 1 \pmod 3$. Either way we reach a contradiction to (2). We have thus shown that $f(3m+4) > f(3m+3)$, which means $f(3m+4) \equiv 1 \pmod 3$.

Finally, assume $f(3m+5) < f(3m+4)$. This means, by condition (1) and Lemma 16, that $f(3m+5) \equiv 0 \pmod 3$, which is a contradiction to (2). Thus, $f(3m+5) > f(3m+4)$ and $f(3m+5) \equiv 2 \pmod 3$. ∎

Theorem 18. *For any function f, $\gamma(C(C_{3k+2}, f)) < 2\gamma(C_{3k+2})$, where $k \in \mathbb{Z}^+$.*

Proof. Combine Theorem 7, Theorem 14, Theorem 15, and Theorem 17. ∎

4.3. Towards a characterization of $\gamma(C(C_{3k}, f))$

Definition. Let f be a function from $S = \{1, 2, \ldots, 3k\}$ to itself. We say f is a *three-translate* if $f(x+3i) = f(x) + 3i$ for $x \in \{1, 2, 3\}$ and $i \in \{0, 1, \ldots, k-1\}$. Let $\widetilde{f} = f|_{\{1,2,3\}}$.

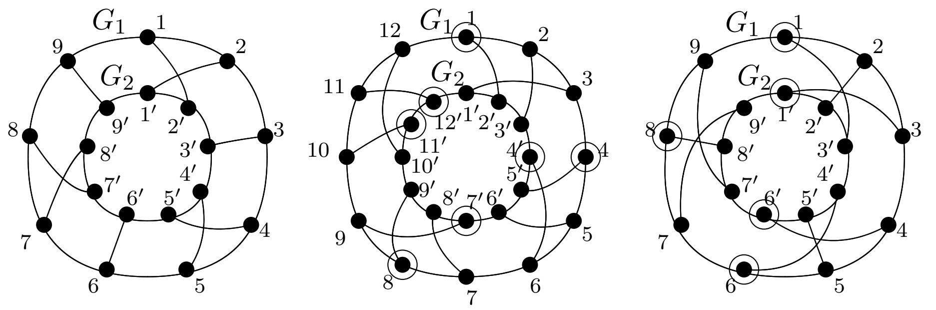
Notation. Denote by $\widetilde{f} = (a_1, a_2, a_3)$ the function such that $\widetilde{f}(1) = a_1$, $\widetilde{f}(2) = a_2$, and $\widetilde{f}(3) = a_3$. We use $C(C_{3k}, f)$ and $C(C_{3k}, \widetilde{f})$ interchangeably when f is a three-translate.

First consider $C(C_{3k}, f)$ for a three-translate permutation f.

Theorem 19. *Let f be a three-translate permutation and let $k \geq 4$. Then $\gamma(C(C_{3k}, f)) = 2k = 2\gamma(C_{3k})$ if and only if $\widetilde{f}$ is $(2, 1, 3)$ or $(1, 3, 2)$.*

Proof. Notice that $\widetilde{f}$ is one of the six permutations: identity, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, and $(3, 2, 1)$. First, the identity does not attain the upper bound for $k \geq 3$ by Corollary 8. Second, the permutations $(2, 3, 1)$ and $(3, 1, 2)$ are inverses of each other and induce isomorphic graphs in $C(C_{3k}, f)$; they do not attain the upper bound for $k \geq 4$: $D = \{1, 4, 8, 4', 7', 11', 12'\}$ is a dominating set of $C(C_{12}, f)$ where $\widetilde{f} = (2, 3, 1)$ (see (B) of Figure 6). Third, the transposition $(3, 2, 1)$ fails to attain the upper bound for $k \geq 3$: $D = \{1, 6, 8, 1', 6'\}$ is a dominating set of $C(C_9, f)$ (see (C) of Figure 6). When $\widetilde{f}$ is $(2, 3, 1)$ or $(3, 1, 2)$ or $(3, 2, 1)$, one can readily see how to extend a dominating set from k to $k+1$. Lastly, the transpositions $(1, 3, 2)$ and $(2, 1, 3)$ induce isomorphic graphs in $C(C_{3k}, f)$.

