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THE k-RAINBOW DOMATIC NUMBER OF A GRAPH

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Abstract

For a positive integer k, a *k-rainbow dominating function* of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1,2,\ldots,k\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N(v)} f(u) = \{1,2,\ldots,k\}$ is fulfilled, where $N(v)$ is the neighborhood of v. The 1-rainbow domination is the same as the ordinary domination. A set $\{f_1, f_2, \ldots, f_d\}$ of k-rainbow dominating functions on G with the property that $\sum_{i=1}^{d} |f_i(v)| \leq k$ for each $v \in V(G)$, is called a *k-rainbow dominating family* (of functions) on G. The maximum number of functions in a k-rainbow dominating family on G is the *k-rainbow domatic number* of G, denoted by $d_{rk}(G)$. Note that $d_{r1}(G)$ is the classical domatic number $d(G)$. In this paper we initiate the study of the k-rainbow domatic number in graphs and we present some bounds for $d_{rk}(G)$. Many of the known bounds of $d(G)$ are immediate consequences of our results.

Keywords: k-rainbow dominating function, k-rainbow domination number, k-rainbow domatic number.

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1. Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \bigcup_{v\in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. The complement of a graph G is denoted by $\overline{G}$. We write K_n for the *complete graph* of order n, C_n for a *cycle* of length n and P_n for a path of order n.

A subset S of vertices of G is a *dominating set* if $N[S] = V$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A domatic partition is a partition of V into dominating sets, and the domatic number $d(G)$ is the largest number of sets in a domatic partition. The domatic number was introduced by Cockayne and Hedetniemi [7]. In their paper, they showed that

$$\gamma(G) \cdot d(G) \leq n. \tag{1}$$

For a positive integer k, a *k-rainbow dominating function* (kRDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1, 2, \ldots, k\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u\in N(v)} f(u) = \{1, 2, \ldots, k\}$ is fulfilled. The *weight* of a kRDF f is the value $\omega(f) = \sum_{v\in V} |f(v)|$. The *k-rainbow domination number* of a graph G, denoted by $\gamma_{rk}(G)$, is the minimum weight of a kRDF of G. A $\gamma_{rk}(G)$-*function* is a k-rainbow dominating function of G with weight $\gamma_{rk}(G)$. Note that $\gamma_{r1}(G)$ is the classical domination number $\gamma(G)$. The k-rainbow domination number was introduced by Brešar, Henning, and Rall [2] and has been studied by several authors (see for example [3, 4, 5, 12]). Rainbow domination of a graph G coincides with ordinary domination of the Cartesian product of G with the complete graph, in particular, $\gamma_{rk}(G) = \gamma(G\Box K_k)$ for any graph G [2]. This implies (cf. [4]) that

$$\gamma_{r1}(G) \leq \gamma_{r2}(G) \leq \cdots \leq \gamma_{rk}(G) \leq n \text{ for any graph } G \text{ of order } n. \tag{2}$$

Furthermore, it was proved in [8] that

$$\min\{|V(G)|, \gamma(G) + k - 2\} \leq \gamma_{rk}(G) \leq k\gamma(G) \text{ for any } k \geq 2 \text{ and any graph } G.$$

A set $\{f_1, f_2, \ldots, f_d\}$ of k-rainbow dominating functions of G with the property that $\sum_{i=1}^{d} |f_i(v)| \leq k$ for each $v \in V(G)$, is called a *k-rainbow dominating family* (of functions) on G. The maximum number of functions in a k-rainbow dominating family (kRD family) on G is the *k-rainbow domatic number* of G, denoted by

$d_{rk}(G)$. The k-rainbow domatic number is well-defined and

$$d_{rk}(G) \geq k, \text{ for all graphs } G \tag{3}$$

since the set consisting of the function $f_i : V(G) \to \mathcal{P}(\{1, 2, \ldots, k\})$ defined by $f_i(v) = \{i\}$ for each $v \in V(G)$ and each $i \in \{1, 2, \ldots, k\}$, forms a kRD family on G.

Our purpose in this paper is to initiate the study of the k-rainbow domatic number in graphs. We first study basic properties and bounds for the k-rainbow domatic number of a graph. In addition, we determine the 2-rainbow domatic number of some classes of graphs.

