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DOUBLE GEODETIC NUMBER OF A GRAPH

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Abstract

For a connected graph G of order n, a set S of vertices is called a double geodetic set of G if for each pair of vertices x, y in G there exist vertices $u, v \in S$ such that $x, y \in I[u, v]$. The double geodetic number $dg(G)$ is the minimum cardinality of a double geodetic set. Any double godetic of cardinality $dg(G)$ is called dg-set of G. The double geodetic numbers of certain standard graphs are obtained. It is shown that for positive integers r, d such that $r < d \leq 2r$ and $3 \leq a \leq b$ there exists a connected graph G with $rad\ G = r$, $diam\ G = d, g(G) = a$ and $dg(G) = b$. Also, it is proved that for integers $n, d \geq 2$ and l such that $3 \leq k \leq l \leq n$ and $n-d-l+1 \geq 0$, there exists a graph G of order n diameter $d, g(G) = k$ and $dg(G) = l$.

Keywords: geodetic number, weak-extreme vertex, double geodetic set, double geodetic number.

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1. Introduction

By a *graph* $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m respectively.

For basic graph theoretic terminology we refer to [4]. For vertices x and y in a connected graph G, the *distance* $d(x, y)$ is the length of a shortest x-y path in G. It is known that the distance is a metric on the vertex set of G. An $x - y$ path of length $d(x, y)$ is called an $x - y$ *geodesic*. A vertex v is said to lie on an $x - y$ geodesic P if v is a vertex of P including the vertices x and y. For any vertex u of G, the *eccentricity* of u is $e(u) = \max\{d(u, v) : v \in V\}$. A vertex v is an *eccentric vertex of* u if $e(u) = d(u, v)$. The radius *rad* G and diameter *diam* G are defined by *rad* $G = \min\{e(v) : v \in V\}$ and *diam* $G = \max\{e(v) : v \in V\}$ respectively. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v is an *extreme vertex* of G if the subgraph induced by its neighbors is complete.

The *closed interval* $I[x, y]$ consists of all vertices lying on some $x - y$ geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x, y]$. A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a *g-set* of G. The geodetic number of a graph was introduced in [1, 5] and further studied in [2, 3, 6]. It was shown in [5] that determining the geodetic number of a graph is an NP-hard problem. Let 2^V denote the set of all subsets of V. The mapping $I : V \times V \to 2^V$ defined by $I[u, v] = \{w \in V : \text{w lies on a } u - v \text{ geodesic in } G\}$ is the *interval function* of G. One of the basic properties of I is that $u,\ v \in I[u, v]$ for any pair $u,\ v \in V$. Hence the interval function captures every pair of vertices and so the problem of double geodetic sets is trivially well-defined while it is clear that this fails in many graphs already for triplets (for example, complete graphs). This motivated us to introduce and study double geodetic sets. The following theorems will be used in the sequel.

Theorem 1.1 [3]**.** *Every geodetic set of a graph G contains its extreme vertices. In particular, if the set of extreme vertices S of G is a geodetic set of G, then S is the unique minimum geodetic set of G.*

Theorem 1.2 [3]**.** *Let G be a connected graph with a cut vertex v. Then every geodetic set of G contains at least one vertex from each component of $G - v$.*

2. Double Geodetic Number of a Graph

Definition. Let G be a connected graph with at least two vertices. A set S of vertices of G is called a *double geodetic set* of G if for each pair of vertices x, y in G there exist vertices u, v in S such that $x, y \in I[u, v]$. The *double geodetic number* $dg(G)$ of G is the minimum cardinality of a double geodetic set. Any double geodetic set of cardinality $dg(G)$ is called *dg-set* of G.

Example 2.1. For the graph G in Figure 2.1, it is clear that no 2-element or no 3-element subset of G is a double geodetic set of G. $S = \{u_1, u_3, u_4, u_5\}$ is a double geodetic set,then it follows that $dg(G) = 4$.