Claim. If $\widetilde{f}$ is $(1,3,2)$ or $(2,1,3)$, then $\gamma(C(C_{3k},f)) = 2k = 2\gamma(C_{3k})$ for each $k \geq 3$.



(A) (B) (C)

Figure 6. Examples of $C(C_{3k},f)$ for three-translate permutations f when $k \geq 3$.

For definiteness, let $\widetilde{f} = (2,1,3)$ (see (A) of Figure 6). For the sake of contradiction, assume $\gamma(C(C_{3k},f)) < 2\gamma(C_{3k}) = 2k$ and consider a minimum dominating set D for $C(C_{3k},f)$. We can partition the vertices into k sets $S_i = \{u_{3i-2}, u_{3i-1}, u_{3i}, v_{3i-2}, v_{3i-1}, v_{3i}\}$ for $1 \leq i \leq k$. By the Pigeonhole Principle, $|D \cap S_i| \leq 1$ for some i. Without loss of generality, we assume that $|D \cap S_1| \leq 1$. Since neither u_2 nor v_2 has a neighbor that is not in S_1, $D \cap S_1$ must be either $\{u_1\}$ or $\{v_1\}$ — in order for both u_2 and v_2 to be dominated by only one vertex.

Notice that u_3 and v_3 are dominated neither by u_1 nor by v_1, so $D \cap S_2$ must contain both u_4 and v_4. But then either $|D \cap S_2| \geq 3$ or u_6 and v_6 are not dominated by any vertex in $D \cap S_2$: if $|D \cap S_2| \geq 3$, we start the argument anew at S_3; thus we may, without loss of generality, assume u_6 and v_6 are not dominated by any vertex in $D \cap S_2$ and $|D \cap S_2| = 2$. This forces u_7 and v_7 to be in D, but this still leaves u_9 and v_9 un-dominated by any vertex in $\cup_{i=1}^{3}(D \cap S_i)$. Again, if $|D \cap S_3| \geq 3$, we start the argument anew at S_4. Thus, we may assume u_9 and v_9 are not dominated by any vertex in $\cup_{i=1}^{3}(D \cap S_i)$.

This pattern (allowing restarts) is forced to persist if $\gamma(C(C_{3k},f)) < 2k$. Now, one of two situations prevails for U_k. First, the argument begins anew at U_k. In this case, even if u_{3k-2} and v_{3k-2} are dominated by vertices outside S_k, one still has $|D \cap S_k| \geq 2$, and hence $|D| \geq 2k$. Second, the vertices u_{3k-2} and v_{3k-2} are already in D. And if $|D \cap S_k| = 2$, then either u_{3k} or v_{3k} is left un-dominated. Therefore, $|D \cap S_k| \geq 3$; this means $|D| \geq 2k$, contradicting the original hypothesis. ■

Remark 20. For $k \in \mathbb{Z}^+$, one can readily check that $\gamma(C(C_{12k},(2,3,1))) = \gamma(C(C_{12k},(3,1,2))) \leq 7k$ and $\gamma(C(C_{9k},(3,2,1))) \leq 5k$.

Next we consider $C(C_{3k}, f)$ for a non-permutation three-translate f. Note that constant three-translates (i.e., $\tilde{f}$ = constant) never achieve the upper bound.

Remark 21. It is easy to check that there are five non-isomorphic and non-constant three-translates which are not permutations for $k \geq 3$. That is,

(i) $C(C_{3k},(1,1,2)) \cong C(C_{3k},(1,1,3)) \cong C(C_{3k},(1,2,2)) \cong C(C_{3k},(2,2,3)) \cong C(C_{3k},(1,3,3)) \cong C(C_{3k},(2,3,3))$;

(ii) $C(C_{3k},(1,2,1)) \cong C(C_{3k},(2,1,2)) \cong C(C_{3k},(2,3,2)) \cong C(C_{3k},(3,2,3))$;