2. Properties of the k-rainbow Domatic Number

In this section we mainly present basic properties of $d_{rk}(G)$ and bounds on the k-rainbow domatic number of a graph. However, we start with a lower and an upper bound on the k-rainbow domination number.

Observation 1. *If G is a graph of order n, then $\gamma_{rk}(G) \leq n - \Delta(G) + k - 1$.*

Proof. Let v be a vertex of maximum degree $\Delta(G)$. Define $f : V(G) \to \mathcal{P}(\{1, 2, \ldots, k\})$ by $f(v) = \{1, 2, \ldots, k\}$ and

$$f(x) = \begin{cases} \emptyset & \text{if } x \in N(v), \\ \{1\} & \text{if } x \in V(G) - N[v]. \end{cases}$$

It is easy to see that f is a k-rainbow dominating function on G and so $\gamma_{rk}(G) \leq n - \Delta(G) + k - 1$. ■

Let $k \geq 1$ be an integer, and let G be a graph of order $n \geq k$ and maximum degree $\Delta(G) = n - 1$. Since $n \geq k$, we observe that $\gamma_{rk}(G) \geq k$. If v is a vertex of maximum degree $\Delta(G)$, then define $f : V(G) \to \mathcal{P}(\{1, 2, \ldots, k\})$ by $f(v) = \{1, 2, \ldots, k\}$, $f(x) = \emptyset$ if $x \in V(G) \setminus \{v\}$. Because of $d(v) = \Delta(G) = n - 1$, f is a k-rainbow dominating function on G and thus $\gamma_{rk}(G) \leq k$. It follows that $\gamma_{rk}(G) = k = n - \Delta(G) + k - 1$. This example shows that Observation 1 is sharp. The case $k = 1$ in Observation 1 is attributed to Berge [1]. In 1979, Walikar, Acharya and Sampathkumar [10] proved $\gamma(G) \geq \lceil n/(\Delta(G) + 1) \rceil$ for each graph of order n. Next we will give an analogues lower bound for $\gamma_{rk}(G)$ when $k \geq 2$.

Theorem 2. *If G is a graph of order n and maximum degree Δ, then*

$$\gamma_{r2}(G) \geq \left\lceil \frac{2n}{\Delta + 2} \right\rceil.$$

Proof. Let f be a $\gamma_{r2}(G)$-function and let $V_i = \{v \mid |f(v)| = i\}$ for $i = 0, 1, 2$. Then $\gamma_{r2}(G) = |V_1| + 2|V_2|$ and $n = |V_0| + |V_1| + |V_2|$. Since each vertex of V_0 is adjacent to at least one vertex of V_2 or at least two vertices of V_1, we deduce that $|V_0| \le \Delta|V_2| + \frac{1}{2}\Delta|V_1|$.
This implies that

$$\begin{aligned}(\Delta + 2)\gamma_{r2}(G) &= 2\gamma_{r2}(G) + \Delta(|V_1| + 2|V_2|) \ge 2\gamma_{r2}(G) + 2|V_0| \\ &= 2|V_1| + 4|V_2| + 2|V_0| = 2n + 2|V_2| \ge 2n,\end{aligned}$$

and this leads to the desired bound. ■

Using inequality (2) and Theorem 2, we obtain the next result immediately.

Theorem 3. *If $k \ge 2$ is an integer, and G is a graph of order n and maximum degree Δ, then*

$$\gamma_{rk}(G) \ge \left\lceil \frac{2n}{\Delta + 2} \right\rceil.$$

Theorem 4. *If G is a graph of order n, then $\gamma_{rk}(G) \cdot d_{rk}(G) \le kn$.*
Moreover, if $\gamma_{rk}(G) \cdot d_{rk}(G) = kn$, then for each kRD *family $\{f_1, f_2, \ldots, f_d\}$ on G with $d = d_{rk}(G)$, each function f_i is a $\gamma_{rk}(G)$-function and $\sum_{i=1}^{d} |f_i(v)| = k$ for all $v \in V$.*

