Discussiones Mathematicae
Graph Theory 32 (2012) 387–401
doi:10.7151/dmgt.1598

STABLE SETS FOR $(P_6, K_{2,3})$-FREE GRAPHS

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Abstract

The Maximum Stable Set (MS) problem is a well known NP-hard problem. However different graph classes for which MS can be efficiently solved have been detected and the augmenting graph technique seems to be a fruitful tool to this aim. In this paper we apply a recent characterization of minimal augmenting graphs [22] to prove that MS can be solved for $(P_6,K_{2,3})$-free graphs in polynomial time, extending some known results.

Keywords: graph algorithms, stable sets, P_6-free graphs.

2010 Mathematics Subject Classification: 05C69, 05C85.

1. Introduction

A *stable set* in a graph G is a set of pairwise nonadjacent vertices of G. The Maximum Stable Set (MS) problem is that of determining a stable set of maximum cardinality of a graph G. The MS problem is NP-hard, even under strong restrictions [13]. The following specific graphs are mentioned later. A P_k has vertices $v_1, v_2, \ldots, v_k$ and edges $v_j v_{j+1}$ for $1 \leq j < k$. A C_k has vertices $v_1, v_2, \ldots, v_k$ and edges $v_j v_{j+1}$ for $1 \leq j \leq k-1$ (index arithmetic modulo k). A $K_{p,q}$, for $p, q \geq 1$, is a complete bipartite graph with sides of cardinality p and q respectively. A $K_{1,3}$ is also called a *claw*. Given two graphs G_1, G_2, let $G_1 + G_2$ denote the graph obtained as a disjoint union of G_1 and G_2.

Let us say that a graph G is *F-free* if no induced subgraph of G is isomorphic to a given graph F. If G is F_1-free and F_2-free for given graphs F_1 and F_2, then let us say that G is *(F_1, F_2)-free*.

Let us say that a graph is of *type T* if it is a subdivided claw or a path, i.e. if it is a tree with at most one vertex of degree 3 and the other vertices of degree less than 3. Then a graph of type T which is different from a path contains three paths, each one from the vertex of degree 3 to respectively the three vertices of degree 1: then it can be denoted as $T_{i,j,k}$, where i, j, k stand for the length of such three paths (e.g. a $T_{1,1,1}$ is a claw).

Alekseev [1, 4] proved that MS remains NP-hard in the class of F-free graphs whenever F is a graph of which at least one component is not of type T.

Notice that if MS is polynomial for F-free graphs, for a given graph F, then MS is polynomial for $P_1 \cup F$ graphs, where $P_1 \cup F$ is the graph formed by the disjoint union of an isolated vertex and F: in fact, for any graph $G = (V, E)$, the MS problem can be solved by solving the same problem on each its subgraph $G[V \setminus N(v)]$, for $v \in V$.

Let us consider the computational complexity of MS for F-free graphs, for every 5-vertex graph F.

Assume that F is connected. If F is not of type T, then MS remains NP-hard for F-free graphs by Alekseev's result. If F is of type T, then F is either a fork (a $fork$ has vertices a, b, c, d, e and edges ab, bc, cd, ce) or a P_5. If F is a fork, then MS is polynomial for F-free graphs [2, 3], also in its weighted version [21]: notice that then MS is polynomial for F'-free graphs, for every induced subgraph F' of a fork. If F is a P_5, then the computational complexity of MS is unknown for F-free graphs.

Assume that F is disconnected. If at least one component of F is not of type T, then MS remains NP-hard for F-free graphs by Alekseev's result. Then assume that every component of F is of type T. If F has an isolated vertex, then the remaining four vertices of F either form an induced subgraph of a fork, or form a $P_2 + P_2$, or form a $4P_1$ (i.e., a stable set of four vertices): then by the above remarks and since MS is polynomial for $P_2 + P_2$-free graphs [11] and clearly for $5P_1$-free graphs, MS is polynomial for F-free graphs. If F has no isolated vertices, i.e., F is a $P_2 + P_3$, then MS is polynomial for F-free graphs [23].

Summarizing, if F is a 5-vertex graph, then the computational complexity of MS is unknown for F-free graphs only in case $F = P_5$. Also the computational complexity of MS is unknown for F-free graphs, where F is a connected graph of type T with more than 5 vertices, in particular for P_t-free graphs for $t \geq 6$.

In this paper we prove that MS can be solved for (P_6,$K_{2,3}$)-free in polynomial time. That extends the following analogous results concerning:

(i) (P_5,$K_{2,3}$)-free graphs, see [15] where the result holds even for (P_5,$K_{m,m}$)-free graphs (see [27] for the weighted case) and

(ii) (P_6,C_4)-free graphs, see [7, 26] (see [7] for the weighted case). Let us recall that, since a $K_{2,3}$ contains a C_4, MS remains NP-hard for $K_{2,3}$-free graphs [29].

Two topics are linked to this paper: the first is the study of P_6-free graphs (with particular reference to MS for subclasses of these graphs); the second is the augmenting graph technique (see e.g. [17] for a survey on this topic), which is a fruitful approach to detect graph classes for which MS can be solved in polynomial time, and which we apply in this paper with particular reference to a recent characterization of minimal augmenting graphs [22].

