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NOTE

PARITY VERTEX COLORINGS OF BINOMIAL TREES [1]

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Abstract

We show for every $k \geq 1$ that the binomial tree of order $3k$ has a vertex-coloring with $2k + 1$ colors such that every path contains some color odd number of times. This disproves a conjecture from [1] asserting that for every tree T the minimal number of colors in a such coloring of T is at least the vertex ranking number of T minus one.

Keywords: binomial tree, parity coloring, vertex ranking.

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1. INTRODUCTION

A *parity vertex coloring* of a graph G is a vertex coloring such that each path in G contains some color odd number of times. For a study of parity vertex and (similarly defined) edge colorings, the reader is referred to [1,2]. A *vertex ranking* of G is a proper vertex coloring by a linearly ordered set of colors such that every path between vertices of the same color contains some vertex of a higher color. The minimum numbers of colors in a parity vertex coloring and a vertex ranking of G are denoted by $\chi_p(G)$ and $\chi_r(G)$, respectively.

Clearly, every vertex ranking is also parity vertex coloring, so $\chi_p(G) \leq \chi_r(G)$ for every graph G. Borowiecki, Budajová, Jendrol', and Krajči [1] conjectured that for trees these parameters behave almost the same.

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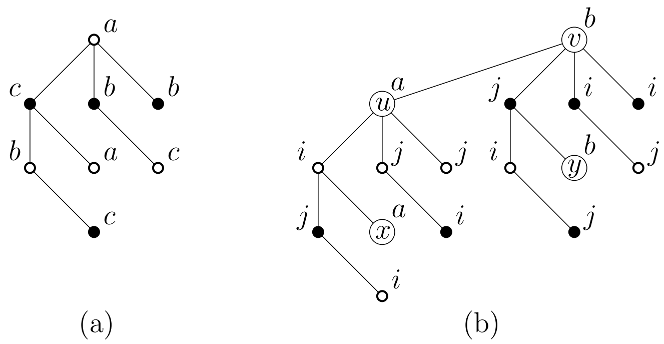

Figure 1. (a). The coloring $g_{(a,b,c)}$ of B_3, (b). the coloring of two subtrees $B_3(u)$ and $B_3(v)$ with $uv \in E(B_{3k})$.

Conjecture 1. *For every tree T it holds $\chi_r(T) - \chi_p(T) \leq 1$.*

In this note we show that the above conjecture is false for every binominal tree of order $n \geq 5$. A *binomial tree* B_n of order $n \geq 0$ is a rooted tree defined recursively. $B_0 = K_1$ with the only vertex as its root. The binomial tree B_n for $n \geq 1$ is obtained by taking two disjoint copies of B_{n-1} and joining their roots by an edge, then taking the root of the second copy to be the root of B_n.

Binomial trees have been under consideration also in other areas. For example, B_n is a spanning tree of the n-dimensional hypercube Q_n that has been conjectured [3] to have the minimum average congestion among all spanning trees of Q_n. In [1] it was shown, in our notation, that $\chi_r(B_n) = n + 1$ for all $n \geq 0$.

We show that $\chi_p(B_{3k}) \leq 2k + 1$ for every $k \geq 1$, which hence disproves the above conjecture. More precisely, for the purpose of induction we prove a stronger statement in the below theorem. Let us say that a color c on a vertex-colored path P is

- *inner*, if c does not appear on the endvertices of P,
- *single*, if c appears exactly once on P.

Moreover, we say that a vertex of B_n is *even* (resp. *odd*) if its distance to the root is even (resp. odd).

Theorem 2. *For every $k \geq 1$ the binomial tree B_{3k} has a parity vertex coloring with $2k + 1$ colors such that every path of length at least 2 has an inner single color.*

Proof. For $k = 1$ we define the coloring $f : V(B_3) \to \{1, 2, 3\}$ by $f = g_{(1,2,3)}$ where $g_{(a,b,c)}$ is defined on Figure 1(a). Observe that f satisfies the statement. In what follows, we assume $k \geq 2$.
The binomial tree B_{3k+3} can be viewed as B_{3k} with a copy of B_3 hanged on each vertex. See Figure 1 for an illustration. For a vertex $v \in V(B_{3k})$, let us denote