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a kRD family on G such that $d = d_{rk}(G)$. Then

$$\begin{aligned}d \cdot \gamma_{rk}(G) &= \textstyle\sum_{i=1}^{d} \gamma_{rk}(G) \le \sum_{i=1}^{d} \sum_{v \in V} |f_i(v)| \\ &= \textstyle\sum_{v \in V} \sum_{i=1}^{d} |f_i(v)| \le \sum_{v \in V} k = kn.\end{aligned}$$

If $\gamma_{rk}(G) \cdot d_{rk}(G) = kn$, then the two inequalities occurring in the proof become equalities. Hence for the kRD family $\{f_1, f_2, \ldots, f_d\}$ on G and for each i, $\sum_{v \in V} |f_i(v)| = \gamma_{rk}(G)$. Thus each function f_i is a $\gamma_{rk}(G)$-function, and $\sum_{i=1}^{d} |f_i(v)| = k$ for all $v \in V$. ■

The case $k = 1$ in Theorem 4 leads to the well-known inequality $\gamma(G) \cdot d(G) \le n$, given by Cockayne and Hedetniemi [7] in 1977.

Corollary 5. *If k is a positive integer, and G is a graph of order $n \ge k$, then*

$$d_{rk}(G) \le n.$$

Proof. The hypothesis $n \ge k$ leads to $\gamma_{rk}(G) \ge k$. Therefore it follows from Theorem 4 that

$$d_{rk}(G) \le \frac{kn}{\gamma_{rk}(G)} \le \frac{kn}{k} = n,$$

and this is the desired inequality. ■

Corollary 6. *If k is a positive integer, and G is isomorphic to the complete graph K_n of order $n \geq k$, then $d_{rk}(G) = n$.*

Proof. In view of Corollary 5, we have $d_{rk}(G) \leq n$. If $\{v_1, v_2, \ldots, v_n\}$ is the vertex set of G, then we define the function $f_i : V(G) \to \mathcal{P}(\{1, 2, \ldots, k\})$ by $f_i(v_j) = \{1, 2, \ldots, k\}$ for $i = j$ and $f_i(v_j) = \emptyset$ for $i \neq j$, where $i, j \in \{1, 2, \ldots, n\}$. Then $\{f_1, f_2, \ldots, f_n\}$ is a kRD family on G and thus $d_{rk}(G) = n$. ■

Theorem 7. *If G is a graph of order $n \geq k$, then*

$$\gamma_{rk}(G) + d_{rk}(G) \leq n + k.$$

Proof. Applying Theorem 4, we obtain

$$\gamma_{rk}(G) + d_{rk}(G) \leq \frac{kn}{d_{rk}(G)} + d_{rk}(G).$$

Note that $d_{rk}(G) \geq k$, by inequality (3), and that Corollary 5 implies that $d_{rk}(G) \leq n$. Using these inequalities, and the fact that the function $g(x) = x + (kn)/x$ is decreasing for $k \leq x \leq \sqrt{kn}$ and increasing for $\sqrt{kn} \leq x \leq n$, we obtain

$$\gamma_{rk}(G) + d_{rk}(G) \leq \max\left\{\frac{kn}{k} + k, \frac{kn}{n} + n\right\} = n + k,$$

and this is the desired bound. ■

If G is isomorphic to the complete graph of order $n \geq k$, then $\gamma_{rk}(G) = k$ and $d_{rk}(G) = n$ by Corollary 6. Thus $\gamma_{rk}(K_n) \cdot d_{rk}(K_n) = nk$ and $\gamma_{rk}(K_n) + d_{rk}(K_n) = n + k$ when $n \geq k$. This example shows that Theorems 4 and 7 are sharp.

Corollary 8 (Cockayne and Hedetniemi, [7], 1977)**.** *If G is a graph of order $n \geq 1$, then $\gamma(G) + d(G) \leq n + 1$*