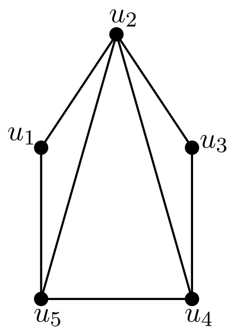

Figure 2.1

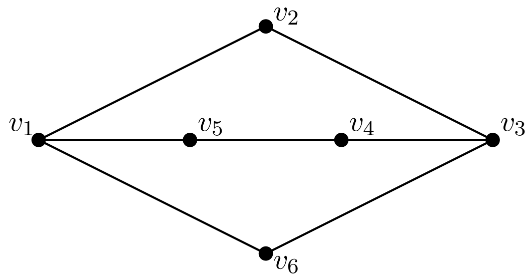

Figure 2.2

Remark 2.2. For the graph G in Figure 2.1 $S = \{u_1, u_3, u_5\}$ is a g-set of G and so $g(G) = 3$. Thus, the double geodetic number and geodetic number of a graph can be different.

Theorem 2.3. *For any graph G of order $n, 2 \leq g(G) \leq dg(G) \leq n$.*

Proof. A geodetic set needs at least two vertices and therefore $g(G) \geq 2$. It is clear that every double geodetic set is also a geodetic set and so $g(G) \leq dg(G)$, since the set of all vertices of G is a double geodetic set of G, $dg(G) \leq n$. ∎

Remark 2.4. The bounds in Theorem 2.3 are sharp. For the complete graph K_n $(n \geq 2)$, we have $dg(K_n) = n$. The set of the two end vertices of a nontrivial path P_n on n vertices is its unique double geodetic set so that $dg(P_n) = 2$. Thus the complete graph K_n has the largest possible double geodetic number n and that the nontrivial paths have the smallest double geodetic number.

Theorem 2.5. *Each extreme vertex of a connected graph G belongs to every double geodetic set of G. In particular, if the set of all end vertices of G is a double geodetic set, then it is the unique dg-set of G.*

Proof. Since every double geodetic set is a geodetic set, the result follows from Theorem 1.1. ∎

Corollary 2.6. *For a graph G of order n with k extreme vertices,* $\max\{2, k\} \leq dg(G) \leq n$.

Proof. This follows from Theorems 2.3 and 2.5. ∎

Theorem 2.7. *Let G be a connected graph with a cut vertex v. Then each double geodetic set of G contains at least one vertex from each component of $G - v$.*

Proof. This follows from Theorem 1.2 and the fact that every double geodetic set is a geodetic set. ∎

Theorem 2.8. *No cut-vertex of a connected graph G belongs to any dg-set of G.*

Proof. Let S be any dg-set of G. Suppose that S contains a cut vertex z of G. Let $G_1, G_2, \ldots, G_r$ $(r \geq 2)$ be the components of $G - z$. Let $S_1 = S - \{z\}$. We claim that S_1 is a double geodetic set of G. Let $x, y \in V(G)$. Since S is a double geodetic set, there exist $u, v \in S$ such that $x, y \in I[u, v]$. If $z \notin \{u, v\}$ then $u, v \in S_1$ and so S_1 is a double geodetic set of G, which is contradiction to the minimality of S. Now, assume that $z \in \{u, v\}$ say $z = u$. Assume without loss of generality that v belongs to S_1. By Theorem 2.7, we can choose a vertex w in G_k $(k \neq 1)$ such that $w \in S$. Now, since z is a cut vertex of G, it follows that $I[z, v] \subseteq I[w, v]$. Hence $x, y \in I[w, v]$ with $w, v \in S_1$. Thus S_1 is a double geodetic set of G which is contradiction to the minimality of S. Thus no cut vertex belongs to any dg-set of G. ∎

Corollary 2.9. *For any tree T, the double geodetic number $dg(T)$ equals the number of end vertices in T. In fact, the set of all end vertices of T is the unique dg-set of T.*

