Discussiones Mathematicae
Graph Theory 32 (2012) 161–175

CHARACTERIZING CARTESIAN FIXERS AND MULTIPLIERS

Stephen Benecke[1]

and

Christina M. Mynhardt[2]

Department of Mathematics and Statistics
University of Victoria, P.O. Box 3060 STN CSC
Victoria, B.C., Canada V8W 3R4

e-mail: {stephen, mynhardt}@math.uvic.ca

Abstract

Let $G \,\Box\, H$ denote the Cartesian product of the graphs G and H. In 2004, Hartnell and Rall [On dominating the Cartesian product of a graph and K_2, Discuss. Math. Graph Theory 24(3) (2004), 389–402] characterized prism fixers, i.e., graphs G for which $\gamma(G \,\Box\, K_2) = \gamma(G)$, and noted that $\gamma(G \Box K_n) \geq \min\{|V(G)|, \gamma(G)+n-2\}$. We call a graph G a consistent fixer if $\gamma(G \Box K_n) = \gamma(G)+n-2$ for each n such that $2 \leq n < |V(G)| - \gamma(G)+2$, and characterize this class of graphs.

Also in 2004, Burger, Mynhardt and Weakley [On the domination number of prisms of graphs, Dicuss. Math. Graph Theory **24**(2) (2004), 303–318] characterized prism doublers, i.e., graphs G for which $\gamma(G \,\Box\, K_2) = 2\gamma(G)$. In general $\gamma(G \,\Box\, K_n) \leq n\gamma(G)$ for any $n \geq 2$. We call a graph attaining equality in this bound a Cartesian n-multiplier and also characterize this class of graphs.

Keywords: Cartesian product, prism fixer, Cartesian fixer, prism doubler, Cartesian multiplier, domination number.

2010 Mathematics Subject Classification: 05C69, 05C99.

[1]Supported by the Skye Foundation and the National Research Foundation of South Africa.
[2]Supported by the Natural Sciences and Engineering Research Council of Canada.

1. Introduction

We generally follow the notation and terminology of [5]. For two graphs G and H, the Cartesian product $G \,\Box\, H$ is the graph with vertex set $V(G) \times V(H)$ and vertex (v_i, u_j) adjacent to (v_k, u_l) if and only if (a) $v_i v_k \in E(G)$ and $u_j = u_l$, or (b) $v_i = v_k$ and $u_j u_l \in E(H)$. The graph $G \,\Box\, K_2$ is called the *prism* of G.

As usual $\gamma(G)$ denotes the domination number of G. A set $D \subseteq V(G)$ is called a *γ-set* if it is a dominating set with $|D| = \gamma(G)$. The domination number $\gamma(G \,\Box\, K_2)$ of the prism of G lies between $\gamma(G)$ and $2\gamma(G)$. The edgeless graph $G = \overline{K_m}$ attains equality in the lower bound, whereas $\gamma(K_m \,\Box\, K_2) = 2\gamma(K_m)$.

In 2004, Hartnell and Rall [4] characterized graphs G, called *prism fixers*, for which $\gamma(G \,\Box\, K_2) = \gamma(G)$. A γ-set D of G is called a *symmetric γ-set* if D can be partitioned into two nonempty subsets D_1 and D_2 such that $V(G) - N[D_1] = D_2$ and $V(G) - N[D_2] = D_1$. We write $D = D_1 \cup D_2$ for convenience. A symmetric γ-set $D = D_1 \cup D_2$ is called *primitive* if $|D_i| = 1$ for at least one i.

Theorem 1 [4]. *A connected graph G is a prism fixer if and only if G has a symmetric γ-set.*

Hartnell and Rall generalized the lower bound for $\gamma(G \,\Box\, K_2)$ to $\gamma(G \,\Box\, K_n)$ by utilizing one of their results in [3]. They confirmed that the lower bound is sharp by providing a family of graphs attaining equality.

Corollary 2 [4]. *For any graph G and $n \geq 2$, $\gamma(G \,\Box\, K_n) \geq \min\{|V(G)|, \gamma(G) + n - 2\}$.*

Note that $\gamma(G \,\Box\, K_n) = |V(G)|$ for the edgeless graph $G = \overline{K_m}$. Also, if $n \geq |V(G)| - \gamma(G) + 2$, then $\min\{|V(G)|, \gamma(G) + n - 2\} = |V(G)|$. A minimum domination strategy is to take all vertices in a single copy of G as a dominating set, hence $\gamma(G \,\Box\, K_n) = |V(G)|$.

For $2 \leq n < |V(G)| - \gamma(G) + 2$, Corollary 2 gives a nontrivial lower bound, and a graph G is called a *Cartesian n-fixer* if $\gamma(G \Box K_n) = \gamma(G) + n - 2$. We henceforth simply refer to a Cartesian n-fixer as an *n-fixer*. Furthermore, if G is an n-fixer for each n such that $2 \leq n < |V(G)| - \gamma(G) + 2$, then G is called a *consistent fixer*. We characterize these graphs in Section 2. In Section 3 we discuss graphs that are n-fixers for only some values of n in the range $2 \leq n < |V(G)| - \gamma(G) + 2$. In 2004, Burger, Mynhardt and Weakley [1] characterized *prism doublers*, i.e., graphs G for which $\gamma(G \,\Box\, K_2) = 2\gamma(G)$. In general $\gamma(G \,\Box\, K_n) \leq n\gamma(G)$ for any $n \geq 2$, and a graph attaining equality in this upper bound is called a *Cartesian n-multiplier*. Once again, we refer to such a graph simply as an *n-multiplier*. In Section 4 we follow a similar argument to that in [1] to characterize n-multipliers.

For $A, B \subseteq V(G)$, we abbreviate "A dominates B" to "$A \succ B$"; if $B = V(G)$ we write $A \succ G$ and if $B = \{b\}$ we write $A \succ b$. Further, $N(v) = \{u \in V(G) :$

$uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$ denote the *open* and *closed neighbourhoods*, respectively, of a vertex v of G. The *closed neighbourhood* of $S \subseteq V(G)$ is the set $N[S] = \bigcup_{s \in S} N[s]$, the *open neighbourhood* of S is $N(S) = \bigcup_{s \in S} N(s)$, while $N\{S\}$ denotes the set $N(S) - S$.