Theorem 9. *For every graph G,*

$$d_{rk}(G) \leq \delta(G) + k.$$

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a kRD family on G such that $d = d_{rk}(G)$, and let v be a vertex of minimum degree $\delta(G)$. Since $\sum_{u \in N[v]} |f_i(u)| \geq 1$ for all $i \in \{1, 2, \ldots, d\}$ and $\sum_{u \in N[v]} |f_i(u)| < k$ for at most k indices $i \in \{1, 2, \ldots, d\}$, we obtain

$$\begin{aligned} kd - k(k-1) &\leq \textstyle\sum_{i=1}^{d} \sum_{u \in N[v]} |f_i(u)| = \sum_{u \in N[v]} \sum_{i=1}^{d} |f_i(u)| \\ &\leq \textstyle\sum_{u \in N[v]} k = k(\delta(G) + 1), \end{aligned}$$

and this leads to the desired bound. ■

To prove sharpness of Theorem 9, let $p \geq 2$ be an integer, and let G_i be a copy of K_{p+k+1} with vertex set $V(G_i) = \{v_1^i, v_2^i, \ldots, v_{p+k+1}^i\}$ for $1 \leq i \leq p$. Now let G be the graph obtained from $\bigcup_{i=1}^p G_i$ by adding a new vertex v and joining v to each v_1^i. Define the k-rainbow dominating functions $f_1, f_2, \ldots, f_{p+k}$ as follows: for $1 \leq i \leq p$ and $1 \leq s \leq k$
$f_i(v_1^i) = \{1, 2, \ldots, k\}$, $f_i(v_{i+1}^j) = \{1, 2, \ldots, k\}$ if $j \in \{1, 2, \ldots, p\} - \{i\}$ and $f(x) = \emptyset$ otherwise,
$f_{p+s}(v) = \{1\}$, $f_{p+s}(v_{p+s+1}^j) = \{1, 2, \ldots, k\}$ if $j \in \{1, 2, \ldots, p\}$ and $f(x) = \emptyset$ otherwise.
It is straightforward to verify that f_i is a k-rainbow dominating function on G for each i and $\{f_1, f_2, \ldots, f_{p+k}\}$ is a k-rainbow dominating family on G. Since $\delta(G) = p$, we have $d_{rk}(G) = \delta(G) + k$.

The special case $k = 1$ in Theorem 9 was done by Cockayme and Hedetniemi [7]. As an application of Theorem 9, we will prove the following Nordhaus-Gaddum type result.

Theorem 10. *For every graph G of order n,*

$$d_{rk}(G) + d_{rk}(\overline{G}) \leq n + 2k - 1.$$

If $d_{rk}(G) + d_{rk}(\overline{G}) = n + 2k - 1$, then G is regular.

Proof. It follows from Theorem 9 that

$$\begin{aligned} d_{rk}(G) + d_{rk}(\overline{G}) &\leq (\delta(G) + k) + (\delta(\overline{G}) + k) \\ &= (\delta(G) + k) + (n - \Delta(G) - 1 + k) \leq n + 2k - 1. \end{aligned}$$

If G is not regular, then $\Delta(G) - \delta(G) \geq 1$, and this inequality chain leads to the better bound $d_{rk}(G) + d_{rk}(\overline{G}) \leq n + 2k - 2$, and the proof is complete. ■

Corollary 11 (Cockayne and Hedetniemi [7] 1977)**.** *If G is a graph of order $n \geq 1$, then $d(G) + d(\overline{G}) \leq n + 1$.*

3. Properties of the 2-rainbow Domatic Number

Let $A_1 \cup A_2 \cup \cdots \cup A_d$ be a domatic partition of $V(G)$ into dominating sets such that $d = d(G)$. Then the set of functions $\{f_1, f_2, \ldots, f_d\}$ with $f_i(v) = \{1, 2\}$ if $v \in A_i$ and $f_i(v) = \emptyset$, otherwise for $1 \leq i \leq d$ is a 2RD family on G. This shows that $d(G) \leq d_{r2}(G)$ for every graph G.

Observation 12. *Let G be a graph of order $n \geq 2$. Then $\gamma_{r2}(G) = n$ and $d_{r2}(G) = 2$ if and only if $\Delta(G) \leq 1$.*

Proof. If $\gamma_{r2}(G) = n$, then, by Theorem 1, $\Delta(G) \leq 1$.

