Discussiones Mathematicae
Graph Theory 32 (2012) 153–160

ON A GENERALIZATION OF THE FRIENDSHIP THEOREM

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Abstract

The Friendship Theorem states that if any two people, of a group of at least three people, have exactly one friend in common, then there is always a person who is everybody's friend. In this paper, we generalize the Friendship Theorem to the case that in a group of at least three people, if every two friends have one or two common friends and every pair of strangers have exactly one friend then there exist one person who is friend to everybody in the group. In particular, we show that the graph corresponding to this problem is of type $G = K_1 \vee (sK_2 + tK_3)$, where s and t are non-negative integers and K_m is the complete graph on m vertices.

Keywords: (λ, μ)-graph, Friendship Theorem.

2010 Mathematics Subject Classification: 05C75.

1. Introduction

In this paper we assume a graph G to be a finite simple graph. We denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. The neighborhood of a vertex $v \in V(G)$ is the set $N(v) = \{u \in V(G) \mid (u,v) \in E(G)\}$. We denote the *degree* of a vertex $v \in G$ by $d_G(v)$, which is the number of edges of G incident to v. It is obvious that $d_G(v) = |N(v)|$. If G_1 and G_2 are two simple graphs, we define the *join* of G_1 and G_2, denoted by $G_1 \vee G_2$, to be the graph with vertex set $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{(u,v) \mid u \in V(G_1), v \in V(G_2)\}$. Also we define the *disjoint union* of G_1 and G_2, denoted by $G_1 + G_2$, to be the graph whose vertex set is $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and edge set is $E(G_1 + G_2) = E(G_1) \cup E(G_2)$.

We say that two vertices u, $v \in V(G)$ are adjacent if $(u, v) \in E(G)$. The friendship theorem can be stated as follows: Suppose that in a group of three people or more, any pair of people have exactly one friend. Then there is one person who is a friend to everybody in the group. This theorem was introduced and proved in [2] by Erdös, Rènyi and Sós.

In a graph theory notation we can state the friendship theorem as follows:

Theorem 1. *If G is a graph in which any two distinct vertices have exactly one common neighbor, then G has a vertex joined to all others.*

Several proofs of the friendship theorem are known. The proof of the theorem in [2] used polarities in finite projective planes. While the proof in [8] is based on computing the eigenvalues of the square of the adjacency matrix of the graph argument. A third proof which is purely combinatorial was given in [6].

We use the notation $\delta(u, v) = |N(u) \cap N(v)|$ to denote the number of common neighbors of the vertices u and v. Using the above notation, we introduce another version of the Friendship Theorem, that was introduced and proved in [2].

Theorem 2. *If $\delta(u, v) = 1$ for any two distinct vertices u, v in a graph G, then $G = K_1 \vee (mK_2)$, where mK_2 denotes the disjoint union of m copies of the complete graph on 2 vertices.*

In [3], a generalization of the Friendship Theorem was introduced by R. Grera and J. Shen, in which they used graphs of type (λ, μ):

Definition. Let G be a graph with n vertices. We say that G is a (λ, μ)-graph if every pair of adjacent vertices have λ common neighbors, and every pair of non-adjacent vertices have μ common neighbors.

We denote such a graph by $SR(n, \lambda, \mu)$. Note that if G is $SR(n, 1, 1)$ then, for some positive integer m, $G = K_1 \vee (mK_2)$ which is the graph that represents the Friendship Theorem.

The generalization of the Friendship Theorem as introduced and proved in [3] is as follows:

Theorem 3. *Suppose G is an irregular (λ, μ)-graph on n vertices. Then one of the following is true:*

(1) $\mu = 0$ *and* $G = mK_{\lambda+2} + tK_1$ *(disjoint union of m copies of $K_{\lambda+2}$ and t copies of K_1), where* $n = m(\lambda + 2) + t$.

(2) $\mu = 1$ *and* $G = K_1 \vee (mK_{\lambda+1})$, *where* $n = m(\lambda + 1) + 1$.

2. Two Important Lemmas

In this paper we generalize the Friendship Theorem to the case that $\lambda = 1$ or 2, and $\mu = 1$. That is, we consider graphs of type $SR(n, 1 \text{ or } 2, 1)$, which is the graph G on n vertices such that every two adjacent vertices have one or two common neighbors, and every two non-adjacent vertices have one common neighbor. To characterize these kind of graphs we need the following two lemmas:

Lemma 4. *If $G = SR(n, 1 \text{ or } 2, 1)$ then G has no subgraph of this form:*