Consider two graphs G and H, with vertex sets labelled $v_1, v_2, \ldots, v_m$ and $u_1, u_2, \ldots, u_n$ respectively. Vertices (v_i, u_j) of the Cartesian product $G \,\Box\, H$ are labelled $v_{i,j}$ for convenience. The subgraph induced by all vertices that differ from a given vertex $v_{i,j}$ only in the first [second] coordinate, is known as the (Cartesian) *G-layer* [*H-layer*] through $v_{i,j}$.

We often consider projections $p_G : V(G \,\Box\, H) \to V(G)$ and $p_H : V(G \,\Box\, H) \to V(H)$. A general vertex $v_{i,j}$ of $G \,\Box\, H$ has as first coordinate the vertex $p_G(v_{i,j}) = v_i \in V(G)$ and second coordinate $p_H(v_{i,j}) = u_j \in V(H)$. The *preimage* $p_G^{-1}(v_i)$ of a vertex v_i in G is the set of vertices in $G \,\Box\, H$ that have v_i as first coordinate, that is, the vertex set of the H-layer through $v_{i,j}$ for any j. The *preimage* of $A \subseteq V(G)$ is the set $p_G^{-1}(A) = \bigcup_{v \in A} p_G^{-1}(v)$. The projection p_G and preimage p_G^{-1} are abbreviated to p and p^{-1} respectively.

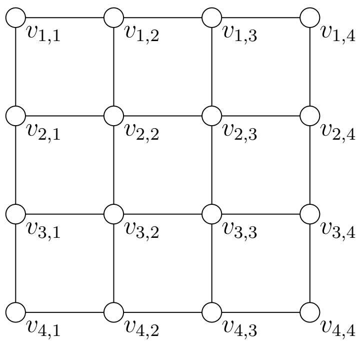

Figure 1. The Cartesian product $P_4 \,\Box\, P_4$.

As an example, consider the graph $P_4 \,\Box\, P_4$ in Figure 1. For this graph we have $p(\{v_{1,3}, v_{3,2}\}) = \{v_1, v_3\}$, while $p^{-1}(\{v_1, v_3\}) = \{v_{i,j} : i = 1, 3,\ j = 1, 2, 3, 4\}$. Lastly, a dominating set W of $G \Box H$ can be partitioned into sets $W_1, W_2, \ldots, W_n$, where W_i is a subset of vertices in the i^{th} G-layer. We write $W = W_1 \cup W_2 \cup \cdots \cup W_n$ when this partition is clear from the context.

2. Consistent Fixers

Hartnell and Rall [4] provided examples of graphs that show that the lower bound in Corollary 2 is sharp. Let G_k be the graph with vertex set $V(G_k) = \{v\} \cup \{x_i, y_i, z_i : i = 1, 2, \ldots, k\}$, and edge set $\{vx_i, x_iy_i, y_iz_i, z_iv : i = 1, 2, \ldots, k\}$.

(The 4-cycles $G_k[\{v, x_i, y_i, z_i\}]$ share a common vertex v, $i = 1, 2, \ldots, k$.) Then $\gamma(G_k) = k + 1$ and $D = \{(y_i, u_1) : i = 1, 2, \ldots, k\} \cup \{(v, u_j) : j = 2, 3, \ldots, n\}$ is a dominating set of $G_k \,\square\, K_n$ of cardinality $k + n - 1 = \gamma(G_k) + n - 2$. The graph G_3 is illustrated in Figure 2. If $k > \frac{n-2}{2}$, then $|V(G_k)| = 3k + 1 > k + n - 1$ and hence $\gamma(G_k \,\square\, K_n) = \gamma(G_k) + n - 2$.

For the graph G_3 in Figure 2, let $D_1 = \{y_1, y_2, y_3\}$ and $D_2 = \{v\}$, and note that $D = D_1 \cup D_2$ is a primitive symmetric γ-set of G_3. In general, any graph G that has a primitive symmetric γ-set satisfies $\gamma(G \,\square\, K_n) = \gamma(G) + n - 2$ for any $2 \leq n < |V(G)| - \gamma(G) + 2$:

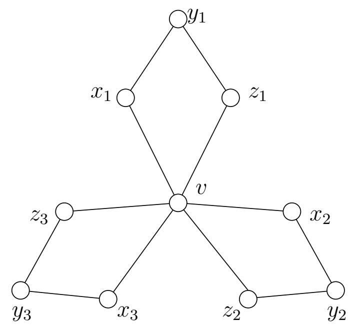

Figure 2. The graph G_3.

Let $V(K_n) = \{u_1, u_2, \ldots, u_n\}$ and $D = D_1 \cup D_2$ be a primitive symmetric γ-set of G with $D_2 = \{x\}$. Figure 3 illustrates the dominating set $W = \{(v, u_1) : v \in D_1\} \cup \{(x, u_i) : i = 2, 3, \ldots, n\}$ of $G \,\square\, K_n$ of cardinality $\gamma(G) + n - 2$. In the first G-layer, the set $Y = V(G) - D$ is dominated by $\{(v, u_1) : v \in D_1\}$, and in the i^{th} G-layer Y is dominated by (x, u_i), $i \geq 2$.

The question now arises whether graphs with primitive symmetric γ-sets are the only n-fixers. Our characterization will show that this is not the case.

We first state some useful properties of a graph having a symmetric γ-set.

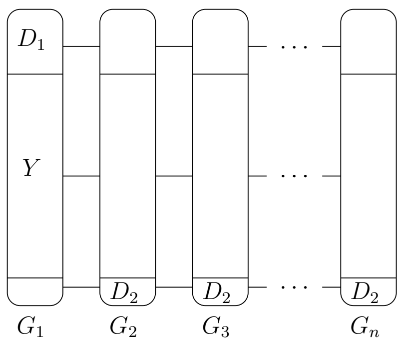

Figure 3. A domination strategy for $G \,\square\, K_n$ if G has a primitive symmetric γ-set.

Observation 3 [4].