Concerning the first topic: the class of P_6-free graphs is a natural extension of that of P_5-free graphs. The first characterization of such graphs was maybe given in [6]. Then further results were introduced also recently, see e.g. [10, 12, 16, 18, 19, 20]. In particular structural properties of P_6-free graphs were directly applied to define polynomial time algorithms to solve the MS problem (also for its weighted version) for subclasses of these graphs, such as (P_6,triangle)-free [9], (P_6,$K_{1,p}$)-free [24], (P_6, C_4)-free [7, 26] and (P_6,diamond)-free graphs [28]. Let us observe that results on MS for subclasses of P_6-free graphs may keep their own interest even if the complexity of MS for P_5-free graphs should be determined. In fact: if MS should (be shown to) remain NP-hard for P_5-free graphs, then MS would remain NP-hard for P_6-free graphs too; if MS should (be shown to) be polynomial for P_5-free graphs, then according to the aforementioned Alekseev's result the class of P_6-free graphs would be one of the three minimal classes (the other ones are that of $T_{1,1,3}$-free graphs and that of $T_{1,2,2}$-free graphs), defined by forbidding a single connected subgraph, for which the computational complexity of MS would be unknown.

Concerning the second topic: the augmenting graph technique to solve the MS problem derives directly from the well-known augmenting technique to solve the Maximum Matching problem, and the first application to MS of such a technique was maybe introduced in [25, 30] for claw-free graphs. Then further results were introduced also recently, see e.g. [5, 14, 22]. Let us observe that in [5] the authors prove that while applying the augmenting graph technique one can treat banner-free graphs (a *banner* has vertices a, b, c, d, e and edges ab, bc, be, cd, de) as C_4-free graphs; in particular the mentioned results of [5, 14, 22] deal with subclasses of banner-free graphs; in this manuscript we consider a subclass of $K_{2,3}$-free graphs (i.e., that is an extension of the application of the augmenting graph technique in a different direction).

2. Preliminaries

For any missing notation or references, let us refer to [8]. Let $G = (V, E)$ be a finite undirected graph and let $|V| = n$, $|E| = m$. For every $u \in V$, let $N(u) = \{v \in V : uv \in E\}$ be the set of *neighbors* of u. Let $N[v] = N(v) \cup \{v\}$. Let U, W be two subsets of V. Let $N(U) = \{v \in V \setminus U :$ there exists $u \in U$ such that $uv \in E\}$. Let $N[U] = N(U) \cup U$. Let $N_W(U) = N(U) \cap W$; if $U = \{u\}$, then let us simply write $N_W(u)$. Let us say that $v \in V$ *dominates* U if v is adjacent to each vertex of U.

Let $G[U]$ denote the subgraph of G induced by $U \subseteq V$. A *component of* G is the vertex-set of a maximal connected subgraph of G. The *distance* $d(v, w)$ between $v, w \in V$ is the number of edges in a shortest path from v to w.

Let S be a stable set of G. A bipartite graph $H = (H_1, H_2, F)$ is called an *augmenting graph* for S if $H_2 \subseteq S$, $H_1 \subseteq V \setminus S$, $N(H_1) \cap (S \setminus H_2) = \emptyset$, and $|H_1| > |H_2|$. The following theorem is well known and not difficult to prove (see e.g. [17]).

Theorem 1. *Let S be a stable set S of a graph G. Then S is not maximum if and only if there exists an augmenting graph for S.*

Replacement of the vertices of H_2 in S by the vertices of H_1 is called the *H-augmentation* of S (in particular, $|H_1| - |H_2|$ is the increment). Then the following algorithm correctly solves the MS problem for any graph G and points out that the difficulty of the problem can be directly linked to that of detecting augmenting graphs for stable sets.

Algorithm Alpha

Input: a graph $G = (V, E)$.
Output: a maximum stable set S of G.

Step 1. Compute any stable set S of G.
Step 2. Check if there exists a (minimal) augmenting graph for S, say H.
Step 3. If the answer is *no*, then return S. STOP.
Step 4. If the answer is *yes*, then apply H-augmentation to S. Go to Step 2.

A *stable system of representatives* (shortly ssr) *of* $U \subseteq V$ is a stable set $T \subseteq V \setminus U$, with $|T| = |U|$, such that $G[T \cup U]$ has a matching of $|T| = |U|$ elements, i.e., one can write $U = \{u_1, \ldots, u_m\}$ and $T = \{t_1, \ldots, t_m\}$ so that $(u_i, t_i) \in E$ for $i = 1, \ldots, m$.

A *minimal augmenting graph* for S is an augmenting graph for S that is not the induced supergraph of any other augmenting graph for S. Notice that every minimal augmenting graph is connected. Let us report the following result from [22].

Lemma 2 [22]**.** *Let $G = (V, E)$ be a graph, S be a maximal stable set of G, and $v \in V \setminus S$. If v belongs to a minimal augmenting graph (H_1, H_2, F) for S, then $H_1 \setminus \{v\}$ admits an* ssr *in H_2.*

Theorem 2 of [6] implies that every connected P_6-free graph $G = (V, E)$ admits a vertex v such that $d(v, u) \leq 3$ for every $u \in V$. Theorem 2 of [20] implies that every connected P_6-free bipartite graph admits two such special vertices, belonging respectively to the two sides of the bipartite graph. The following observation points out that, in a connected P_6-free bipartite graph G, a sufficient condition for a vertex to enjoy the above property is to have maximum degree in G among the vertices of its side.