Conversely, let $\Delta(G) \leq 1$. If $\Delta(G) = 0$, then obviously $\gamma_{r2}(G) = n$ and $d_{r2}(G) = 2$. Let $\Delta(G) = 1$. Then $G = rK_1 \cup \frac{n-r}{2} K_2$ with $n - r \geq 2$ even, and we have

$$\gamma_{r2}(G) = r\gamma_{r2}(K_1) + \frac{n-r}{2}\gamma_{r2}(K_2) = r + (n-r) = n.$$

By (3) and Theorem 4, we obtain $d_{r2}(G) = 2$. This completes the proof. ∎

Using Theorem 9 and the following proposition, we determine the 2-rainbow domatic number of paths.

Proposition A [3]. *For $n \geq 2$,*

$$\gamma_{r2}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

Proposition 13. *For $n \geq 3$,*

$$d_{r2}(P_n) = \begin{cases} 2 & \text{if } n = 4, \\ 3 & \text{otherwise.} \end{cases}$$

Proof. Let $G = P_n$. If $n = 4$, then Proposition 3 implies $\gamma_{r2}(G) = 3$, and the result follows from Theorem 4 and (3). Assume now that $n \neq 4$. By Theorem 4 and Proposition 3, we have $d_{r2}(G) \leq 3$. Consider four cases.

Case 1. $n \equiv 3 \pmod 4$. Define the 2-rainbow dominating functions f_1, f_2, f_3 as follows:

$f_1(v_{4i+1}) = \{1\}, f_1(v_{4i+3}) = \{2\}$ for $0 \leq i \leq (n-3)/4$, and
$f_1(x) = \emptyset$ otherwise,

$f_2(v_{4i+1}) = \{2\}, f_2(v_{4i+3}) = \{1\}$ for $0 \leq i \leq (n-3)/4$, and
$f_2(x) = \emptyset$ otherwise,
$f_3(v_{2i+2}) = \{1, 2\}$ for $0 \leq i \leq (n-3)/2$, and $f_3(x) = \emptyset$ otherwise.

It is easy to see that f_i is a 2-rainbow dominating function on G for each i and $\{f_1, f_2, f_3\}$ is a 2-rainbow dominating family on G.

Case 2. $n \equiv 1 \pmod 4$. Define the 2-rainbow dominating functions f_1, f_2, f_3 as follows:

$f_1(v_n) = \{1\}, f_1(v_{4i+1}) = \{1\}, f_1(v_{4i+3}) = \{2\}$ for $0 \leq i \leq (n-1)/4 - 1$ and
$f_1(x) = \emptyset$ otherwise,

$f_2(v_n) = \{2\}, f_2(v_{4i+1}) = \{2\}, f_2(v_{4i+3}) = \{1\}$ for $0 \leq i \leq (n-1)/4 - 1$ and
$f_2(x) = \emptyset$ otherwise,

$f_3(v_{2i}) = \{1, 2\}$ for $1 \leq i \leq (n-1)/2$, and $f_3(x) = \emptyset$ otherwise.

Clearly, f_i is a 2-rainbow dominating function on G for each i and $\{f_1, f_2, f_3\}$ is a 2-rainbow dominating family on G.

Case 3. $n \equiv 0 \pmod 4$. Define the 2-rainbow dominating functions f_1, f_2, f_3 as follows:

$f_1(v_1) = f_1(v_{4i+6}) = \{1\}, f_1(v_3) = f_1(v_4) = f_1(v_{4i+8}) = \{2\}$ for $0 \leq i \leq n/4 - 2$, and $f_1(x) = \emptyset$ otherwise,

$f_2(v_1) = f_2(v_{4i+6}) = \{2\}, f_2(v_3) = f_2(v_4) = f_2(v_{4i+8}) = \{1\}$ for $0 \leq i \leq n/4 - 2$, and $f_2(x) = \emptyset$ otherwise,

$f_3(v_2) = f_3(v_{2i+1}) = \{1, 2\}$ for $2 \leq i \leq n/2 - 1$, and $f_3(x) = \emptyset$ otherwise.

It is easy to see that f_i is a 2-rainbow dominating function on G for each i and $\{f_1, f_2, f_3\}$ is a 2-rainbow dominating family on G.