(i) *Let G be a connected graph with symmetric γ-set $D = D_1 \cup D_2$ and let $Y = V(G) - D$. Then*

(a) $N[D_i] = D_i \cup Y$, $i = 1, 2$,

(b) *D is an independent set,*

(c) *the sets $\{N(x)\}_{x \in D_i}$ are disjoint, and these sets form a partition of Y,*

(d) *each vertex in D is adjacent to at least two vertices in Y.*

(ii) *Let G be a graph with at least one symmetric γ-set, but no primitive symmetric γ-set, and let $Y = V(G) - D$. Then $\gamma(G[Y]) > 1$.*

(iii) *If G is a 2-fixer and $W = W_1 \cup W_2$ is a γ-set of $G \,\square\, K_2$, then $p(W_1) \cup p(W_2)$ is a symmetric γ-set of G.*

Suppose G is a 2-fixer with no primitive symmetric γ-set and $\gamma(G \square K_3) = \gamma(G)+1$. Then a minimum domination strategy for the Cartesian product $G \square K_3$ will never be to take a γ-set of $G \,\square\, K_2$ and select one vertex in the third G-layer, as we show next.

Lemma 4. *Let G be a connected 3-fixer with symmetric γ-set $D = D_1 \cup D_2$, but no primitive symmetric γ-set. Then no γ-set $W = W_1 \cup W_2 \cup W_3$ of $G \,\square\, K_3$ has $p(W_1) = D_1$, $p(W_2) = D_2$ and $|W_3| = 1$.*

Proof. Let $D = D_1 \cup D_2$ be a symmetric γ-set of G with $|D_1|, |D_2| \geq 2$ and let $Y = V(G) - D$. Suppose $W = W_1 \cup W_2 \cup W_3$ is a γ-set of $G \,\square\, K_3$, with $p(W_1) = D_1$, $p(W_2) = D_2$ and $W_3 = \{(x, u_3)\}$. Then $x \succ Y$. If $x \notin D$, then $x \in Y$ and so $\gamma(G[Y]) = 1$, contradicting Observation 3(ii). So assume $x \in D$, say $x \in D_2$, and let $z \in D_2 - \{x\}$. Then z is adjacent to some vertex in Y, hence x and z have a common neighbour in Y, contradicting Observation 3(i)(c). ■

We now provide a characterization of consistent fixers. We only consider connected graphs and also require G to have at least three vertices; since $\gamma(G) \leq \frac{1}{2}|V(G)|$ for any connected graph G, this requirement ensures that a value $n \geq 3$ is included in the range $2 \leq n < |V(G)| - \gamma(G) + 2$.

Theorem 5. *Let G be a connected graph of order at least 3. Then G is a consistent fixer if and only if*

(i) *G has a primitive symmetric γ-set, or*

(ii) *G has symmetric γ-sets, none of which are primitive, and G has a dominating set $X = X_1 \cup X_2 \cup X_3$ with the following properties:*

(a) $X_i \succ V(G) - X$, $i = 1, 2, 3$,

(b) *for each $i = 1, 2, 3$, the sets $\{N(x) - X\}_{x \in X_i}$ are disjoint and form a partition of $V(G) - X$,*

(c) *the sets* X_i *are disjoint and* $|X| = |X_1| + |X_2| + |X_3| = \gamma(G) + 1$,

(d) $|X_2| = |X_3| = 1$.

Proof. Let G be a consistent fixer. Then by Theorem 1, G has a symmetric γ-set $D = D_1 \cup D_2$. Suppose $|D_1|, |D_2| \geq 2$ for any such set D. We show that (ii) holds.

Since G is also a Cartesian 3-fixer, there exists a minimum dominating set $W = W_1 \cup W_2 \cup W_3$ of $G \square K_3$ of cardinality $\gamma(G)+1$. Let $X_i = p(W_i)$, $i = 1, 2, 3$, $X = X_1 \cup X_2 \cup X_3$ and $Y = V(G) - X$.

Then $X \subseteq V(G)$ is a dominating set of G of cardinality at most $\gamma(G)+1$, i.e., $\gamma(G) \leq |X| \leq \gamma(G) + 1$. If $Y = \emptyset$, then $|V(G)| = |X| \leq \gamma(G) + 1$, contradicting the statement $3 < |V(G)| - \gamma(G) + 2$. Therefore $Y \neq \emptyset$, and so to dominate $p^{-1}(Y)$, $W_i \neq \emptyset$ for each i. Hence $X_i \neq \emptyset$ and, moreover, $X_i \succ Y$ for each $i = 1, 2, 3$. Thus (a) holds.
Without loss of generality, assume that $|X_1| \geq |X_2| \geq |X_3|$ and that W has been chosen so that $|X_1|$ is as large as possible. Since $\gamma(G) \leq |X| \leq \gamma(G) + 1$,

(1) at most one vertex of X occurs in more than one set X_i.

Similarly, no vertex occurs in all three X_i, i.e.,

$$X_1 \cap X_2 \cap X_3 = \emptyset. \tag{2}$$

We now prove the following statement:

(3) Each vertex in $X_2 \cup X_3$ is adjacent to some vertex in Y.

Suppose there exists $x \in X_2$ that is not adjacent to any vertex in Y, and w_2 is a vertex of W_2 such that $p(w_2) = x$. (The argument is the same if $x \in X_3$.) If $x \in X_1$ and w_1 is a vertex of W_1 such that $p(w_1) = x$, then $W - \{w_1\}$ is a dominating set of $G \square K_3$ of cardinality $\gamma(G)$, which is impossible by Corollary 2. Thus $x \notin X_1$. But then $W' = (W_1 \cup \{w_1\}) \cup (W_2 - \{w_2\}) \cup W_3$ is a minimum dominating set of $G \square K_3$ such that $X_1' = p(W_1 \cup \{w_1\}) = X_1 \cup \{x\}$ has larger cardinality than X_1, contradicting the choice of W. Thus (3) holds.

(b) Suppose two distinct vertices $u, v \in X_i$ are both adjacent to some vertex $y \in Y$. By (a), y is adjacent to a vertex in each X_i. By (1) and (2), at least one X_j, $j \neq i$, contains a neighbour w of y such that $w \notin \{u, v\}$. But $X_k \succ Y$, $k \neq i, j$, so $(X - \{u, v, w\}) \cup \{y\}$ is a dominating set of G that has cardinality at most $\gamma(G) - 1$, a contradiction. Hence each vertex $y \in Y$ is dominated by exactly one vertex from X_i, and (b) follows.