Observation 3. *Let $H = (H_1, H_2, E)$ be a connected bipartite P_6-free graph. Let $v \in H_1$ be a vertex such that v has maximum degree in H among the vertices of H_1. Then $d(v, h) \leq 3$ for every $h \in H_1 \cup H_2$.*

Proof. By contradiction assume that there exists $h \in H_1 \cup H_2$ such that $d(v, h) = 4$. Since G is connected bipartite, $h \in H_1$. Let v, a, u, b, h be the vertices inducing a shortest path from v to h. By the maximum degree of v (and since u is adjacent to b), there exists a vertex $a' \in H_2$ such that a' is adjacent to v and nonadjacent to u. Notice that a' is also nonadjacent to h, since $d(v, h) = 4$. Then a', v, a, u, b, h induce a P_6 (contradiction). ■

Let G be a connected P_6-free graph. Let S be a maximal but not maximum stable set of G, and let $H = (H_1, H_2, F)$ be a minimal augmenting graph for S. Let us say that a vertex $v \in H_1$ such that v has maximum degree in H among the vertices of H_1 is a *nail* of H. By Observation 3 and the aforementioned observation that H is connected, if v is a nail of H, then $d(v, h) \leq 3$ for every $h \in H_1 \cup H_2$.

3. Stable Sets for $(P_6, K_{2,3})$-free Graphs

Throughout this section let $G = (V, E)$ be a connected $(P_6, K_{2,3})$-free graph, and S be a maximal stable set of G. To solve MS for G we apply Algorithm Alpha. Then let us prove that Step 2 of Algorithm Alpha, referring to minimal augmenting graphs, can be efficiently executed. To this end, since every minimal augmenting graph for S contains at least one nail, let us proceed as follows.

Let us show that if a vertex v of G is a nail of a minimal augmenting graph $H = (H_1, H_2, F)$ for S, then H can be efficiently detected. Then let us fix a vertex $v \in V \setminus S$ and assume that v is a nail of a minimal augmenting graph $H = (H_1, H_2, F)$ for S (then H is connected). Let us write $A = N_S(v)$, $B = N(A) \setminus N[v]$, and $C = (S \setminus A) \cap N(B)$. Then by the definition of a nail and by Observation 3 one can assume that:

(1) H is a subgraph of $G[A \cup B \cup C \cup \{v\}]$, i.e., $H_1 \subseteq B \cup \{v\}$ and $H_2 \subseteq A \cup C$.

(2) No vertex of B has in $A \cup C$ more neighbors than v in A: if this does not happen, then one can delete all the vertices of B which have in $A \cup C$ more neighbors than v in A (since v is a nail of H).

Furthermore, since G is $K_{2,3}$-free, the following fact holds:

(3) Each vertex of B has degree 1 or 2 in A.

Let $A = \{a_1, \dots, a_h\}$ and $C^* = C \cap H_2 = \{c_1, \dots, c_k\}$.

To show that H can be efficiently detected, let us distinguish between the case in which $C^* = \emptyset$ and the case in which $C^* \neq \emptyset$.

3.1. The case in which $C^* = \emptyset$

In this case, the difficulty is to check if A admits an ssr in B.

Lemma 4. *Let $\bar{b}_i \in B \cap N(a_i)$ for $i = 1, 2, 3$ be pairwise nonadjacent. Assume that $\bar{b}_1$ and $\bar{b}_2$ are nonadjacent to any vertex of $\{a_4, \dots, a_h\}$. Then one can check if $\{a_4, \dots, a_h\}$ admits an ssr in $B \setminus N[\{\bar{b}_1, \bar{b}_2, \bar{b}_3\}]$ in $O(n + m)$ time.*

Proof. First let us prove a claim.

Claim 5. *Let $\bar{p}, \bar{q} \in B \setminus N[\{\bar{b}_1, \bar{b}_2, \bar{b}_3\}]$. Let $\bar{p} \in N(a_p)$ and $\bar{q} \in N(a_q)$ for any $p, q \in \{4, \dots, h\}$. If $\bar{p}$ is nonadjacent to a_q, then $\bar{p}$ is nonadjacent to $\bar{q}$.*

Proof. By contradiction assume that $\bar{p}$ is adjacent to $\bar{q}$. By (3), to avoid a P_6 formed by either $\bar{b}_1, a_1, v, a_q, \bar{q}, \bar{p}$ or $\bar{b}_2, a_2, v, a_q, \bar{q}, \bar{p}$ one may without loss of generality that $\bar{p}$ is adjacent to a_1, and $\bar{q}$ is adjacent to a_2. By (3): $\bar{q}$ is nonadjacent to a_p, and both $\bar{p}$ and $\bar{q}$ are nonadjacent to a_3. Then, since by (3) $\bar{b}_3$ can not be adjacent to both a_p and a_q, either $\bar{b}_3, a_3, v, a_p, \bar{p}, \bar{q}$ or $\bar{b}_3, a_3, v, a_q, \bar{q}, \bar{p}$ induce a P_6 (contradiction). □

Let us write $A^* = \{a_4, \dots, a_h\}$ and $B^* = B \setminus N[\{\bar{b}_1, \bar{b}_2, \bar{b}_3\}]$.