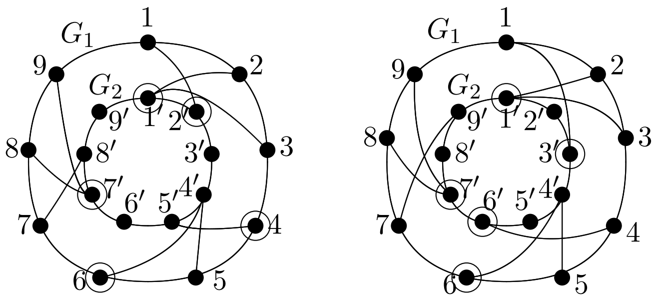
(iii) $C(C_{3k},(2,1,1)) \cong C(C_{3k},(2,2,1)) \cong C(C_{3k},(3,2,2)) \cong C(C_{3k},(3,3,2))$;

(iv) $C(C_{3k},(1,3,1)) \cong C(C_{3k},(3,1,3))$;

(v) $C(C_{3k},(3,1,1)) \cong C(C_{3k},(3,3,1))$.

Theorem 22. *Let f be a three-translate which is not a permutation and let $k \geq 3$. Then $\gamma(C(C_{3k}, \widetilde{f})) = 2k = 2\gamma(C_{3k})$ if and only if $C(C_{3k}, \widetilde{f}) \cong C(C_{3k},(1,1,2))$ or $C(C_{3k}, \widetilde{f}) \cong C(C_{3k},(1,2,1))$ or $C(C_{3k}, \widetilde{f}) \cong C(C_{3k}, (1,3,1))$.*

Proof. There are 21 functions which are not permutations from $S = \{1,2,3\}$ to itself. The three constant functions obviously fail to achieve the upper bound (if $\widetilde{f} \equiv$ constant, then $\gamma(C(C_{3k}, \widetilde{f})) = \gamma(C_{3k}) = k$); so there are 18 non-permutation functions to consider. By Remark 21, we need to consider five non-isomorphic classes.



$D = \{4, 6, 1', 2', 7'\}$ (A)

$D = \{6, 1', 3', 6', 7'\}$ (B)

Figure 7. Examples of $\gamma(C(C_{3k}, f))$ such that $\gamma(C(C_{3k}, f)) < 2\gamma(C_{3k})$ for non-permutation three-translates f and for $k \geq 3$.

First, we consider when the domination number of $C(C_{3k}, f)$ is less than $2\gamma(C_{3k}) = 2k$. If $C(C_{3k}, \widetilde{f}) \cong C(C_{3k},(2,1,1))$, then $D = \{4, 6, 1', 2', 7'\}$ is a dominating set of $C(C_9,(2,1,1))$ (see (A) of Figure 7).
If $C(C_{3k}, \widetilde{f}) \cong C(C_{3k},(3,1,1))$, then $D = \{6, 1', 3', 6', 7'\}$ is a dominating set

of $C(C_9,(3,1,1))$ (see (B) of Figure 7). In each case, $|D| = 5 < 2\gamma(C_9)$, and one can readily see how to extend a dominating set from k to $k+1$ such that $\gamma(C(C_{3k},\widetilde{f})) < 2\gamma(C_{3k}) = 2k$.

Second, we consider $C(C_{3k},\widetilde{f}) \cong C(C_{3k},(1,1,2))$ or $C(C_{3k},\widetilde{f}) \cong C(C_{3k},(1,2,1))$ or $C(C_{3k},\widetilde{f}) \cong C(C_{3k},(1,3,1))$ (see Figure 8). In all three cases, $\gamma(C(C_{3k},\widetilde{f})) = 2\gamma(C_{3k})$ and our proofs for the three cases agree in the main idea but differ in details.