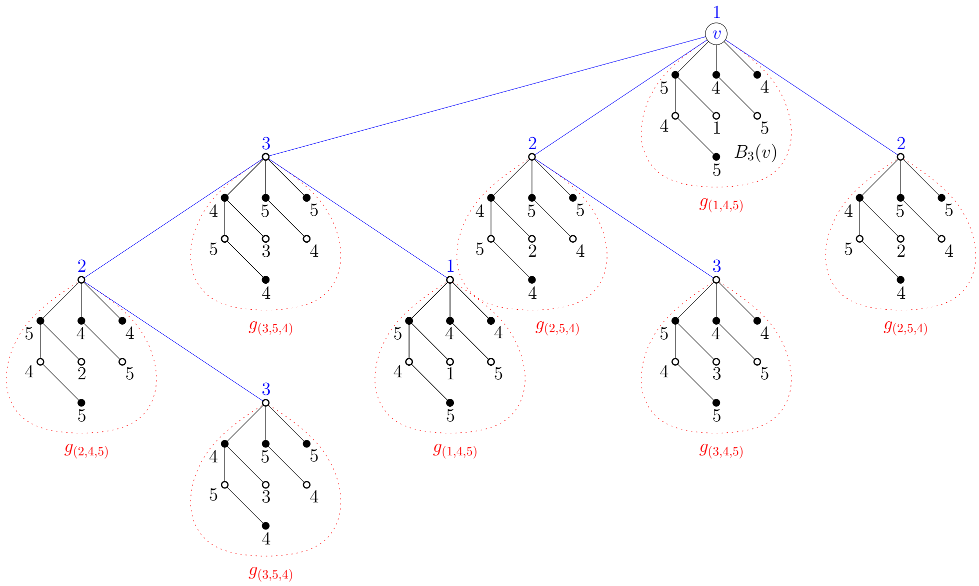

Figure 2. The constructed coloring of B_6 with 5 colors.

the copy of B_3 hanged on v by $B_3(v)$. Let f' be the coloring of B_{3k} with colors $\{1, 2, \ldots, 2k+1\}$ obtained by induction and let $i = 2k+2$, $j = 2k+3$ be the new colors. We define the coloring $f : V(B_{3k+3}) \to \{1, 2, \ldots, j\}$ by

$$f(B_3(v)) = \begin{cases} g_{(f'(v),i,j)} & \text{if } v \text{ is even,} \\ g_{(f'(v),j,i)} & \text{if } v \text{ is odd.} \end{cases}$$

for every vertex $v \in V(B_{3k})$. See Figure 1 for an illustration. Obviously, it is a proper coloring.

Now we show that the constructed coloring f satisfies the statement. Let P be a path in B_{3k+3} with endvertices in subtrees $B_3(u)$ and $B_3(v)$, respectively. We distinguish three cases.

Case 1. $u = v$. Then P is inside $B_3(u)$ and we are done since the statement holds for $k = 1$.

Case 2. $uv \in E(B_{3k+3})$. Without lost of generality, we assume that u is odd and u is a child of v, see Figure 1(b). Clearly, the path P contains the vertices u and v. Moreover, if none of the colors $a = f'(u)$, $b = f'(v)$ is inner and single on P, then both endvertices of P are in $\{u, v, x, y\}$ where x, y are the vertices as on Figure 1(b). Observe that then in all possible cases, i or j is an inner single color on P or $P = (u, v)$. *Case* 3. $u \neq v$ and $uv \notin E(B_{3k+3})$. Let $P = (P_1, P_2, P_3)$ where P_1, P_2, and P_3 are subpaths of P in $B_3(u)$, B_{3k}, and $B_3(v)$ respectively.

As the length of P_2 is at least 2, it contains an inner single color d by induction. Since d is inner, it does not appear neither on P_1 nor P_2. Therefore, the color d is also inner and single on P. ■

From Theorem 2 we obtain the following upper bound.

Corollary 3. $\chi_p(B_n) \leq \lceil \frac{2n+3}{3} \rceil$ *for every* $n \geq 0$.

Proof. It is enough to show that $\chi_p(B_{n+1}) \leq \chi_p(B_n)+1$ for every $n \geq 0$. To this end, if we color both copies of B_n in B_{n+1} by (the same) parity vertex coloring with $\chi_p(B_n)$ colors, and we give the root of B_{n+1} a new color, we obtain a parity vertex coloring of B_{n+1} with $\chi_p(B_n) + 1$ colors. ■

On the other hand, Borowiecki et al. [1] showed that $\chi_p(P_n) = \lceil \log_2(n+1) \rceil$ for every n-vertex path P_n. This gives us a trivial lower bound $\chi_p(B_n) \geq \lceil \log_2(2n+1) \rceil$ as B_n contains a $2n$-vertex path. We ask if the following linear upper bound holds.

Question 4. Is it true that $\chi_p(B_n) \geq \frac{n}{2}$ for every $n \geq 0$?

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