For $i = 4, \dots, h$ let $D_i = \{b \in B^* : N_{A^*}(b) = \{a_i\}\}$. Notice that by Claim 5 all the vertices in D_i, for $i = 4, \dots, h$, have no neighbors in $B^* \setminus D_i$. Then, since one has to check if A^* admits an ssr in B^*, one can proceed as follows. For every $D_i \neq \emptyset$: delete all the vertices of D_i except from one. Denote as B^*_{one} what remains of B^*.

For $i, j = 4, \dots, h$ let $D_{i,j} = \{b \in B^*_{one} : N_{A^*}(b) = \{a_i, a_j\}\}$. Notice that the vertices in $D_{i,j}$ are mutually adjacent (since G is $K_{2,3}$-free), and that by Claim 1 all the vertices in $D_{i,j}$, for $i, j = 4, \dots, h$, have no neighbors in $B^*_{one} \setminus D_{ij}$. Then, since one has to check if A^* admits an ssr in B^*_{one}, one can proceed as follows. For every $D_{i,j} \neq \emptyset$: delete all the vertices of $D_{i,j}$ except from one. Denote as B^*_{two} what remains of B^*_{one}.

Now by (3) and by Claim 5 B^*_{two} is a stable set. Then to check if A^* admits an ssr in B^*_{two} it is enough to check if the bipartite graph $G[A^* \cup B^*_{two}]$ admits a matching of $h - 3$ elements. Since G is P_6-free, that can be done in linear time as shown in [12]. Then the lemma follows. ■

Lemma 6. *Assume that $C^* = \emptyset$. Then H can be detected in $O(n^3 m)$ time.*

Proof. Since $C^* = \emptyset$, by Lemma 2 one has to check if A admits an ssr in B. If $|A| \leq 3$, then the assertion can be easily proved. Then assume that $|A| \geq 4$. If A admits an ssr in B, then there exists a vertex $b \in N(a_1)$ belonging to such an ssr. For every $\bar{b}_1 \in N(a_1)$ one can check if $\bar{b}_1$ belongs to such an ssr, as follows.

First assume that $\bar{b}_1$ has degree 1 in A. Then for every $\bar{b}_2 \in N(a_2) \setminus N[\bar{b}_1]$ do:

1. if $\bar{b}_2$ has degree 1 in A, then for every $\bar{b}_3 \in N(a_3) \setminus N[\{\bar{b}_1, \bar{b}_2\}]$ check if $\{a_4, \ldots, a_m\}$ admits an ssr in $B \setminus N[\{\bar{b}_1, \bar{b}_2, \bar{b}_3\}]$, according to Lemma 4;

2. if $\bar{b}_2$ has degree 2 in A, then: if $\bar{b}_2$ is adjacent to a_1, then one can proceed similarly to the previous case; if $\bar{b}_2$ is adjacent to a_i, with $i \neq 1, 2$, then one can assume without loss of generality that $i = 3$ and proceed similarly to the previous case.

Then assume $\bar{b}_1$ has degree 2 in A. Then $\bar{b}_1$ is adjacent to some a_i with $i \neq 1$. Then one can assume without loss of generality that $i = 2$ and proceed similarly to the case in which $\bar{b}_1$ has degree 1 in A. ∎

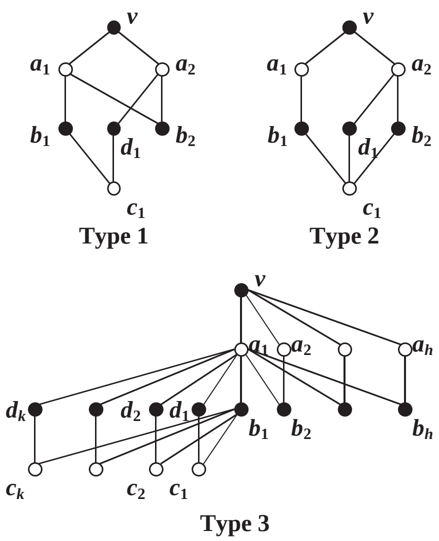

Figure 1

3.2. The case in which $C^* \neq \emptyset$

In this case, let us show that H can be just of three types each of which can be efficiently detected. By Lemma 2, let $\{\{b_1, \ldots, b_h\}, \{d_1, \ldots, d_k\}\}$ be a partition of $H_1 \setminus \{v\}$, such that $\{b_1, \ldots, b_h\}$ is an ssr of A, and $\{d_1, \ldots, d_k\}$ is an ssr of C^* (in $H_1 \setminus \{v\}$). Referring to Figure 1, let us say that H is of:

- Type 1 if: $A = \{a_1, a_2\}$, $C^* = \{c_1\}$; b_1 is nonadjacent to a_2; b_2 is adjacent to a_1; d_1 is adjacent to a_2 and nonadjacent to a_1; c_1 is adjacent to b_1 and nonadjacent to b_2;
- Type 2 if: $A = \{a_1, a_2\}$, $C^* = \{c_1\}$; b_1 is nonadjacent to a_2; b_2 is nonadjacent to a_1; d_1 is adjacent to a_2 and nonadjacent to a_1; c_1 is adjacent to b_1, b_2;
- Type 3 if: a_1 is adjacent to b_i for every $i \geq 2$; a_1 is adjacent to d_j for every $j \geq 2$; a_i is nonadjacent to b_t for every $i \geq 2$ and $t \neq i$; a_i is nonadjacent to d_j for every $i \geq 2$ and $j \geq 1$; b_1 is adjacent to c_i for every $i \geq 1$; c_j is nonadjacent to b_t for every $j \geq 2$ and $t \geq 1$; c_j is nonadjacent to d_t for every $j \geq 2$ and $t \neq i$.