Case 4. $n \equiv 2 \pmod 4$. Define the 2-rainbow dominating functions f_1, f_2, f_3 as follows:

$f_1(v_1) = f_1(v_n) = f_1(v_{4i+6}) = \{1\}, f_1(v_3) = f_1(v_4) = f_1(v_{4i+8}) = \{2\}$ for $0 \leq i \leq (n-2)/4 - 2$, and $f_1(x) = \emptyset$ otherwise,

$f_2(v_1) = f_2(v_n) = f_2(v_{4i+6}) = \{2\}, f_2(v_3) = f_2(v_4) = f_2(v_{4i+8}) = \{1\}$ for $0 \leq i \leq (n-2)/4 - 2$, and $f_2(x) = \emptyset$ otherwise,

$f_3(v_2) = f_3(v_{2i+1}) = \{1, 2\}$ for $2 \leq i \leq n/2 - 1$, and $f_3(x) = \emptyset$ otherwise.

Clearly f_i is a 2-rainbow dominating function on G for each i and $\{f_1, f_2, f_3\}$ is a 2-rainbow dominating family on G. This completes the proof. ■

Using Theorem 4 and the following proposition, we determine the 2-rainbow domatic number of cycles.

Proposition B [3]**.** *For $n \geq 3$,*

$$\gamma_{r2}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor.$$

Proposition 14. *If C_n is the cycle on $n \geq 4$ vertices, then*

$$d_{r2}(C_n) = \begin{cases} 4 & n \equiv 0 \pmod 4, \\ 3 & \text{otherwise.} \end{cases}$$

Proof. Let $C_n = (v_1, v_2, \ldots, v_n)$. Consider four cases.

Case 1. $n \equiv 0 \pmod 4$. Define the 2-rainbow dominating functions f_1, f_2, f_3, f_4 as follows:

$f_1(v_{4(i-1)+1}) = \{1\}, f_1(v_{4(i-1)+3}) = \{2\}$ for $0 \leq i \leq n/4 - 1$, and $f_1(x) = \emptyset$ otherwise,

$f_2(v_{4(i-1)+1}) = \{2\}, f_2(v_{4(i-1)+3}) = \{1\}$ for $0 \leq i \leq n/4 - 1$, and $f_2(x) = \emptyset$ otherwise,

$f_3(v_{4(i-1)+2}) = \{1\}, f_3(v_{4(i-1)+4}) = \{2\}$ for $0 \leq i \leq n/4 - 1$, and $f_3(x) = \emptyset$ otherwise,

$f_4(v_{4(i-1)+2}) = \{2\}, f_4(v_{4(i-1)+4}) = \{1\}$ for $0 \leq i \leq n/4 - 1$, and
$f_4(x) = \emptyset$ otherwise.

It is easy to see that f_i is a 2-rainbow dominating function on G for each i and $\{f_1, f_2, f_3, f_4\}$ is a 2-rainbow dominating family on G. Thus $d_{r2}(C_n) = 4$.

Case 2. $n \equiv 1 \pmod 4$. Then by Theorem 4 and Proposition 3, $d_{r2}(C_n) \leq 3$. Define the 2-rainbow dominating functions f_1, f_2, f_3 as follows:

$f_1(v_{4(i-1)+1}) = \{1\}, f_1(v_{4(i-1)+3}) = \{2\}$, for $0 \leq i \leq (n-1)/4 - 1$,
$f_1(v_n) = \{1\}$ and $f_1(x) = \emptyset$ otherwise,

$f_2(v_{4(i-1)+1}) = \{2\}, f_2(v_{4(i-1)+3}) = \{1\}$, for $0 \leq i \leq (n-1)/4 - 1$,
$f_2(v_n) = \{2\}$ and$f_2(x) = \emptyset$ otherwise,

$f_3(v_{4(i-1)+2}) = f_3(v_{4(i-1)+4}) = \{1, 2\}$ for $0 \leq i \leq (n-1)/4 - 1$, and
$f_3(x) = 0$ otherwise.

Clearly, f_i is a 2-rainbow dominating function on G for each i and $\{f_1, f_2, f_3\}$ is a 2-rainbow dominating family on G. Thus $d_{r2}(C_n) = 3$.