Proof. This follows from Theorems 2.5 and 2.8. ∎

Corollary 2.10. *For every pair k, n of integers with $2 \leq k \leq n$, there exists a connected graph G of order n such that $dg(G) = k$.*

Proof. For $k = n$, let $G = K_n$. Then, by Theorem 2.5 $dg(G) = n = k$. Also, for each pair of integers with $2 \leq k < n$, there exists a tree of order n with k end vertices. Hence the result follows from Corollary 2.9. ∎

Proposition 2.11. *For a nontrivial connected graph G, $g(G) = 2$ if and only if $dg(G) = 2$.*

Proof. If $dg(G) = 2$, then by Theorem 2.3, $g(G) = 2$. Suppose that $g(G) = 2$. Let $S = \{u, v\}$ be a g-set of G. Then it is clear that $x, y \in I[u, v]$ for any pair x, y of vertices of G. Thus S is a dg-set of G and so $dg(G) = 2$. ∎

Corollary 2.12. *For the cycle C_{2n} $(n \geq 2)$, $dg(C_{2n}) = 2$.*

Proof. Since $g(C_{2n}) = 2$, the result follows from Proposition 2.11. ∎

Definition. A vertex v in a connected graph G is said to be a *weak extreme vertex* if there exists a vertex u in G such that $u, v \in I[x, y]$ for a pair of vertices x, y in G, then $v = x$ or $v = y$.

Equivalently, a vertex v in a connected graph is a weak extreme vertex if there exists a vertex u in G such that v is either an initial vertex or a terminal vertex of any interval containing both u and v.

Example 2.13. Each extreme vertex of a graph is weak extreme. Also, for the graph G in Figure 2.2, it is clear that the pair v_2, v_5 lies only on the $v_2 - v_5$ geodesic and so v_2 and v_5 are weak extreme vertices of G. Similarly, the vertices v_4 and v_6 are also weak extreme vertices of G. It is easily seen that v_1 and v_3 are also weak extreme vertices of G.

Proposition 2.14. *Every double geodetic set of a connected graph G contains all the weak extreme vertices of G. In particular, if the set W of all weak extreme vertices is a double geodetic set, then W is the unique dg-set of G.*

Proof. Let S be a double geodetic set of G and v a weak extreme vertex such that $v \notin S$. Let u be a vertex in G such that $u \neq v$. Since S is a double geodetic set of G, we have $u, v \in I[x, y]$ for some $x, y \in S$. Also, since v is a weak extreme vertex of G, we have $v = x$ or $v = y$. Thus $v \in S$, which is a contadiction. ■

Example 2.15. For the graph G in Figure 2.3, the set $S = \{v_1, v_5, v_{10}\}$ of end vertices is the unique minimum geodetic set of G so that $g(G) = 3$. Since the pair of vertices v_3, v_9 do not lie on any geodesic of a pair vertices from S, S is not a double geodetic set of G. It is clear that the vertex v_3 is the only weak extreme vertex, which is not extreme. Since $S' = \{v_1, v_3, v_5, v_{10}\}$ is a double geodetic set of G, it follows from Proposition 2.14 that $dg(G) = 4$.

Theorem 2.16. *For the cycle C_{2n+1} $(n \geq 1)$, $dg(C_{2n+1}) = 2n + 1$.*

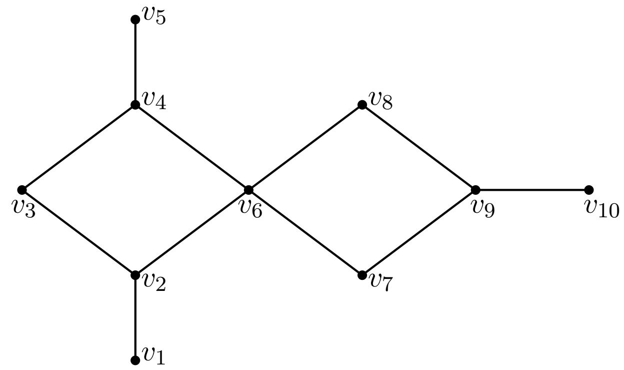

Figure 2.3

Proof. Let v be a vertex of C_{2n+1} and u an eccentric vertex of v. It is clear that the pair u, v of vertices lie only on the interval $I[u, v]$ and so the vertex v is weak extreme. Hence it follows from Proposition 2.14 that the set of all vertices of C_{2n+1} is the unique double geodetic set of C_{2n+1}. Thus $dg(C_{2n+1}) = 2n + 1$. ■