Lemma 7. *Assume that $C^* \neq \emptyset$. Then H is of* Type 1*, or* 2*, or* 3.

Proof. Since $C^* \neq \emptyset$ and H is a minimal augmenting graph for S, there is a vertex in $\{b_1, \ldots, b_h\}$ adjacent to a vertex in $\{c_1, \ldots, c_k\}$, otherwise $\{v\} \cup \{a_1, \ldots, a_h\} \cup \{b_1, \ldots, b_h\}$ is an augmenting graph for S.

Assume without loss of generality that b_1 is adjacent to c_1. Then by (2) with respect to b_1, one has $|A| \geq 2$.

Claim 8. *Exactly one of the following cases holds:*
(i) *d_1 dominates $A \setminus \{a_1\}$, or*
(ii) *d_1 is adjacent to a_1.*

Proof. By (2) with respect to d_1, statements (i) and (ii) can not hold at the same time. Then let us assume that d_1 is nonadjacent to a_1, and prove that d_1 dominates $A \setminus \{a_1\}$. By contradiction assume that there exists a vertex in $A \setminus \{a_1\}$ nonadjacent to d_1, say a_2 without loss of generality. To avoid that $d_1, c_1, b_1, a_1, v, a_2$ induce a P_6, b_1 is adjacent to a_2. Then by (2) with respect to b_1, one has $A \setminus \{a_1, a_2\} \neq \emptyset$. Furthermore by (3), b_1 is nonadjacent to any vertex in $A \setminus \{a_1, a_2\}$. Then to avoid that d_1, c, b_1, a_1, v, a_3 induce a P_6, d_1 is adjacent to a_3. Let us consider b_2. Notice that b_2 is nonadjacent to a_1 (otherwise a_1, a_2, v, b_1, b_2 induce a $K_{2,3}$) and to c_1 (otherwise $a_1, v, a_2, b_2, c_1, d_1$ induce a P_6). Furthermore b_2 is nonadjacent to a_3: in fact otherwise to avoid that $a_1, b_1, a_2, b_2, a_3, b_3$ induce a P_6, one has that either b_3 is adjacent to a_2 (but then b_2, b_3, v, a_2, a_3 induce a $K_{2,3}$) or b_3 is adjacent to a_1 (but then $a_2, b_1, a_1, b_3, a_3, d_1$ induce a P_6). Then $b_2, a_2, b_1, c_1, d_1, a_3$ induce a P_6 (contradiction). □

According to Claim 8 let us consider the following cases.

Case 1. d_1 dominates $A \setminus \{a_1\}$ (and is nonadjacent to a_1). Then by (3), $|A \setminus \{a_1\}| \leq 2$.

Case 1.1. $|A \setminus \{a_1\}| = 1$. Then d_1 is adjacent to a_2. By (2) with respect to b_1, b_1 is nonadjacent to a_2. To avoid that $b_2, a_2, d_1, c_1, b_1, a_1$ induce a P_6, b_2 is adjacent either to a_1 or to c_1 (not to both by (2)).

Assume that b_2 is adjacent to a_1 (and is nonadjacent to c_1). Let us show that $v, a_1, a_2, b_1, b_2, c_1, d_1$ induce a minimal augmenting graph of Type 1. To this end, let us show that no extension of this graph is possible, i.e., that $C^* = \{c_1\}$ (and thus $D = \{d_1\}$). By contradiction assume that $C^* \setminus \{c_1\} \neq \emptyset$. Then every vertex of $C^* \setminus \{c_1\}$ is nonadjacent to any vertex of $\{b_1, b_2, d_1\}$, by (2). But then $v, a_1, a_2, b_1, b_2, c_1, d_1$ induce an augmenting graph, i.e., this possible extension would not be a minimal augmenting graph. Then H is of Type 1.

Assume that b_2 is adjacent to c_1 (and is nonadjacent to a_1). Let us show that $v, a_1, a_2, b_1, b_2, c_1, d_1$ induce a minimal augmenting graph of Type 2. To this end, let us show that no extension of this graph is possible, i.e., that $C^* = \{c_1\}$ (and thus $D = \{d_1\}$). By contradiction assume that $C^* \setminus \{c_1\} \neq \emptyset$. Then every vertex of $C^* \setminus \{c_1\}$ is nonadjacent to any vertex of $\{b_1, b_2, d_1\}$, by (2). But then $v, a_1, a_2, b_1, b_2, c_1, d_1$ induce an augmenting graph, i.e., this possible extension would not be a minimal augmenting graph. Then H is of Type 2.

Case 1.2. $|A \setminus \{a_1\}| = 2$. Then d_1 is adjacent to a_2 and a_3. Then to avoid a $K_{2,3}$: b_2 is nonadjacent to a_3, and b_3 is nonadjacent to a_2. Furthermore, by (2) let us assume without loss of generality that b_1 is nonadjacent to a_3.