Case 3. $n \equiv 3 \pmod 4$. Then by Theorem 4 and Proposition 3, $d_{r2}(C_n) \leq 3$. Define the 2-rainbow dominating functions f_1, f_2, f_3 as follows:

$f_1(v_{4(i-1)+1}) = \{1\}, f_1(v_{4(i-1)+3}) = \{2\}$, for $0 \leq i \leq (n+1)/4 - 1$, and
$f_1(x) = \emptyset$ otherwise,

$f_2(v_{4(i-1)+1}) = \{2\}, f_2(v_{4(i-1)+3}) = \{1\}$, for $0 \leq i \leq (n+1)/4 - 1$, and
$f_2(x) = \emptyset$ otherwise,

$f_3(v_{4(i-1)+2}) = f_3(v_{4(i-1)+4}) = \{1, 2\}$ for $0 \leq i \leq (n-3)/4 - 1$,
$f_3(v_{n-1}) = 1$ and $f_3(x) = 0$ otherwise.

Clearly, f_i is a 2-rainbow dominating function on G for each i and $\{f_1, f_2, f_3\}$ is a 2-rainbow dominating family on G. Thus $d_{r2}(C_n) = 3$.

Case 4. $n \equiv 2 \pmod 4$. Then by Theorem 4 and Proposition 3, $d_{r2}(C_n) \leq 3$. Define the 2-rainbow dominating functions f_1, f_2, f_3 as follows:

$f_1(v_1) = f_1(v_2) = f_1(v_{4i+3}) = \{1\}, f_1(v_4) = f_1(v_5) = f_1(v_{4i+5}) = \{2\}$ for
$1 \leq i \leq \frac{n-6}{4}$ and $f_1(x) = \emptyset$ otherwise,

$f_2(v_1) = f_2(v_2) = f_2(v_{4i+3}) = \{2\}$, $f_2(v_4) = f_2(v_5) = f_2(v_{4i+5}) = \{1\}$ for
$1 \leq i \leq \frac{n-6}{4}$ and $f_2(x) = \emptyset$ otherwise,

$f_3(v_3) = f_3(v_{4i+2}) = \{1, 2\}$ for $1 \leq i \leq \frac{n-2}{4}$ and $f_3(x) = \emptyset$ otherwise.

Clearly, f_i is a 2-rainbow dominating function on G for each i and $\{f_1, f_2, f_3\}$ is a 2-rainbow dominating family on G. Thus $d_{r2}(C_n) = 3$. ■

Theorem 2 and its proof lead immediately to the next result.

Corollary 15. *Let G be a graph of order n and maximum degree Δ. Then*

$$\gamma_{r2}(G) \geq \begin{cases} \lceil \frac{2n+2}{\Delta+2} \rceil & \text{if there is a } \gamma_{r2}(G)\text{-function } f \text{ with } V_2 \neq \emptyset, \\ \lceil \frac{2n}{\Delta+2} \rceil & \text{otherwise.} \end{cases}$$

Using Corollary 15, we will improve the upper bound on $d_{r2}(G)$ given in Theorem 9 for some regular graphs.

Theorem 16. *If G is a δ-regular graph of order n with $\delta \geq 1$ and a $\gamma_{r2}(G)$-function f such that $V_2 \neq \emptyset$ or $2n \not\equiv 0 \pmod{(\delta+2)}$, then*

$$d_{r2}(G) \leq \delta + 1.$$

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a 2RD family on G such that $d = d_{r2}(G)$. It follows that

$$\sum_{i=1}^{d} \omega(f_i) = \sum_{i=1}^{d} \sum_{v \in V} |f_i(v)| = \sum_{v \in V} \sum_{i=1}^{d} |f_i(v)| \leq \sum_{v \in V} 2 = 2n. \tag{4}$$

Suppose to the contrary that $d \geq \delta + 2$. If $V_2 \neq \emptyset$, then Corollary 15 leads to

$$\sum_{i=1}^{d} \omega(f_i) \geq \sum_{i=1}^{d} \gamma_{r2}(G) \geq d \left\lceil \frac{2n+2}{\delta+2} \right\rceil \geq (\delta+2) \left(\frac{2n+2}{\delta+2} \right) > 2n,$$

a contradiction to the inequality (4). If $2n \not\equiv 0 \pmod{(\delta+2)}$, then it follows from Corollary 15 that

$$\sum_{i=1}^{d} \omega(f_i) \geq \sum_{i=1}^{d} \gamma_{r2}(G) \geq d \left\lceil \frac{2n}{\delta+2} \right\rceil > (\delta+2) \left(\frac{2n}{\delta+2} \right) = 2n,$$

a contradiction to (4) again. Therefore $d \leq \delta + 1$ and the proof is complete. ■

By Theorem 14, $d_{r2}(C_4) = 4$ and therefore $d_{r2}(C_4) = \delta(C_4) + 2$. This 2-regular graph demonstrates that the bound in Theorem 16 is not valid in general in the case that $2n \equiv 0 \pmod{(\delta+2)}$.