Here is the main idea. Since one can explicitly check the few cases when $k < 3$, assume $k \geq 3$. In all three cases, we view $C(C_{3k},\widetilde{f})$ as the union of k subgraphs $\langle U_i \rangle$ for $1 \leq i \leq k$, where $U_i = \{u_{3i-2}, u_{3i-1}, u_{3i}, v_{3i-2}, v_{3i-1}, v_{3i}\}$, together with two additional edges between U_i and U_j exactly when $i - j \equiv -1$ or 1 (mod k). For each i, the presence of internal vertices in U_i (vertices which can not be dominated from outside of U_i) imply the inequality $|D \cap U_i| \geq 1$. Assuming, for the sake of contradiction, that there exists a minimum dominating set D with $|D| < 2k$, we conclude, by the pigeonhole principle, the existence of a "deficient U_p" (i.e., $|D \cap U_p| = 1 < 2$). Starting at this U_p and sequentially going through each U_i, we can argue that this deficient U_p is necessarily compensated (or "paired off") by an "excessive U_q" (i.e., $|D \cap U_q| > 2$). Going through all indices in $\{1,2,\ldots,k\}$, we are forced to conclude that $|D| \geq 2k$, contradicting our hypothesis. To avoid undue repetitiveness, we provide a detailed proof only in one of the three cases, the case of $C(C_{3k},(1,3,1))$, which is isomorphic to $C(C_{3k},(3,1,3))$.

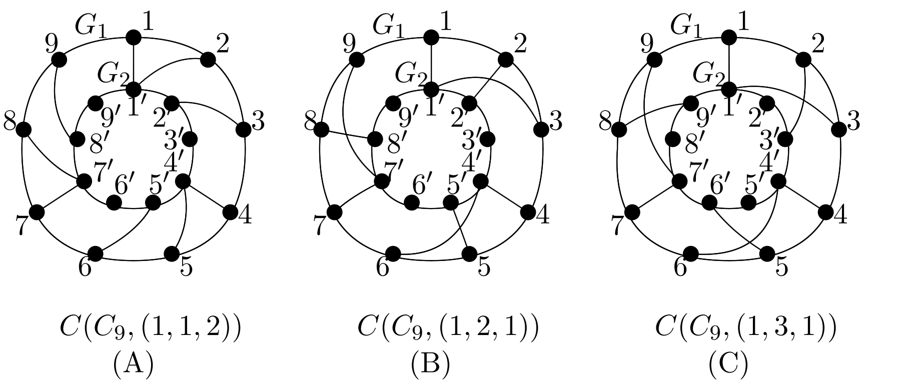


Figure 8. Examples of $C(C_{3k}, f)$ such that $\gamma(C(C_{3k}, f)) = 2\gamma(C_{3k})$ for non-permutation three-translates f and for $k \geq 3$.

Claim. If $C(C_{3k},\widetilde{f}) \cong C(C_{3k},(3,1,3))$, then $\gamma(C(C_{3k}, f)) = 2k = 2\gamma(C_{3k})$.

Proof of Claim. The assertion may be explicitly verified for $k < 4$; so let $k \geq 4$. For the sake of contradiction, assume $\gamma(C(C_{3k}, f)) < 2k$ and consider a minimum dominating set D for $C(C_{3k}, f)$. We can partition the vertices into

k sets $U_i = \{u_{3i-2}, u_{3i-1}, u_{3i}, v_{3i-2}, v_{3i-1}, v_{3i}\}$ for $1 \le i \le k$. By the Pigeonhole Principle, $|D \cap U_i| \le 1$ for some i. Without loss of generality, we assume that $|D \cap U_1| \le 1$. Since neither u_2 nor v_2 has a neighbor that is not in U_1, $D \cap U_1$ must be $\{v_1\}$ — the only vertex to dominate both u_2 and v_2.

Notice that u_3 and v_3 are not dominated by v_1, the only vertex in $D \cap U_1$, so $D \cap U_2$ must contain both u_4 and v_4. But then either $|D \cap U_2| \ge 3$ or u_6 is not dominated by any vertex in $D \cap U_2$, if $|D \cap U_2| \ge 3$, we start the argument anew at U_3; thus we may, without loss of generality, assume u_6 is not dominated by any vertex in $D \cap U_2$. This forces u_7, which dominates u_6, u_8, and v_9, to be in D. Now, for v_7 and v_8 to be dominated, one of them must be in D. But this still leaves u_9 un-dominated by any vertex in $\cup_{i=1}^3 U_i$. Again, if $|D \cap U_3| \ge 3$, we start the argument anew at U_4. Thus, we may, without loss of generality, assume u_9 is not dominated by any vertex in $\cup_{i=1}^3 U_i$.