To avoid that $b_3, a_3, v, a_1, b_1, c_1$ induce a P_6, b_3 is adjacent either to c_1 or to a_1. If b_3 is adjacent to a_1, then b_2 is adjacent to a_1. Otherwise $a_1, b_3, a_3, d_1, a_2, b_2$ induce a P_6, then b_1 is nonadjacent to a_2. Otherwise a_1, a_2, v, b_1, b_2 induce a $K_{2,3}$ but then $b_1, a_1, b_2, a_2, d_1, a_3$ induce a P_6. If b_3 is adjacent to c_1, then b_2 is adjacent to c_1. Otherwise $b_2, a_2, v, a_3, b_3, c_1$ induce a P_6, then b_1 is nonadjacent to a_2. Otherwise a_2, c_1, b_1, b_2, d_1 induce a $K_{2,3}$ but then $a_2, v, a_3, b_3, c_1, b_1$ induce a P_6.

Case 2. d_1 is adjacent to a_1 (and does not dominate $A \setminus \{a_1\}$). By (2) with respect to b_1, b_1 is nonadjacent to at least one vertex of $A \setminus \{a_1\}$, say a_h. To avoid that $b_h, a_h, v, a_1, b_1, c_1$ induce a P_6, b_h is adjacent either to c_1 or to a_1 (not to both, otherwise a_1, c_1, d_1, b_1, b_h induce a $K_{2,3}$).

Case 2.1. b_h is adjacent to c_1 (and is nonadjacent to a_1). Then to avoid that $a_i, v, a_h, b_h, c_1, b_1$ induce a P_6, for all $i = 2, \ldots, h-1$, a_i is adjacent either to b_1 or to b_h. By (3) this implies that $|A| \leq 4$.

Assume that $|A| = 2$, i.e., $h = 2$. Then b_1 and d_1 are nonadjacent to a_2, by (2). Let us show that $v, a_1, a_2, b_1, b_2, c_1, d_1$ induce a minimal augmenting graph of Type 2, up to symmetry. By symmetry the proof is similar to that given in Case 1.1. Then H is of Type 2.

Assume that $|A| = 3$, i.e., $h = 3$. To avoid that $a_2, v, a_3, b_3, c_1, b_1$ induce a P_6, a_2 is adjacent either to b_1 or to b_3. If a_2 is adjacent to b_1 and nonadjacent to b_3, then: to avoid that $b_2, a_2, b_1, c_1, b_3, a_3$ induce a P_6, b_2 is adjacent either to a_3 or to c_1; if b_2 is adjacent to a_3 (and then is nonadjacent to a_1 by (3)), then $a_1, b_1, a_2, b_2, a_3, b_3$ induce a P_6; if b_2 is adjacent to c_1, then $a_1, v, a_2, b_2, c_1, b_3$

induce a P_6. If a_2 is adjacent to b_3 and nonadjacent to b_1, then by symmetry one obtains a contradiction as well. If a_2 is adjacent to both b_1 and b_3, then: d_1 is nonadjacent to a_2 (otherwise a_1, a_2, d_1, b_1, v induce a $K_{2,3}$), b_2 is nonadjacent to a_1 (otherwise a_1, a_2, b_2, b_1, v induce a $K_{2,3}$), b_2 is nonadjacent to c_1 (otherwise a_2, c_1, b_1, b_2, b_3 induce a $K_{2,3}$); then $b_2, a_2, v, a_1, d_1, c_1$ induce a P_6.

Assume that $|A| = 4$, i.e., $h = 4$. Then one can apply an argument similar to that of the previous paragraph, with b_4 instead of b_3, to obtain a contradiction.

Case 2.2. b_h is adjacent to a_1 (and is nonadjacent to c_1). By (3), b_h is nonadjacent to any vertex of $\{a_2, \ldots, a_{h-1}\}$.

Claim 9. *c_1 is nonadjacent to any vertex of $\{b_2, \ldots, b_{h-1}\}$ (and then of $\{b_2, \ldots, b_h\}$).*

Proof. By contradiction, assume that c_1 is adjacent to a vertex of $\{b_2, \ldots, b_{h-1}\}$, say b_i, for some $i \in \{2, \ldots, h-1\}$; by (2) b_i can not be adjacent to both a_1 and a_h; then either $c_1, b_i, a_i, v, a_1, b_h$ (if b_i is nonadjacent to a_1) or $c_1, b_i, a_i, v, a_h, b_h$ (if b_i is nonadjacent to a_h) induce a P_6 (contradiction). □

Claim 10. *a_1 is adjacent to every vertex of $\{b_2, \ldots, b_{h-1}\}$ (and then of $\{b_2, \ldots, b_h\}$).*

Proof. By contradiction assume that a_1 is nonadjacent to a vertex of $\{b_2, \ldots, b_{h-1}\}$, say b_i for some $i \in \{2, \ldots, h-1\}$: then a_i is adjacent to b_1, otherwise $c_1, b_1, a_1, v, a_i, b_i$ induce a P_6. It follows, by (3) with respect to b_1, that at most one vertex of $\{b_2, \ldots, b_{h-1}\}$ is nonadjacent to a_1, namely b_i. Without loss of generality let us say that $b_i = b_2$: then a_2 is adjacent to b_1. Then by (2) with respect to b_1, one has $|A| \geq 3$. Notice that for all $t = 3, \ldots, h$, a_t is adjacent to b_2, otherwise $a_t, b_t, a_1, b_1, a_2, b_2$ induce a P_6. Then by (3) with respect to b_2, one has $|A| = 3$. Then $b_2, a_3, b_3, a_1, b_1, c_1$ induce a P_6 (contradiction). □