Theorem 2.17. *For the wheel W_n $(n \geq 5)$,* $dg(W_n) = \begin{cases} 2 & \text{if } n = 5, \\ n-1 & \text{if } n \geq 6. \end{cases}$

Proof. Since $g(W_5) = 2$, it follows from Proposition 2.11 that $dg(W_5) = 2$. Let $W_n = K_1 + C_{n-1}$ $(n \geq 6)$ with x the vertex of K_1. Let v be any vertex of C_{n-1}. First we prove that v is a weak extreme vertex of W_n. Let v' be an eccentric vertex of v in W_n. Then $v \neq x$ and v, v' lie only on $I[v, v']$ so that v is a weak extreme vertex of W_n. Hence it follows from Proposition 2.14 that $dg(W_n) \geq n - 1$. It is clear that the set of all vertices of C_{n-1} is a double geodetic set of W_n and so $dg(W_n) = n - 1$. ■

Theorem 2.18. *For the complete bipartite graph $G = K_{m,n}$ $(m, n \geq 2)$, $dg(G) = \min\{m, n\}$.*

Proof. Let X and Y be the partite sets of $K_{m,n}$. Let S be a double geodetic set of $K_{m,n}$. We claim that $X \subseteq S$ or $Y \subseteq S$. Otherwise, there exist vertices x, y such that $x \in X$, $y \in Y$ and $x, y \notin S$. Now, since the pair of vertices x, y lie only on the intervals $I[x, y], I[x, t]$ and $I[s, y]$ for some $t \in X$ and $s \in Y$, it follows that $x \in S$ or $y \in S$, which is a contradiction to $x, y \notin S$. Thus $X \subseteq S$ or $Y \subseteq S$. Also it is clear that both X and Y are double geodetic sets of $K_{m,n}$ and so the result follows. ■

Theorem 2.19. *Let G be a connected graph of order $n \geq 2$. If G has exactly one vertex v of degree $n - 1$, then $dg(G) \leq n - 1$.*

Proof. Let v be the vertex of degree $n - 1$. Let $S = V(G) - \{v\}$. We claim that S is double geodetic set of G. Let $x, y \in V(G)$. If $x, y \in S$, then it is clear that $x, y \in I[x, y]$. So assume that $x \in S$ and $y = v$. Since v is the only vertex of degree $n-1$, there exists a vertex $x' \neq v$ which is non adjacent to x. Hence $x, y \in I[x, x']$ and it follows that S is a double geodetic set of G and so $dg(G) \leq |S| = n - 1$. ■

Remark 2.20. For the graph given in Figure 2.1 $dg(G) = 4$ so that the bound in Theorem 2.19 is sharp. For the wheel $W_5, dg(W_5) = 2$ (see Theorem 2.17) and so the inequality in Theorem 2.19 can also be strict.

The following theorem gives a necessary condition for a graph G of order n to have $dg(G) = n - 1$.

Theorem 2.21. *Let G be a connected graph such that G has a full degree vertex v and $G - v$ has radius at least* 3. *Then $dg(G) = n - 1$.*

Proof. Let $u \neq v$ be a vertex of G and u' an eccentric vertex of u. Since $G - v$ has radius at least 3, we have $d(u, u') \geq 3$. Now, since *diam* $G = 2$, it follows that $P : u, v, u'$ is the unique $u - u'$ geodesic containing the vertices u and u' in G. Hence u is a weak extreme vertex of G. Thus, all the vertices of G except v

are weak extreme vertices. Since $G - v$ has radius at least 3, it follows that v is not a weak extreme vertex of G. Now, $V(G) - \{v\}$ is a double geodetic set of G and hence by Proposition 2.14, $dg(G) = n - 1$. ■

Problem 2.22. Characterize graphs G of order n for which

(i) $dg(G) = n - 1$,

(ii) $dg(G) = n$.