(c) We only prove that $X_2 \cap X_3 = \emptyset$; the proofs that $X_1 \cap X_2 = \emptyset$ and $X_1 \cap X_3 = \emptyset$ are similar. It will follow that $|X| = |X_1| + |X_2| + |X_3| = \gamma(G) + 1$. Suppose there exists a vertex $z \in X_2 \cap X_3$. Then $|X| = \gamma(G)$ and, by (1) and (2), $X_1 \cap (X_2 \cup X_3) = \emptyset$, so that $X = X_1 \cup (X_2 \cup X_3)$ is a symmetric γ-set of G.

If $|X_3| = 1$, then $X_3 = \{z\} \subseteq X_2$ and $X = X_1 \cup X_2$. By (a), z dominates all of Y. But $z \in X_2$, and so (b) implies that $X_2 = \{z\}$, i.e., $|X_2| = 1$. Then X is a primitive symmetric γ-set, which is not the case under consideration. Therefore $|X_3| \geq 2$; say $w, z \in X_3$. By (1), $w \notin X_1 \cup X_2$, and by (3), w is adjacent to some vertex in Y. Since $X_2 \succ Y$, there exists $v \in X_2$ such that v and w have a common neighbour in Y. This contradicts Observation 3(i)(c) for the symmetric γ-set $X = X_1 \cup (X_2 \cup X_3)$. Therefore $X_2 \cap X_3 = \emptyset$.

(d) Suppose that $|X_2| \geq 2$. Then $|X_1| \geq 2$. Let $y_1 \in Y$ and choose $x_1 \in X_1$, $x_2 \in X_2$ such that x_1 and x_2 are both adjacent to y_1. Since $X_3 \succ Y$, the set $X' = (X - \{x_1, x_2\}) \cup \{y_1\}$ is a dominating set of G of cardinality $\gamma(G)$, i.e., a γ-set of G. We show that

$$\{x_1, x_2\} \succ Y. \tag{4}$$

Suppose to the contrary that $y \in Y$ is not adjacent to either x_1 or x_2. Then there exist $x_1' \in X_1 - \{x_1\}$ and $x_2' \in X_2 - \{x_2\}$ adjacent to y, so that $(X' - \{x_1', x_2'\}) \cup \{y\}$ is a dominating set of G of cardinality $\gamma(G) - 1$, which is impossible.

Let $v \in X_2 - \{x_2\}$. By (3) there exists a vertex $y_2 \in Y$ adjacent to v. By (b) y_2 is not adjacent to x_2 and so, by (4), y_2 is adjacent to x_1. It follows similar to (4) that $\{x_1, v\} \succ Y$. But then any vertex in Y not adjacent to x_1 is adjacent to both x_2 and v, which is impossible by (b). Thus $x_1 \succ Y$, and (b) implies that $|X_1| = 1$, a contradiction. Therefore $|X_2| = 1$ which, by the choice of the X_i, also implies that $|X_3| = 1$.

Conversely, let G be a graph that satisfies the conditions of the statement, $2 \leq n < |V(G)| - \gamma(G) + 2$ and $V(K_n) = \{u_1, u_2, \ldots, u_n\}$. If G has a symmetric γ-set $D = D_1 \cup D_2$ with $D_2 = \{x\}$, then the set $W = \{(v, u_1) : v \in D_1\} \cup \{(x, u_i) : i = 2, 3, \ldots, n\}$ is a dominating set of $G \,\square\, K_n$ of cardinality $\gamma(G) + n - 2$, as illustrated in Figure 2.

Suppose that $|D_1|, |D_2| \geq 2$ and that G has a set $X = X_1 \cup X_2 \cup X_3$ with the stated properties. Let $X_2 = \{x_2\}$ and $X_3 = \{x_3\}$. Then the set

$$W = \{(v, u_1) : v \in X_1\} \cup \{(x_2, u_2)\} \cup \{(x_3, u_i) : i = 3, 4, \ldots, n\}$$

is a dominating set of $G \,\square\, K_n$ of cardinality $\gamma(G) + n - 2$. ■

The dominating set $X = X_1 \cup X_2 \cup X_3$ in Theorem 5(ii) has the following additional properties.

Proposition 6. *Let G be a connected graph of order at least 3. If G is a consistent fixer with no primitive symmetric γ-set, then the dominating set $X = X_1 \cup X_2 \cup X_3$ in Theorem 5(ii) has the following properties:*

(i) $X_1 \cup X_2$ *and* $X_1 \cup X_3$ *are independent sets,*

(ii) $\gamma(G[N(x)]) \geq 2$ *for every* $x \in X_1$,

(iii) *for some* $x \in X_1$, $G[N(x)]$ *has a* γ*-set,* $\{y_1, y_2\}$ *say, such that for every* $x' \in X_1 - \{x\}$,

(a) $y_1 \succ N(x')$ *and* $N(y_2) \cap N(x') = \emptyset$, *or*

(b) $y_2 \succ N(x')$ *and* $N(y_1) \cap N(x') = \emptyset$.

Proof. Say $X_2 = \{x_2\}$, $X_3 = \{x_3\}$, $Y = V(G) - X$, and note that

$$x_i \succ Y, \ i = 2, 3. \tag{5}$$

(i) Consider any symmetric γ-set $D = D_1 \cup D_2$ of G and recall that $|D_i| \geq 2$. Define $Y' = V(G) - D$. We compare D and X, and show that

$$\begin{aligned} &|D_i \cap Y| = 1 \text{ for } i = 1, 2, \quad |D \cap X_1| = \gamma(G) - 2 = |X_1| - 1, \\ &\text{and} \quad |X_1 \cap Y'| = 1. \end{aligned} \tag{6}$$

We begin by showing that $\{x_2, x_3\} \cap D = \emptyset$. Suppose $x_2 \in D$; without loss of generality say $x_2 \in D_2$. Then (5) and Observation 3(i)(b) imply that $Y \cap D = \emptyset$. Now if $x_3 \in D$, then Observation 3(i)(c) implies that $x_3 \in D_1$ and that the only vertices in $X_1 \cap D$ are vertices that are nonadjacent to all vertices in Y. But $|X| = \gamma(G) + 1$, $|X_1| = \gamma(G) - 1$ and $|D| = \gamma(G)$, so that $\gamma(G) - 2$ vertices in X_1 are in D. Therefore exactly one vertex in X_1, say x_1, is adjacent to vertices in Y. By Theorem 5(ii)(a), $x_1 \succ Y$. Furthermore, $x_1 \in Y'$ by Observation 3(i)(c). If there exists a $v \in X_1 - \{x_1\}$, then $v \in D$, hence v is adjacent to at least two vertices in Y' by Observation 3(i)(d). Since $Y' - \{x_1\} = Y$, this is a contradiction. So $X_1 = \{x_1\}$ and it follows that D is a primitive symmetric γ-set, a contradiction. Therefore $x_3 \notin D$ and so $D = X_1 \cup X_2$ and $V(G) - D = Y \cup \{x_3\}$.