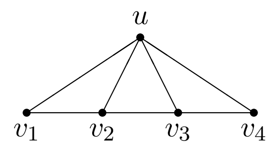

Figure 1

Proof. Suppose G has a subgraph of the type given in Figure 1. Since $\delta(v_1, v_3) = 2$ (u and v_2 are common neighbors to v_1 and v_3) then v_1 must be adjacent to v_3, so that $(v_1, v_3) \in E(G)$. By the same reasoning $(v_1, v_4) \in E(G)$ and $(v_2, v_4) \in E(G)$. That is, the following is a subgraph of G

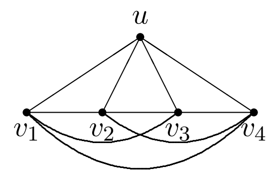

Figure 2

This implies that v_2, v_3 and u are common neighbors to v_1 and v_4, so that $\delta(v_2, v_4) \geqslant 3$, a contradiction to the fact that $\delta(v_i, v_j) \leqslant 2$ for all $v_i \neq v_j$. ■

Lemma 5. *If $G = SR(n, 1 \text{ or } 2, 1)$, and G has a subgraph of this type:*

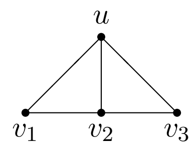

Figure 3

then G must have a subgraph of type $K_1 \vee K_3$.

Proof. Suppose that G has a subgraph of the type given in Figure 3. Since u and v_2 are common neighbors to v_1 and v_3 then $\delta(v_1, v_3) = 2$. It follows that $(v_1, v_3) \in E(G)$. We can regraph S as: $S = K_1 \vee K_3$ where $K_1 = (\{u\}, \emptyset)$ and $K_3 = (\{v_1, v_2, v_3\}, \{(v_1, v_2), (v_1, v_3), (v_2, v_3)\})$, as shown in Figure 4.

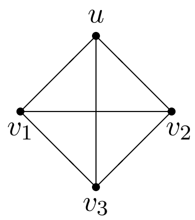

Figure 4

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3. Generalization of the Friendship Theorem

In this section we generalize the friendship theorem to the following theorem:

Theorem 6. *Suppose that for all vertices of $V(G)$ we have $\delta(u,v) = 1$ or 2 if $(u,v) \in E(G)$ and $\delta(u,v) = 1$ if $(u,v) \notin E(G)$. Then $G = K_1 \vee (sK_2 + tK_3)$.*

Proof. Let $u \in V(G)$ such that u has the largest degree among all the vertices of G. Let $N_G(u)$ and $\overline{N}_G(u)$ be the neighbors and non-neighbors of u respectively. Let $N_G(u) = \{v_1, v_2, \ldots, v_k\}$. That is, $\deg(u) = k \geqslant \deg(w)$ for any $w \in V(G)$.

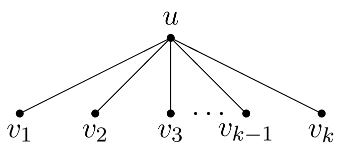

Figure 5

Since G cannot have an induced subgraph of the form given in Lemma 4, then without loss of generality we can rearrange subsets of $\{v_1, v_2, \ldots, v_k\}$ into three induced subgraphs of G:

(I) G has induced subgraphs each isomorphic to the one presented in Figure 6, such that $(v_{i-1}, v_i) \notin E(G)$ and $(v_{i+2}, v_{i+3}) \notin E(G)$.

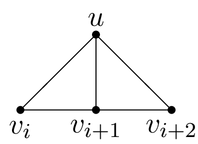

Figure 6

(II) G has induced subgraphs each isomorphic to the one presented in Figure 7, such that $(v_{i-1}, v_i) \notin E(G)$ and $(v_{i+1}, v_{i+2}) \notin E(G)$.

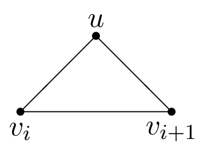

Figure 7

(III) G has induced subgraphs each isomorphic to the one presented in Figure 8, such that $(v_{i-1}, v_i) \notin E(G)$ and $(v_i, v_{i+1}) \notin E(G)$.

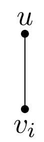

Figure 8

It follows that the set of neighbors of u can be rearranged as follows: where s, t and ℓ are non-negative integers. Using Lemma 5 we can rewrite the graph in Figure 9 as:

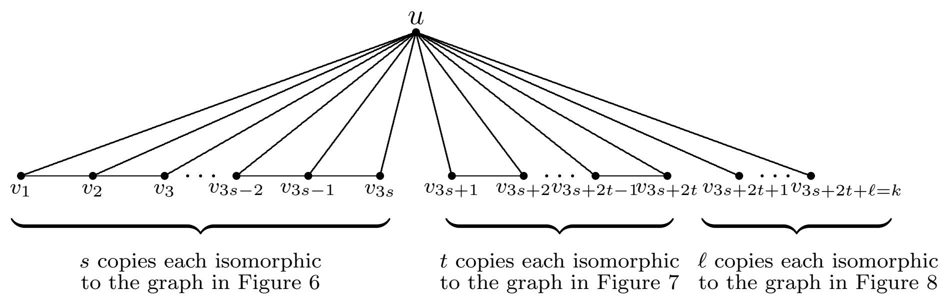

Figure 9

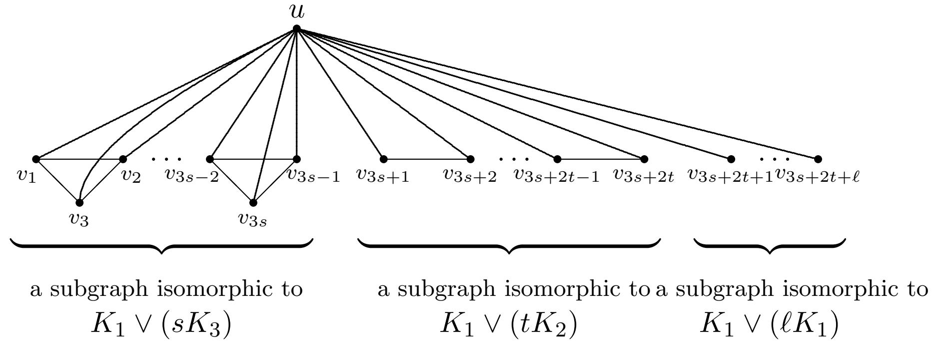

Figure 10

We claim that no vertex of the third type is possible to exist. Suppose otherwise. Then each vertex in the set $\{v_{3s+2t+1}, \ldots, v_k\}$ must have one or two common neighbors with u since $(v_j, u) \in E(G)$ for $j = 3s + 2t + 1, \ldots, k$. That is, there must be a vertex from the neighbors of u which is adjacent to v_j, a contradiction. It follows that we have the following induced subgraph of G: $K_1 \vee (sK_3 + tK_2)$.

If $\deg(u) = n - 1$ then all vertices of G are in $N_G(u)$ and therefore $G = K_1 \vee (sK_3 + tK_2)$.

Suppose otherwise, that $\deg(v) < n - 1$. Then there exists $w \in V(G)$ such that $(u, w) \notin E(G)$. That is, $w \notin \overline{N}_G(u)$ as shown in Figure 11 below.

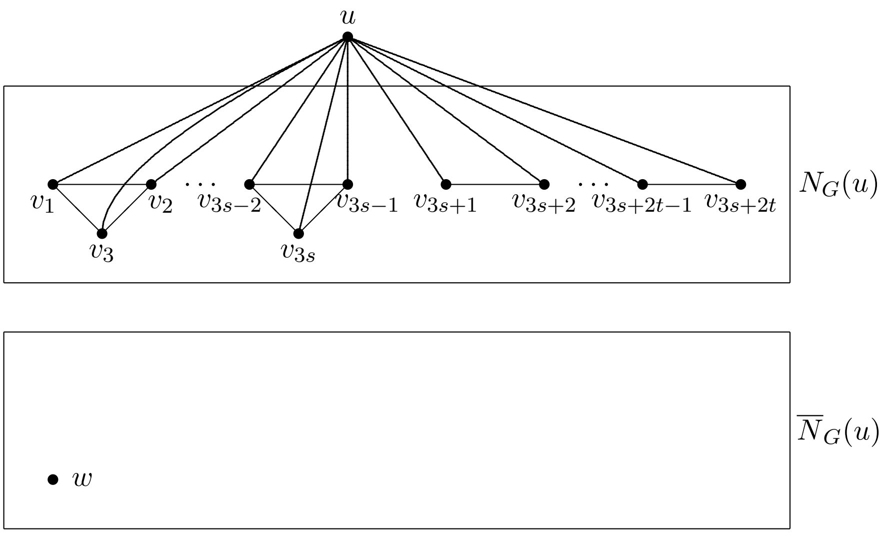

Figure 11

Since $(u, w) \notin E(G)$ then $\delta(u, w) = 1$. It follows that w is adjacent only to one vertex from the set $N_G(u)$. Since we have two types of induced subgraphs connected to u, we divide the discussion into two cases:

Case 1. w is adjacent to a vertex of the induced subgraphs of type 1. Without loss of generality we assume that w is adjacent to v_1. That is $(w, v_1) \in E(G)$. It follows that $(w, v_j) \notin E(G)$ for all $j \geqslant 2$ since, otherwise, w and u would be adjacent to more that one vertex, a contradiction to the fact that $\delta(u, w) = 1$. Since $(v_1, w) \in E(G)$ we have $\delta(v_1, w) = 1$ or 2. That is there exist a vertex $w_1 \in \overline{N}_G(u)$ such that $(v_1, w_1) \in E(G)$ and $(w_1, w) \in E(G)$ since no other vertex from $N_G(u)$ is adjacent to w. Also since w is not adjacent to any v_j for $j \geqslant 2$, then there exist $w_j \in \overline{N}_G(w)$ such that $(v_j, w_j) \in E(G)$ and $(w_j, w) \in E(G)$ as shown in Figure 12 below.