3. The Double Geodetic Number and Diameter of a Graph

If G is connected graph of order n and diameter d, it is proved in [2] that $g(G) \leq n - d + 1$. However, the same is not true for the double geodetic number of a graph. For the graph G given in Figure 3.1, $n = 6, d = 3$ and $dg(G) = 4$ so that $dg(G) = n - d + 1$. Similarly, for the graph G given in Figure 3.2, $dg(G) = 5$ so that $dg(G) > n - d + 1$ and for the graph G given in Figure 3.3, $dg(G) = 3$ so that $dg(G) < n - d + 1$.

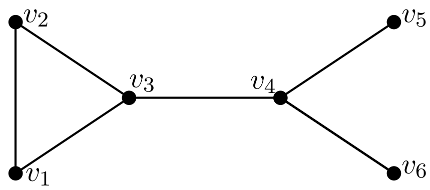

Figure 3.1

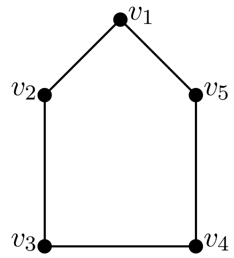

Figure 3.2

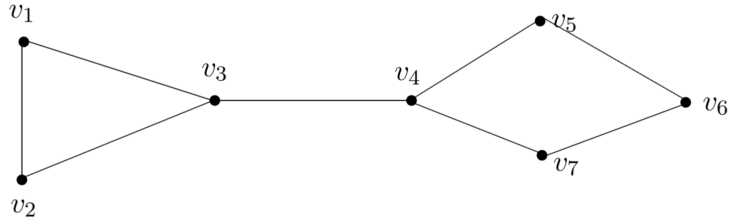

Figure 3.3

A *caterpillar* is a tree, the removal of whose end-vertices leaves a path.

Theorem 3.1. *For every nontrivial tree T of order n with diameter d, $dg(G) = n - d + 1$ if and only if T is a caterpillar.*

Proof. Let T be any nontrivial tree. Let $P : u = v_0, v_1, v_2, \ldots, v_{d-1}, v_d = v$ be a diametral path and let $d = d(u, v)$. Let k be the number of end vertices of T and l the number of internal vertices of T other than $v_1, v_2, \ldots, v_{d-1}$. Then $d-1+l+k = n$. By Theorem 2.5, $dg(T) = k = n-d+1-l$. Hence $dg(T) = n-d+1$ if and only if $l = 0$, if and only if all the internal vertices of T lie on the diameteral path P, if and only if T is caterpillar. ■

Corollary 3.2. *For a non trivial tree T of order n with diameter d, the following are equivalent:*

(i) $g(T) = n - d + 1$,

(ii) $dg(T) = n - d + 1$,

(iii) *T is a caterpillar.*

Proof. This follows from Theorem 3.1 and the fact that $g(T) = dg(T)$ for any tree T. ■

For every connected graph G, $rad\ G \leq diam\ G \leq 2\ rad\ G$. Ostrand [7] showed that every two positive integers a and b with $a \leq b \leq 2a$ are realizable as the radius and diameter respectively, of some connected graph. Now, Ostrand's theorem can be extended so that the geodetic number and double geodetic number can also be priscribed, when $r < d \leq 2r$.