Let $u \in D_2 - \{x_2\}$. By Observation 3(i)(d), u is adjacent to at least two vertices in Y', so u is adjacent to some $y \in Y$. But then y is adjacent to the two vertices $x_2, u \in D_2$, contradicting Observation 3(i)(c). Hence $x_2 \notin D$. Similarly, $x_3 \notin D$, i.e., $\{x_2, x_3\} \subseteq Y'$.

Since $|X_1| = \gamma(G) - 1$, it follows that $Y \cap D \neq \emptyset$. If $|D_i \cap Y| \geq 2$ for some i, then by (5), two vertices in D_i have $x_2 \in Y'$ as common neighbour, contrary to Observation 3(i)(c). Thus $|D_i \cap Y| \leq 1$ for each i, so $|Y \cap D| \leq 2$. If $Y \cap D = \{y\}$, then $D = X_1 \cup \{y\}$. But by Theorem 5(ii)(a), y is adjacent to some vertex in X_1, contradicting Observation 3(i)(b). Therefore $|Y \cap D| = 2$ and (6) follows.

Let $X_1 \cap Y' = \{x_1\}$ and $D_i \cap Y = \{y_i\}$, $i = 1, 2$. Then $X_1 - \{x_1\} \subseteq D$ and so $X_1 - \{x_1\}$ is independent (Observation 3(i)(b)).

Suppose x_1 is not adjacent to y_1. Since $X_1 \succ Y$, y_1 is adjacent to some $x' \in X_1 - \{x_1\} \subseteq D$. But $y_1 \in D$ and D is independent, a contradiction. Hence

x_1 is adjacent to y_1 and, similarly, to y_2. It now follows from Observation 3(i)(c) that x_1 is not adjacent to any vertex in X_1 and so X_1 is independent.

By (5), x_2 and x_3 are adjacent to y_1 and y_2, hence as in the case of x_1, neither x_2 nor x_3 is adjacent to any vertex in $X_1 - \{x_1\}$. Since G is connected, each vertex in $X_1 - \{x_1\}$ is therefore adjacent to a vertex in Y; since D is independent this vertex is necessarily in $Y - \{y_1, y_2\}$. Since $|D_1| \geq 2$, there exists $x_4 \in D_1 - \{y_1\}$; necessarily $x_4 \subseteq X_1 - \{x_1\}$. Let $y_4 \in Y - \{y_1, y_2\}$ be adjacent to x_4 and consider the set $X' = (X - \{x_1, x_3, x_4\}) \cup \{y_4\}$. Then $x_2 \succ Y$, $y_4 \succ x_4$ and $y_4 \succ x_3$ by (5). Therefore $X' \succ G - x_1$. But $|X'| < \gamma(G)$ and so $X' \not\succ G$, i.e., $X' \not\succ x_1$. In particular, x_2 is not adjacent to x_1. Similarly, x_3 is not adjacent to x_1, and the proof of (i) is complete.

(ii) Since $\gamma(G) \geq 4$, $|X_1| \geq 3$. Say $X_1 = \{x_1, x_4, x_5, \ldots, x_k\}$ and define $Y_i = N(x_i)$, $i = 1, 4, 5, \ldots, k$. By (i), no vertex in X_1 is adjacent to any vertex in X, so $Y_i \subseteq Y$ for each i, and since G is connected, $Y_i \neq \emptyset$. By Theorem 5(ii)(a) and (b), the sets $Y_1, Y_4, \ldots, Y_k$ partition Y. Suppose that for some i there exists a vertex $y \in Y_i$ that is adjacent to all other vertices in Y_i and consider $X' = (X - \{x_i, x_2, x_3\}) \cup \{y\}$. Then by (5), $y \succ Y_i \cup \{x_i, x_2, x_3\}$, while $X_1 - \{x_i\} \succ Y - Y_i$, so that $X' \succ G$. But $|X'| = \gamma(G) - 1$, which is impossible. This proves (ii).

(iii) As shown above, $D = \{y_1, y_2, x_4, \ldots, x_k\}$ and $Y' = \{x_1, x_2, x_3\} \cup (Y - \{y_1, y_2\})$. By Observation 3(i)(c), each vertex in Y' is adjacent to exactly one vertex in each D_i. In particular, since X_1 is independent, x_1 is adjacent to y_1 and y_2. Since the Y_i partition Y, no vertex in Y is adjacent to two vertices in X_1. But for each $i = 4, \ldots, k$, x_i is in exactly one of D_1 or D_2, so if $x_i \in D_1 - \{y_1\}$, then each vertex in $Y_i = N(x_i)$ is also adjacent to y_2 but not to y_1, and if $x_i \in D_2 - \{y_2\}$, then each vertex in Y_i is also adjacent to y_1 but not to y_2. Moreover, $\{y_1, y_2\} \succ Y \supseteq Y_1 = N(x_1)$ and so, by (ii), $\{y_1, y_2\}$ is a γ-set of $N(x_1)$. Therefore (iii) holds with $x = x_1$. ∎

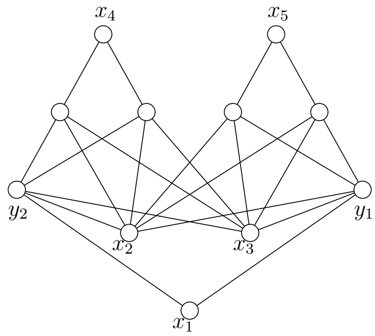

Figure 4. A consistent fixer with no primitive symmetric γ-set.