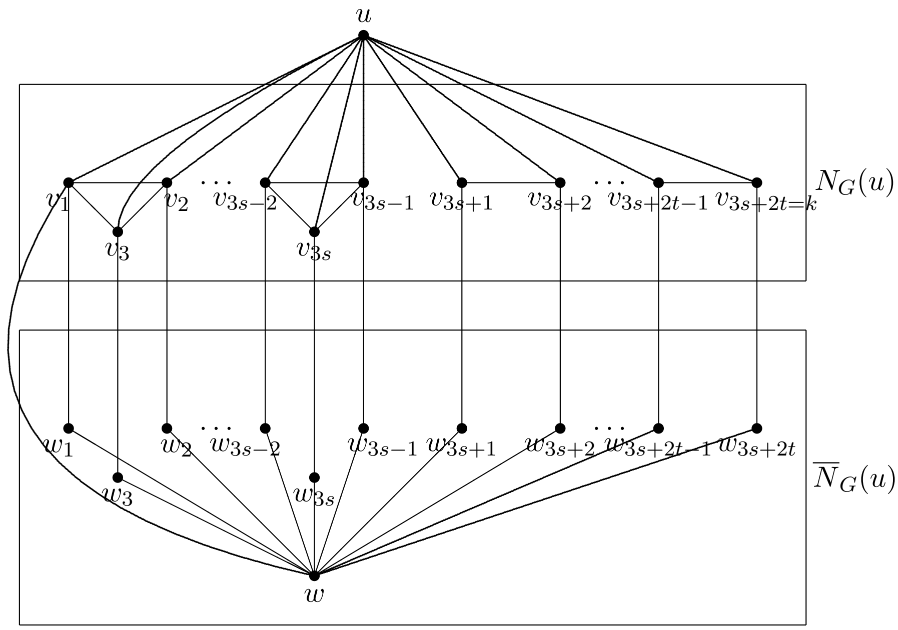

Figure 12

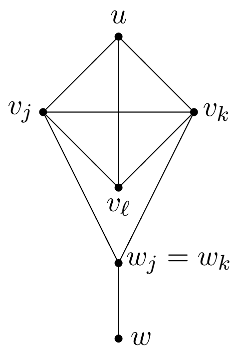

Figure 13 (a)

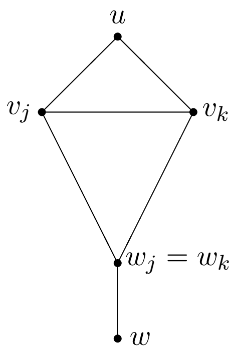

Figure 13 (b)

We claim that if $j \neq k$ then $w_j \neq w_k$. For this suppose otherwise. That is, there exists $j \neq k$ such that $w_j = w_k$. It follows that u and w_j are two common neighbors for both v_j and v_k, so that $\delta(v_j, v_k) = 2$. Therefore $(v_j, v_k) \in E(G)$, and we have only one of the induced subgraphs in Figure 13.

Note that in Figure 13 (a), w_j, v_ℓ and u are common neighbors for v_j and v_k so that $\delta(v_j, v_k) \geq 3$, a contradiction to the fact that $\delta(v_j, v_k) = 1$ or 2. In Figure 13 (b) since v_j and v_k are neighbors for both u and w_j then $\delta(u, w_j) = 2$, so that $(u, w_j) \in E(G)$, a contradiction to the fact that $w_j \in \overline{N}_G(u)$. It follows, from both cases, that $w_j \neq w_k$. We conclude that $deg(w) \geqslant k + 1$, a contradiction to the fact that u is the vertex with highest degree equals to k. Therefore, this case cannot happen.

Case 2. The vertex w is adjacent to a vertex of the subgraph of type 2. Without loss of generality, we assume that w is adjacent to v_k. If we follow the same argument as in Case 1, we conclude that $\deg(w) \geqslant k+1$ a contradiction.

It follows, from both cases, that such w does not exist, so that $V(G) = \{u, v_1, v_2, \ldots, v_k\}$ and, therefore, $G = K_1 \vee (sK_3 + tK_2)$. ■

We can rewrite Theorem 6 as follows:

Theorem 7. *In a group of at least three people, suppose each pair of friends have one or two common friends and each pair of strangers have exactly one common friend. Then there is a person who is a friend to everybody.*

Acknowledgement

I would like to thank the referees for their valuable comments that has resulted in improvements to this paper.

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Received 21 May 2010
Revised 1 April 2011
Accepted 1 April 2011