Using Theorems 9, 10 and 16, we will improve the upper bound given in Theorem 10 in the case that $k = 2$.

Theorem 17. *If G is a graph of order n, then*

$$d_{r2}(G) + d_{r2}(\overline{G}) \leq n + 2.$$

Proof. If G is not regular, then Theorem 10 implies the desired result. Now let G be δ-regular.

Assume that G has a $\gamma_{r2}(G)$-function f such that $V_2 \neq \emptyset$ or $V_2 = \emptyset$ and $2|V_0| < \delta|V_1|$. Then we deduce from Theorem 16 that $d_{r2}(G) \leq \delta + 1$. Using Theorem 9, we obtain the desired result as follows

$$\begin{aligned} d_{r2}(G) + d_{r2}(\overline{G}) &\leq (\delta(G) + 1) + (\delta(\overline{G}) + 2) \\ &= (\delta(G) + 1) + (n - \delta(G) - 1 + 2) = n + 2. \end{aligned}$$

It remains the case that G has a $\gamma_{r2}(G)$-function f such $V_2 = \emptyset$ and $2|V_0| = \delta|V_1|$. Note that $n = |V_0| + |V_1|$ and $|V_1| \geq 2$. Since $\delta(G) + \delta(\overline{G}) = n - 1$, it follows that $\delta(G) \geq (n-1)/2$ or $\delta(\overline{G}) \geq (n-1)/2$. We assume, without loss of generality, that $\delta(G) \geq (n-1)/2$.

If $|V_1| \geq 4$, then $2|V_0| = \delta|V_1| \geq 4\delta$ and thus $|V_0| \geq 2\delta$. This leads to the contradiction

$n = |V_0| + |V_1| \geq 2\delta + 4 \geq n - 1 + 4 = n + 3.$

In the case $|V_1| = 3$, we define $V_1' = \{v \mid f(v) = \{1\}\}$ and $V_1'' = \{v \mid f(v) = \{2\}\}$. We assume, without loss of generality, that $|V_1'| = 1 < 2 = |V_1''|$. Since each vertex of V_0 is adjacent to at least one vertex of V_1', we deduce that $|V_0| \leq \delta < 2\delta$. This implies that

$2|V_0| = |V_0| + |V_0| < \delta + 2\delta = \delta|V_1'| + \delta|V_1''| = \delta|V_1|,$

a contradiction to the assumption $2|V_0| = \delta|V_1|$.

If $|V_1| = 2$, then $|V_0| = \delta$ and so $n = \delta + 2$. Hence $\delta(\overline{G}) = n - \delta - 1 = 1$ and so $d_{r2}(\overline{G}) = 2$. Now Theorem 9 implies that

$d_{r2}(G) + d_{r2}(\overline{G}) \leq (\delta(G) + 2) + 2 = n + 2,$

the desired bound. Since we have discussed all possible cases, the proof is complete. ∎

If G is isomorphic to the complete graph K_n with $n \geq 2$, then Corollarry 6 implies $d_{r2}(G) = n$. Since $d_{r2}(\overline{G}) = 2$, we obtain $d_{r2}(G) + d_{r2}(\overline{G}) = n + 2$. This example demonstrates that Theorem 17 is sharp.

We conclude this paper with a conjecture.

Conjecture 18. *For every integer $k \geq 2$ and every graph G of order n,*

$$d_{rk}(G) + d_{rk}(\overline{G}) \leq n + 2k - 2.$$

Note that Theorem 17 shows that this conjecture is valid for $k = 2$. In addition, the complete graph K_n demonstrates that Conjecture 1 does not hold for $k = 1$.

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