The properties of the dominating set $X = X_1 \cup X_2 \cup X_3$ given in Theorem 5 and Proposition 6 allow us to easily construct consistent fixers without primitive symmetric γ-sets. Figure 4 shows a consistent fixer G that has a symmetric γ-set $D = D_1 \cup D_2$ with $|D_1| = |D_2| = 2$. In this example, $D_1 = \{y_1, x_4\}$, $D_2 = \{y_2, x_5\}$, $X_1 = \{x_1, x_4, x_5\}$, $X_2 = \{x_2\}$ and $X_3 = \{x_3\}$. Since $\Delta(G) = 6$, G has no primitive symmetric γ-set.

If G is a consistent fixer, then $G \,\square\, K_n$, $n \geq 3$, has a minimum dominating set that contains exactly one vertex in all but one of the G-layers of $G \,\square\, K_n$, as stated in the following corollary.

Corollary 7. *If G is a consistent fixer and $3 \leq n < |V(G)| - \gamma(G) + 2$, then $G \,\square\, K_n$ has a γ-set $X = X_1 \cup \cdots \cup X_n$ with $|X_i| = 1$ for $i = 2, \ldots, n$, where X_i lies in the i^{th} G-layer of $G \,\square\, K_n$, $i = 1, \ldots, n$.*

3. Other Fixers

For any integer $t \geq 4$ there exist graphs that are 2-fixers and n-fixers for $t \leq n < |V(G)| - \gamma(G) + 2$, but not for $2 < n < t$. Figure 5 shows a graph G that is a 2-fixer and a 4-fixer, but not a 3-fixer. Each vertex x_2, x_3 and x_6 is adjacent only to the vertices y_1, y_2, a, b, c and d, but these edges are omitted in the figure for the sake of clarity. The graph has a symmetric γ-set $D = D_1 \cup D_2$ with $D_1 = \{x_4, y_1\}$ and $D_2 = \{x_5, y_2\}$. Since $\Delta(G) = 6$, G does not have a primitive symmetric γ-set. Furthermore, it is easy to verify that G does not have a set $X = X_1 \cup X_2 \cup X_3$ with the properties stated in Theorem 5, and therefore is not a 3-fixer. However, for $n \geq 4$, the set

$$W = \{(x_1, u_1), (x_4, u_1), (x_5, u_1), (x_2, u_2), (x_3, u_3)\} \cup \{(x_6, u_i) : i \geq 4\}$$

is a dominating set of $G \,\square\, K_n$ of cardinality $\gamma(G) + n - 2$, so that G is an n-fixer.

The characterization of these n-fixers is similar to that of Theorem 5 and the proof is therefore omitted.

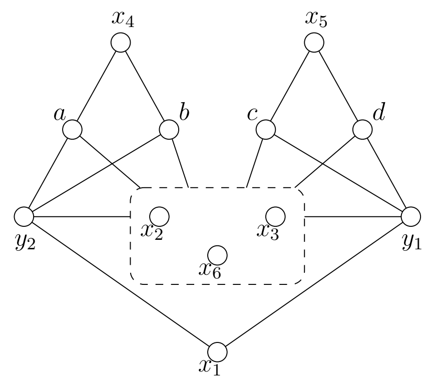

Figure 5. A graph that is a 2-fixer and a 4-fixer, but not a 3-fixer.

Theorem 8. *Let G be a connected graph and $t \geq 4$. Then G is a 2-fixer and an n-fixer for $n \geq t$, but not for $2 < n < t$, if and only if*

(i) *G has symmetric γ-sets, none of which are primitive, and*

(ii) *t is the smallest integer such that G has a dominating set $X = X_1 \cup \cdots \cup X_t$ with the following properties:*

 (a) *$X_i \succ V(G) - X$, $i = 1, 2, \ldots, t$,*

 (b) *for each $i = 1, 2, \ldots, t$, the sets $\{N(x) - X\}_{x \in X_i}$ are disjoint and form a partition of $V(G) - X$,*

 (c) *the sets X_i are disjoint and $|X| = \sum_{i=1}^{t} |X_i| = \gamma(G) + t - 2$,*

 (d) *$|X_i| = 1$ for $i \geq 2$.*

Similar to Proposition 6, the set $X = X_1 \cup \cdots \cup X_t$ has the following additional properties.

Proposition 9. *Let G be a connected graph of order at least 3, and $t \geq 3$. If G is a 2-fixer and an n-fixer, $n \geq t$, that has no primitive symmetric γ-set, then the dominating set $X = X_1 \cup \cdots \cup X_t$ in Theorem 8(ii) has the following properties:*

(i) *$X_1 \cup X_i$ is an independent set, $i = 2, \ldots, t$,*

(ii) *$\gamma(G[N(x)]) \geq 2$ for every $x \in X_1$,*

(iii) *for some $x \in X_1$, $G[N(x)]$ has a γ-set, $\{y_1, y_2\}$ say, such that for every $x' \in X_1 - \{x\}$,*

 (a) *$y_1 \succ N(x')$ and $N(x') \cap N(y_2) = \emptyset$, or*

 (b) *$y_2 \succ N(x')$ and $N(x') \cap N(y_1) = \emptyset$.*

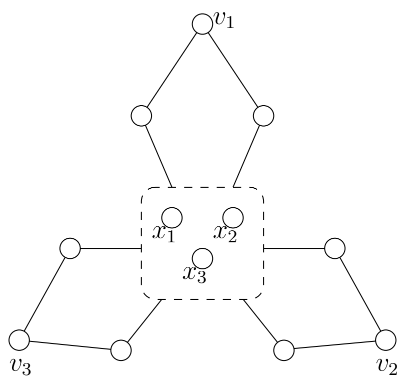

Figure 6. An n-fixer only for $n \geq 4$.

Lastly, we consider graphs that are n-fixers for $n \geq t \geq 3$, but not for $n < t$. As an example, Figure 6 shows a graph G that is an n-fixer for $n \geq 4$ only. In this graph, each vertex x_1, x_2 and x_3 is adjacent only to the neighbours of v_1, v_2 and v_3. It is easy to verify that $\gamma(G) = 4$, the graph does not have a symmetric γ-set, and that it is not a 3-fixer.