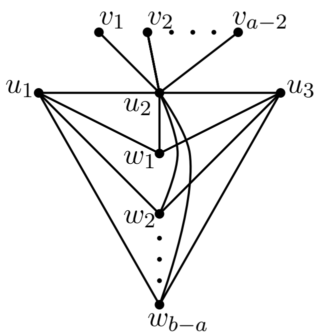

Figure 3.4

Theorem 3.3. *For positive integers r, d, a and b such that $r < d \leq 2r$ and $3 \leq a \leq b$, there exists a connected graph G with $rad\ G = r$, $diam\ G = d, g(G) = a$ and $dg(G) = b$.*

Proof. *Case* 1. $r = 1$. Then $d = 2$. Construct a graph G as follows: Let $P_3 = u_1, u_2, u_3$ be a path of order 3. Add $a-2$ new vertices $v_1, v_2, \ldots, v_{a-2}$ to P_2 and join each v_i $(1 \leq i \leq a-2)$ to the vertex u_2 and obtain the graph H. Also, add $(b-a)$ new vertices $w_1, w_2, \ldots, w_{b-a}$ to H and join each w_i $(1 \leq i \leq b-a)$ to u_1, u_2 and u_3 and obtain the graph G in Figure 3.4. Then G has radius 1 and diameter 2. It is clear that $S_1 = \{v_1, v_2, \ldots, v_{a-2}, u_1, u_3\}$ is a minimum geodetic set of G and so by Theorem 1.1, $g(G) = a$. It is clear that $S_2 = \{v_1, v_2, \ldots, v_{a-2}, u_1, u_3, w_1, w_2, \ldots, w_{b-a}\}$ is the set of all weak extreme vertices of G and since S_2 is a double geodetic set of G, it follows from Proposition 2.14 that $dg(G) = b$.

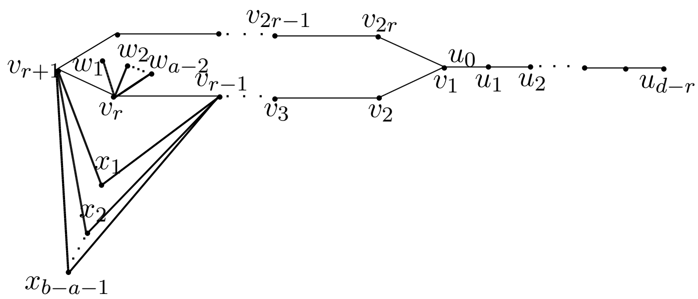

Figure 3.5

Case 2. $r \geq 2$. Construct a graph G as follows: Let $C_{2r} : v_1, v_2, \ldots, v_{2r}, v_1$ be a cycle of order $2r$ and let $P_{d-r+1} : u_0, u_1, \ldots, u_{d-r}$ be a path of order $d - r + 1$. Let H be the graph obtained from C_{2r} and P_{d-r+1} by identifying v_1 in C_{2r} and u_0 in P_{d-r+1}. First, assume that $b > a$. Add $a - 2$ new vertices $w_1, w_2, \ldots, w_{a-2}$ to H and join each vertex w_i $(1 \leq i \leq a - 2)$ to the vertex v_r and add $b - a - 1$ new vertices $x_1, x_2, \ldots, x_{b-a-1}$ to H and join each x_i $(1 \leq i \leq b - a - 1)$ to both v_{r+1} and v_{r-1} and obtain the graph G of Figure 3.5. Then G has radius r and diameter d. Let $S = \{w_1, w_2, \ldots, w_{a-2}, u_{d-r}\}$ be the set of all extreme vertices of G. Since the vertices v_i $(r + 1 \leq i \leq 2r)$ and x_i $(1 \leq i \leq b - a - 1)$ do not lie on a geodesic joining any pair of vertices S, we see that S is not a geodetic set of G. Since $T = S \cup \{v_{r+1}\}$ is a geodetic set of G, it follows from Theorem 1.1 that $g(G) = a$. It is clear that the vertex x_i $(1 \leq i \leq b - a - 1)$ is either an initial vertex or a terminal vertex of any geodesic containing the vertices x_i and w_1 and so each x_i is weak extreme. Similarly, v_{r+1} is either an initial vertex or a terminal vertex of any geodesic containing the vertices v_{r+1} and u_1. Similarly, $v_{2r}$is either an initial vertex or a terminal vertex of any geodesic containing the vertices v_{2r} and w_1. Hence $x_1, x_2, \ldots, x_{b-a}, v_{r+1}, v_{2r}$ are weak extreme vertices. Let $S' = \{w_1, w_2, \ldots, w_{a-2}, u_{d-r}, x_1, x_2, \ldots, x_{b-a-1}, v_{r+1}, v_{2r}\}$. It is easily verified that S' is the set of all weak extreme vertices of G. Since S' is a double geodetic set of G, it follows from Proposition 2.14 that $dg(G) = b$.