This pattern (allowing restarts) is forced to persist if $\gamma(C(C_{3k}, f)) < 2k$. Now, one of two situations prevails for U_k: first, the argument begins anew at U_k. In this case, even if u_{3k-2} and v_{3k-2} are dominated by vertices outside of U_k, one still has $|D \cap U_k| \ge 2$, and hence $|D| \ge 2k$. Second, the vertices u_{3k-2} and either v_{3k-2} or v_{3k-1} are already in D. And if $|D \cap U_k| = 2$, then u_{3k} (and, for that matter, u_1) is left un-dominated. Therefore, $|D \cap U_k| \ge 3$ and $|D| \ge 2k$, contradicting the original hypothesis. ■

Now, we consider sufficient conditions for $\gamma(C(C_{3k}, f)) < 2\gamma(C_{3k})$ in terms of the maximum and the average degree of $C(C_{3k}, f)$, respectively.

Proposition 23. *If $\Delta(C(C_{3k}, f)) \ge k + 5$, then $\gamma(C(C_{3k}, f)) < 2\gamma(C_{3k})$.*

Proof. Suppose $C(C_{3k}, f)$ is a functigraph with maximum degree at least $k+5$. Without loss of generality, we assume that the degree of v_1 is at least $k+5$. Partition the vertices of G_1 into k sets $U_i = \{u_{3i-2}, u_{3i-1}, u_{3i}\}$, where $1 \le i \le k$. If $N[v_1]$ contains any set U_i, say $U_1 \subseteq N[v_1]$, then $\{u_i \mid i \ge 5 \text{ and } i \equiv 2 \pmod 3\} \cup \{v_i \mid i \equiv 1 \pmod 3\}$ is a dominating set of $C(C_{3k}, f)$ with $2k-1$ vertices. Thus, we may assume that $|N[v_1] \cap U_i| \le 2$ for each i. It follows that $|N[v_1] \cap U_i| = 2$ for at least 3 different values of i, say $i = p, q$, and r. Let x, y, and z be the vertices in G_1 that are in U_p, U_q, U_r (respectively) and not in $N[v_1]$.

Suppose one of x, y, and z, say x, maps to a vertex v_{3j+1} for some j. Then $\{u_\ell \mid \ell \equiv 2 \pmod 3 \text{ and } \ell \ne 3p-1\} \cup \{v_\ell \mid \ell \equiv 1 \pmod 3\}$ is a dominating set of $C(C_{3k}, f)$ with $2k-1$ vertices. Otherwise, two of x, y, and z, say x and y, map to vertices v_s and v_t such that $s \equiv t \pmod 3$, say $s \equiv t \equiv 0 \pmod 3$, without loss of generality. But then the set $\{u_\ell \mid \ell \equiv 2 \pmod 3, \ell \ne 3p-1, \text{ and } \ell \ne 3q-1\} \cup \{v_1\} \cup \{v_\ell \mid \ell \equiv 0 \pmod 3\}$ is a dominating set of $C(C_{3k}, f)$ with $2k-1$ vertices. ■

The following example shows that the bound provided in Proposition 23 is nearly sharp. Namely, there exists a function $f : V(C_{3k}) \to V(C_{3k})$ such that the resulting functigraph has $\Delta(C(C_{3k}, f)) = k+3$ and $\gamma(C(C_{3k}, f)) = 2\gamma(C_{3k}) = 2k$.

Example 24. For $k \in \mathbb{Z}^+$, let $f : V(C_{3k}) \to V(C_{3k})$ be a function defined by

$$f(u_i) = \begin{cases} v_i & \text{if } i \equiv 1 \pmod 3, \\ v_{i+1} & \text{if } i \equiv 2 \pmod 3, \\ v_{3k} & \text{if } i \equiv 0 \pmod 3. \end{cases}$$

Then $\gamma(C(C_{3k}, f)) = 2k = 2\gamma(C_{3k})$.