The following characterization describes such fixers. The proof is also similar to that of Theorem 5 and is omitted.

Theorem 10. *Let G be a connected graph and $t \geq 3$. Then G is an n-fixer for $n \geq t$, but not for $2 < n < t$, if and only if G does not have a symmetric γ-set, and t is the smallest integer such that G has a dominating set $X = X_1 \cup \cdots \cup X_t$ with the following properties:*

(a) $X_i \succ V(G) - X$, $i = 1, 2, \ldots, t$,

(b) *for each $i = 1, 2, \ldots, t$, the sets $\{N(x) - X\}_{x \in X_i}$ are disjoint and form a partition of $V(G) - X$,*

(c) *the sets X_i are disjoint and $|X| = \sum_{i=1}^{t} |X_i| = \gamma(G) + t - 2$,*

(d) $|X_i| = 1$ *for $i \geq 2$.*

4. Cartesian n-multipliers

Consider n such that $\gamma(G) + n - 2 < |V(G)|$ and recall that $\gamma(G) + n - 2 \leq \gamma(G \,\square\, K_n) \leq n\gamma(G)$. We observe that, for any positive integer m and for any $0 \leq i \leq (m-1)(n-1) + 1$, there exists a graph G such that $\gamma(G) = m$ and $\gamma(G \,\square\, K_n) = m + n - 2 + i$. (The upper bound on i ensures that $\gamma(G) + n - 2 + i \leq n\gamma(G)$.) Consider the complete bipartite graph $G = K_{l,k}$ with $l \leq k$ and let $x_1, x_2, \ldots, x_l$ be the vertices in the smaller partite set. With notation as in Theorem 8, let $X_i = \{x_i\}$ and $X = \{x_1, x_2, \ldots, x_l\}$. If $l = 2$, then X is a primitive symmetric γ-set of G, which is a consistent fixer by Theorem 5. If $l = n \geq 3$, then X satisfies the conditions in Theorem 10, so G is an n-fixer. If $l = n + i$, then $\gamma(G \,\square\, K_n) = \gamma(G) + n - 2 + i$, up to values of i for which $\gamma(G \,\square\, K_n) = n\gamma(G)$, in which case G is an n-multiplier (or a prism doubler if $n = 2$).

Burger, Mynhardt and Weakley [1] characterized prism doublers as follows.

Proposition 11 [1]**.** *A graph G is a prism doubler if and only if for each set $X \subseteq V(G)$ with $0 < |X| < \gamma(G)$, and $Y = V(G) - N[X]$, either*

(i) $|Y| \geq 2\gamma(G) - |X|$, *or*

(ii) $|Y| = 2\gamma(G) - |X| - d$ *for some $1 \leq d \leq |X|$, and at least d vertices (necessarily in $N[X]$) are required to dominate $N\{X\} - N[Y]$.*

Following a similar argument to that used in [1], we provide a characterization of n-multipliers. In $G \,\square\, K_n$ we denote the i^{th} G-layer of G by G_i and $V(G_i)$

by V_i. For $S \subseteq V(G)$, let $\langle S \rangle_i$ denote the counterpart of S in G_i. Note that if $|V(G)| < n\gamma(G)$, then G is not an n-multiplier since V_1 is a dominating set of $G \,\Box\, K_n$. Thus we only consider graphs G of order at least $n\gamma(G)$.

Proposition 12. *A graph G is an n-multiplier if and only if for each set $X \subseteq V(G)$ with $0 < |X| < \gamma(G)$, and $Y = V(G) - N[X]$, either*

(i) $|Y| \geq n\gamma(G) - |X|$, *or*

(ii) $|Y| = n\gamma(G) - |X| - d$ *for some* $1 \leq d \leq (n-1)|X|$, *and for any partition* $Y_2, Y_3, \ldots, Y_n$ *of* Y, *the subgraph of* $G \,\Box\, K_n$ *induced by* $\bigcup_{i=2}^{n} \langle N\{X\} - N[Y_i] \rangle_i$ *has domination number at least* d.

Proof. Suppose G is an n-multiplier and consider any set $X \subseteq V(G)$, where $0 < |X| < \gamma(G)$, and $Y = V(G) - N[X]$.

If $|Y| \geq n\gamma(G) - |X|$, then (i) holds. If $|Y| < n\gamma(G) - n|X|$, then $(\bigcup_{i=1}^{n} \langle X \rangle_i) \cup \langle Y \rangle_1$ is a dominating set of $G \,\Box\, K_n$ of cardinality $n|X| + |Y| < n\gamma(G)$ — a contradiction.
Hence we assume that $|Y| = n\gamma(G) - |X| - d$ for some $1 \leq d \leq (n-1)|X|$. Suppose there exists a partition $Y_2, Y_3, \ldots, Y_n$ of Y such that the subgraph of $G \,\Box\, K_n$ induced by $\bigcup_{i=2}^{n} \langle N\{X\} - N[Y_i] \rangle_i$ is dominated by some set D of cardinality less than d. Then $\langle X \rangle_1 \cup (\bigcup_{i=2}^{n} \langle Y_i \rangle_i) \cup D$ is a dominating set of $G \,\Box\, K_n$ of cardinality less than $|X| + |Y| + d = n\gamma(G)$ — a contradiction.

Conversely, suppose that $\gamma(G \,\Box\, K_n) < n\gamma(G)$, and consider any minimum dominating set $D = D_1 \cup \cdots \cup D_n$ of $G \,\Box\, K_n$. Let $B_i = p(D_i)$, $i = 1, \ldots, n$. Then $|B_i| < \gamma(G)$ for some i; without loss of generality assume $|B_1| < \gamma(G)$. Then $|B_1| > 0$, otherwise at least $|V(G)|$ vertices are needed to dominate G_1 in $G \,\Box\, K_n$. But then $|V(G)| \leq |D| < n\gamma(G)$ and these graphs are not considered. Thus $0 < |B_1| < \gamma(G)$. We show that neither (i) nor (ii) holds for the set $X = B_1$.