Next, assume that $a = b$. Add $(a - 2)$ new vertices $y_1, y_2, \ldots, y_{a-2}$ to H and join each y_i $(1 \leq i \leq a - 2)$ to the vertex v_{r+1} in H, and obtain a graph G'. Then G' has radius r and diameter d. Let $S_1 = \{y_1, y_2, \ldots, y_{a-2}, u_{d-r}, v_{r+1}\}$ be the set of all weak extreme vertices of G'. Since S_1 is a geodetic set as well as a double geodetic set of G', it follows from Theorem 1.1 and Proposition 2.14 that $g(G') = dg(G') = a$. Thus the proof is complete. ■

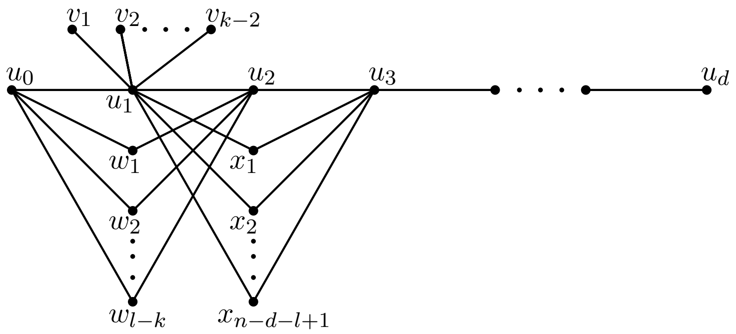

Figure 3.6

Theorem 3.4. *If $n, d \geq 2$ and l are integers such that $3 \leq k \leq l \leq n$ and $n-d-l+1 \geq 0$, then there exists a graph G of order n diameter d with $g(G) = k$ and $dg(G) = l$.*

Proof. Let $P_{d+1} = u_0, u_1, \ldots, u_d$ be a path of length d. Add $k-2$ new vertices $v_1, v_2, \ldots, v_{k-2}$ to P_d and join each v_i $(1 \leq i \leq k-2)$ to u_1, there by producing a tree T. Let H be the graph obtained from T by adding $(l-k)$ new vertices $w_1, w_2, \ldots, w_{l-k}$ to T and joining each w_i $(1 \leq i \leq l-k)$ to both u_0 and u_2. Now, let G be the graph in Figure 3.6 obtained from H by adding $n-d-l+1$ new vertices $x_1, x_2, \ldots, x_{n-d-l+1}$ to H and joining each x_i $(1 \leq i \leq n-d-l+1)$ to both u_1 and u_3. Then G has order n and diameter d. Let $S = \{u_d, v_1, v_2, \ldots, v_{k-2}\}$ be the set of all extreme vertices of G. It is not a geodetic set of G. Since $S' = S \cup \{u_0\}$ is geodetic set of G it follows from Theorem 1.1 that $g(G) = k$. Now, it is clear that each vertex w_i $(1 \leq i \leq l-k)$ is an end of any geodesic containing the vertex w_i and v_1 and so each w_i is weak extreme. Similarly, u_0 is an end of any geodesic containing the vertices u_0 and u_d. Now, it is easily verified that $S_1 = \{u_d, v_1, v_2, \ldots, v_{k-2}, w_1, w_2, \ldots, w_{l-k}, u_0\}$ is the set of all weak extreme vertices of G. Since S_1 is double geodetic set of G, it follows from Proposition 2.14 that $dg(G) = l$. ■

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