Proof. Notice that $\Delta(C(C_{3k}, f)) = \deg(v_{3k}) = k + 3$. For $1 \le i \le k$, define $S_i = \{u_{3i}, u_{3i-1}, u_{3i-2}, v_{3i}, v_{3i-1}, v_{3i-2}\}$, and notice that $\cup_{i=1}^{k} S_i$ is a partition of $V(C(C_{3k}, f))$. Let D be any dominating set of $C(C_{3k}, f)$; we need to show that $|D| \ge 2k$. Observe that $|D \cap S_i| \ge 1$ since neither u_{3i-1} nor v_{3i-1} can be dominated from outside of S_i for $1 \le i \le k$. We will argue in an inductive fashion starting at k and descending to 1.

Suppose $|D| < 2k$; choose the biggest $j \le k$ such that $|D \cap S_j| = 1$. Of necessity $v_{3j} \in D$, as it is the only vertex in S_j dominating both u_{3j-1} and v_{3j-1}. Then $|D \cap S_{j-1}| \ge 2$, since to dominate u_{3j-2} and v_{3j-2} in S_j, D must contain both u_{3j-3} and v_{3j-3} in S_{j-1}.

Now, if $|D \cap S_{j-1}| \ge 3$, then it is "paired off" with S_j. We will choose the biggest $\ell < j$ such that $|D \cap S_\ell| = 1$ and restart at S_ℓ our inductive argument. Of course, S_j may be paired off with S_q where $j > q \ge 1$ and $|D \cap S_q| \ge 3$; in this case, of necessity, $|D \cap S_p| = 2$ for $j > p > q$, and we restart the argument after S_q when $q > 1$. Therefore, one of the following cases must hold for S_1.

(i) $|D \cap S_1| \ge 3$, then S_1 may be paired off with the least j such that $|D \cap S_j| = 1$, if necessary.

(ii) $|D \cap S_1| = 2$ and every S_j with $|D \cap S_j| = 1$ is paired off with S_q such that $q < j$ and $|D \cap S_q| \ge 3$.

(iii) $|D \cap S_1| = 2$ and there exists $j > 1$ with $|D \cap S_j| = 1$ which is not paired off with some S_q such that $q < j$ and $|D \cap S_q| \ge 3$. If $j = k$, then by examining S_k, S_{k-1}, and S_1, we will readily see that the assumption is impossible (u_1 is not dominated). If $j < k$, then there must exist $q > j$ such that $|D \cap S_q| \ge 3$ (in order to dominate $u_{3(j+1)-2}$).

(iv) $|D \cap S_1| = 1$, then there must exist $q > 1$ such that $|D \cap S_q| \ge 3$ (in order to dominate u_4).

In each case, we conclude $|D| \ge 2k$, contradicting our original supposition. ∎

Proposition 25. *Suppose $C(C_{3k}, f)$ is a functigraph with domain G_1 and codomain G_2. Partition G_2 into three sets V_1, V_2, and V_3 such that $V_i = \{v_j \mid j \equiv i \pmod 3\}$. If there is some i such that the average degree over all vertices in V_i is strictly greater than 4, then $\gamma(C(C_{3k}, f)) < 2\gamma(C_{3k})$.*

Proof. Suppose $C(C_{3k}, f)$ is a functigraph with codomain G_2 and that there is some i, say $i = 1$, such that the average degree over all vertices in V_1 is strictly greater than 4. Then $|N[V_1] \cap V(G_1)| \geq 2k + 1$. Let U_1 be the vertices in $V(G_1)$ that are not in $N[V_1]$ and notice that $|U_1| \leq k - 1$. Then $U_1 \cup V_1$ is a dominating set of $C(C_{3k}, f)$. ■

Remark 26. The result obtained in Proposition 25 is sharp as shown in Example 24. In the example, the average degree of the vertices in V_3 is exactly 4.

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