Let $B = B_1 \cup B_2 \cup \cdots \cup B_n$ and $Y = V(G) - N[B_1]$. In the layer G_1, $V_1 - N[D_1]$ is dominated by $D_2 \cup \cdots \cup D_n$. Therefore in G, $Y \subseteq \bigcup_{i=2}^{n} B_i$ and so $|Y| \leq |B| - |B_1| < n\gamma(G) - |B_1|$. Thus (i) does not hold. If $|Y| < n\gamma(G) - n|B_1|$, then (ii) does not hold either and we are done. Hence we assume that $|Y| = n\gamma(G) - |B_1| - d$ for some $1 \leq d \leq (n-1)|B_1|$.

Let $Y_2, Y_3, \ldots, Y_n$ be a partition of Y such that $Y_i \subseteq B_i$, $i = 2, 3, \ldots, n$, and let $Z_i = B_i - Y_i$. Then the set $D' = \bigcup_{i=2}^{n} \langle Z_i \rangle_i$ dominates the subgraph of $G \,\Box\, K_n$ induced by $\bigcup_{i=2}^{n} \langle N\{B_1\} - N[Y_i] \rangle_i$. But

$$|D'| \leq \sum_{i=2}^{n} |B_i| - \sum_{i=2}^{n} |Y_i| < n\gamma(G) - |B_1| - |Y| = d.$$

Therefore (ii) does not hold. ∎

We construct a family of multipliers with domination number 2. Let $n \geq 2$ and consider disjoint complete graphs K_{n+1} and K_{2n}, with vertex sets $A =$

$\{v_1, v_2, \ldots, v_{n+1}\}$ and $B = \{w_1, w_2, \ldots, w_{2n}\}$, respectively. Let G_n be the graph obtained by adding the edges $v_i w_i$, $i = 1, \ldots, n+1$. We use Proposition 12 to show that G_n is an n-multiplier. Since $\gamma(G) = 2$, we only consider sets X of cardinality 1. There are three possibilities for X.

- If $X = \{v_i\}$, then $Y = B - \{w_i\}$ and $|Y| = 2n - 1 = n\gamma(G_n) - |X|$.
- If $X = \{w_i\}$ with $i \le n+1$, then $Y = A - \{v_i\}$ and $|Y| = n = n\gamma(G_n) - |X| - d$ with $d = n-1$. For any $Y' \subseteq Y$, $N(w_i) - N[Y']$ contains the vertices $w_{n+2}, \ldots, w_{2n}$. Thus, for any partition $Y_2, Y_3, \ldots, Y_n$ of Y, the subgraph of $G_n \square K_n$ induced by $\bigcup_{j=2}^{n} \langle N(w_i) - N[Y_j] \rangle_j$ has a subgraph isomorphic to $K_{n-1} \square K_{n-1}$, which has domination number $d = n-1$. Hence Proposition 12(ii) holds.
- If $X = \{w_i\}$, $i > n+1$, a similar argument shows that Proposition 12(ii) also holds.

It follows that G is an n-multiplier.

5. Conclusion

We conclude with open problems for future research. Let G and H be graphs of order m and n respectively. The Cartesian product $G \square H$ possesses a so-called *layer-partition* property, in that its vertex set allows two partitions $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ and $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_m\}$ such that (a) each $P_i \in \mathcal{P}$ induces a copy of G, called a *G-layer*, (b) each $Q_j \in \mathcal{Q}$ induces a copy of H, called an *H-layer*, (c) any P_i and Q_j intersect in exactly one vertex, and (d) any edge in the product is in either exactly one G-layer or exactly one H-layer.

In 1967, Chartrand and Harary [2] defined the *generalized prism* πG of G as the graph consisting of two copies of G, with edges between the copies determined by a permutation π acting on $V(G)$. For any permutation π, $\gamma(G) \le (\pi G) \le 2\gamma(G)$.

We now define a *generalized Cartesian product* $G \mathbin{\boxed{\pi}} H$ that corresponds to $G \square H$ when π is the identity, πG when H is the graph K_2, and that retains a layer-partition property. For two labelled graphs G and H and permutation π acting on $V(G)$, the product $G \mathbin{\boxed{\pi}} H$ is the graph with vertex set $V(G) \times V(H)$, and vertex (v_i, u_j) is adjacent to (v_k, u_l), $j \le l$, if and only if (a) $v_i v_k \in E(G)$ and $u_j = u_l$, or (b) $v_k = \pi^{l-j}(v_i)$ and $u_j u_l \in E(H)$.

Note that $\gamma(G) \le \gamma(G \mathbin{\boxed{\pi}} H) \le \gamma(G)|V(H)|$ for any G, H and permutation π. Burger, Mynhardt and Weakley [1] investigated graphs G for which $\gamma(\pi G) = 2\gamma(G)$ for any π.

Question 1. *For some graph H of order n, is it possible to characterize graphs G for which $\gamma(G \mathbin{\boxed{\pi}} H) = n\gamma(G)$ for every π?*

In 2006, Mynhardt and Xu [6] investigated graphs G for which $\gamma(\pi G) = \gamma(G)$ for any π, and conjectured that only the edgeless graphs have this property.

Question 2. *For some graph H of order n, does there exist a nontrivial graph G such that $\gamma(G \boxtimes_\pi H) = \gamma(G) + n - 2$ for every π?*

Acknowledgements

This paper is based upon part of the first author's Ph.D. dissertation and he gratefully acknowledges financial assistance from the South African NRF prestige awards programme and the Skye Foundation. Work towards this paper was also supported financially by NSERC.

REFERENCES

[1] A.P. Burger, C.M. Mynhardt and W.D. Weakley, *On the domination number of prisms of graphs*, Dicuss. Math. Graph Theory **24** (2004) 303–318.

[2] G. Chartrand and F. Harary, *Planar permutation graphs*, Ann. Inst. H. Poincaré Sect. B (N.S.) **3** (1967) 433–438.

[3] B.L. Hartnell and D.F. Rall, *Lower bounds for dominating Cartesian products*, J. Combin. Math. Combin. Comput. **31** (1999) 219–226.

[4] B.L. Hartnell and D.F. Rall, *On dominating the Cartesian product of a graph and K_2*, Discuss. Math. Graph Theory **24** (2004) 389–402.

[5] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, New York, 1998).

[6] C.M. Mynhardt and Z. Xu, *Domination in prisms of graphs: Universal fixers*, Utilitas Math. **78** (2009) 185–201.

Received 26 February 2009
Revised 15 March 2011
Accepted 4 April 2011