Let us write $B_1 = \{b_2, \ldots, b_h\}$. By Claim 10, a_1 dominates B_1. Then by (3) every vertex $b_i \in B_1$ is adjacent in A only to vertices a_1, a_i. Then b_1 and d_1 are nonadjacent to any vertex of $\{a_2, \ldots, a_h\}$, otherwise a $K_{2,3}$ arises. Let us show that the possible extensions of this graph lead to the conclusion that H is of Type 3.

Then let us assume that $C^* \setminus \{c_1\} \neq \emptyset$. Since $C^* \setminus \{c_1\} \neq \emptyset$ and H is a minimal augmenting graph for S, there is a vertex in $C^* \setminus \{c_1\}$ adjacent to a vertex in $B_1 \cup \{b_1, d_1\}$, otherwise $\{v\} \cup \{a_1, \ldots, a_h\} \cup \{b_1, \ldots, b_h\} \cup \{c_1, d_1\}$ is an augmenting graph for S.

Claim 11. *Every vertex of $C^* \setminus \{c_1\}$ is nonadjacent to any vertex of B_1.*

Proof. By contradiction assume without loss of generality by symmetry that c_k is adjacent to b_h. Then c_k is adjacent to each vertex of $B_1 \setminus \{b_h\}$, otherwise a P_6 arises (namely, $c_k, b_h, a_h, v, a_i, b_i$ for every $b_i \in B_1 \setminus \{b_h\}$). Then $|B_1| \leq 2$ otherwise a $K_{2,3}$ arises involving a_1 and c_k. If $|B_1| = 1$, then one has a contradiction to (2) with respect to b_h. If $|B_1| = 2$, then: b_1 is nonadjacent to c_k, otherwise $c_1, b_1, c_k, b_h, a_h, v$ induce a P_6; then d_k is nonadjacent to a_1, otherwise a_1, c_k, b_2, b_h, d_k induce a $K_{2,3}$; then d_k is adjacent to c_1, otherwise $d_k, c_k, b_h, a_1, b_1, c_1$ induce a P_6; then $v, a_1, b_1, c_1, d_k, c_k$ induce a P_6 (contradiction). □

By the above and by Claim 11, at least one vertex of $C^* \setminus \{c_1\}$ is adjacent to b_1 or to d_1: without loss of generality by symmetry, let us say to b_1. Let $C_1^* = \{c \in C^* \setminus \{c_1\} : c \text{ is adjacent to } b_1\}$. Then $C_1^* \neq \emptyset$.

Claim 12. *For every pair (c_j, d_j) with $c_j \in C_1^*$ one has that: d_j is adjacent to a_1, d_j is nonadjacent to any vertex of $A \setminus \{a_1\}$, d_j is nonadjacent to c_1, c_j is nonadjacent to d_1.*

Proof. First let us show that d_j is adjacent to a_1. By contradiction assume that d_j is nonadjacent to a_1. To avoid that $d_j, c_j, b_1, a_1, v, a_i$ for $i = 2, \ldots, h$ induce a P_6, d_j dominates $A \setminus \{a_1\}$. Then by (2) d_j is nonadjacent to c_1. Then $c_1, b_1, c_j, d_j, a_i, b_i$, for $i > 1$, induce a P_6 (contradiction). Then d_j is adjacent to a_1. Since G is $K_{2,3}$-free one obtains: d_j is nonadjacent to any vertex of $A \setminus \{a_1\}$; d_j is nonadjacent to c_1; c_j is nonadjacent to d_1. □

Finally let us prove that $C_1^* = C^* \setminus \{c_1\}$, i.e., that $(C^* \setminus \{c_1\}) \setminus C_1^* = \emptyset$. By contradiction assume that $(C^* \setminus \{c_1\}) \setminus C_1^* \neq \emptyset$. Since H is a minimal augmenting graph, there exists a vertex $c_q \in (C^* \setminus \{c_1\}) \setminus C_1^*$ adjacent to some vertex d_p such that $c_p \in C_1^* \cup \{c_1\}$ (also by Claim 10). In particular c_q is adjacent to d_1, otherwise $c_p \in C^* \setminus \{c_1\}$ and then $c_q, d_p, c_p, b_1, c_1, d_1$ induce a P_6 (also by Claim 11). Then d_q is adjacent to a_1: in fact otherwise to avoid that $d_q, c_q, d_1, a_1, v, a_2$ induce a P_6, d_q is adjacent to a_2; then to avoid that $b_2, a_2, d_q, c_q, d_1, c_1$ induce a P_6, d_q is adjacent to c_1; then $c_q, d_q, c_1, b_1, a_1, v$ induce a P_6. Furthermore d_q is nonadjacent to c_1, otherwise a_1, c_1, d_q, d_1, b_1 induce a $K_{2,3}$. Now, recalling that $C_1^* \neq \emptyset$, let us consider a vertex $c_j \in C_1^*$. Then d_q is adjacent to c_j, otherwise $d_q, c_q, d_1, c_1, b_1, c_j$ induce a P_6 (also by Claim 12). Then a_1, c_j, d_q, d_j, b_1 induce a $K_{2,3}$ (contradiction).

Then $C_1^* = C^* \setminus \{c_1\}$. Then by the above claims, H is of Type 3. This completes the proof of the lemma. ■

Lemma 13. *Assume that $C^* \neq \emptyset$. Then H can be detected in $O(n^3 m)$ time.*

Proof. By Lemma 7, H is of Type 1, or 2, or 3. Let us observe that one can easily determine the sets A, B, and C.

If H is of Type 1, see Figure 1, then let us proceed as follows. Clearly it is necessary that $|A| = 2$. Then for each vertex $b \in B \setminus N(a_2)$ (where b represents b_1) such that b has exactly one neighbor in C, say c_1, one has to check if there exists a stable set of $B \setminus N(b)$, say x, y (where x and y represent b_2 and d_1 respectively) with x adjacent to a_1, a_2 and nonadjacent to c_1, and with y adjacent to a_2, c_1 and nonadjacent to a_1 (then one should proceed similarly by interchanging a_1 with a_2, for a symmetry check).

If H is of Type 2, see Figure 1, then one can proceed in a similar way. Then assume that H is of Type 3. Then let us proceed as follows. Let us describe the procedure in case $|C^*| \geq 2$. The case in which $|C^*| = 1$ can be similarly treated. Let us say that a vertex of $H_1 \setminus \{v\}$ is *critical* for H if it has more than two neighbors in H. Then H contains one critical vertex, namely vertex b_1.

Let us say that a vertex $b \in B$ is *green* if it is a candidate to be critical for H, i.e., if $|N(b) \cap A| = 1$ and $|N(b) \cap C| \geq 2$. Thus there exists at least one green vertex which is critical for H. Let $b \in B$ be a green vertex. Let $N(b) \cap A = \{a_1\}$ (without loss of generality), and $N(b) \cap C = \{\tilde{c}_1, \ldots, \tilde{c}_m\}$. For every vertex $s \in A \cup C$ with $s \neq a_1$ let $M(s) = \{b' \in B : N(b') \cap (A \cup C) = \{s, a_1\}\}$.

Let $\tilde{d}_j \in M(\tilde{c}_j)$ for some $j \in \{1, \ldots, m\}$. Then every vertex $\tilde{d}_r \in M(\tilde{c}_r) \setminus (N(b) \cup N(\tilde{d}_j))$ in nonadjacent to any vertex $\tilde{d}_t \in M(\tilde{c}_t) \setminus (N(b) \cup N(\tilde{d}_j))$ for every $r, t \neq j$, otherwise $\tilde{d}_r, \tilde{d}_t, \tilde{c}_t, b, \tilde{c}_j, \tilde{d}_j$ induce a P_6.

Let $\tilde{b}_i \in M(a_i)$ for some $i \in \{2, \ldots, h\}$. Then every vertex $\tilde{b}_r \in M(a_r) \setminus N(\tilde{b}_i)$ is nonadjacent to any vertex $\tilde{b}_t \in M(a_t) \setminus N(\tilde{b}_i)$ for every $r, t \neq i$, otherwise $\tilde{b}_r, \tilde{b}_t, a_t, v, a_i, \tilde{b}_i$ induce a P_6.

Furthermore, if $|A| \geq 3$, then every vertex $\tilde{d}_j \in M(\tilde{c}_j)$ for $j = 1, \ldots, m$ is nonadjacent to any vertex $\tilde{b}_i \in M(a_i)$ for $i = 2, \ldots, h$.

Otherwise $\tilde{c}_j, \tilde{d}_j, \tilde{b}_i, a_i, v, a_{i+i}$ (or a_{i-i}) induce a P_6. Then by the above a green vertex b is critical for H if and only if there exists a pair of nonadjacent vertices, namely $\tilde{b}_2$ and $\tilde{d}_1$, with $\tilde{b}_2 \in M(a_2)$ and $\tilde{d}_1 \in M(\tilde{c}_1)$, such that $[M(a_i) \setminus (N(b) \cup N(\tilde{b}_2) \cup N(\tilde{d}_1)) \neq \emptyset$, for all $i = 3, \ldots, h]$ AND $[M(\tilde{c}_j) \setminus (N(b) \cup N(\tilde{b}_2) \cup N(\tilde{d}_1)) \neq \emptyset$, for all $j = 2, \ldots, k]$. Since that can be checked in $O(n^2 m)$ for every green vertex b, the lemma follows. ■

3.3. Summarizing

Then to solve MS for (P_6,$K_{2,3}$)-free graphs one can apply Algorithm Alpha, referring to minimal augmenting graphs, whose Step 2 can be handled by Lemmas 6, 7 and 13.

Theorem 14. *The MS problem can be solved for* (P_6,$K_{2,3}$)*-free graphs in* $O(n^4 m)$ *time.*

Acknowledgement

I would to thank the referees for their valuable and helpful comments to improve the manuscript, in particular for having detected errors and suggested remedies, in the arguments and in the presentation.

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Received 13 May 2010
Revised 30 June 2011
Accepted 4 July 2011