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# THE PROJECTIVE PLANE CROSSING NUMBER OF THE CIRCULANT GRAPH $C(3k;\{1,k\})$

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**Abstract**

In this paper we prove that the projective plane crossing number of the circulant graph $C(3k;\{1,k\})$ is $k-1$ for $k \geq 4$, and is 1 for $k = 3$.

**Keywords:** crossing number, circulant graph, projective plane.

**2010 Mathematics Subject Classification:** 05C10.

## 1. Introduction

The crossing number is an important measure of the non-planarity of a graph. Bhatt and Leighton [1] showed that the crossing number of a network (graph) is closely related to the minimum layout area required for the implementation of a VLSI circuit for that network. In general, determining the crossing number of a graph is hard. Garey and Johnson [3] showed that it is NP-complete. In fact, Hliněný [6] has proved that the problem remains NP-complete even when restricted to cubic graphs. Moreover, the exact crossing number is not known even for specific graph families, such as complete graphs [16], complete bipartite graphs [11, 22], and circulant graph [8, 12, 13, 14, 20, 23]. For more about crossing number, see [2, 21] and references therein.

Attention has been paid to the crossing number of graphs on surfaces [4, 5, 7, 9, 10, 17, 18, 19]. However, exact values are known only for very restricted classes of graphs. In this paper, we compute the projective plane crossing number of the circulant graph $C(3k;\{1,k\})$.

**Theorem 1.** *The projective plane crossing number of the circulant graph $C(3k;\{1,k\})$ is given by*

$$cr_1(C(3k;\{1,k\})) = \begin{cases} k-1 & \text{for } k \geq 4, \\ 1 & \text{for } k = 3. \end{cases}$$

Note that there are only few infinite classes of graphs whose projective plane crossing number are known exactly. See [9, 19].

Here are some definitions. Let $G$ be a simple graph with the vertex set $V = V(G)$ and the edge set $E = E(G)$. The *circulant graph* $C(n;S)$ is the graph with the vertex set $V(C(n;S)) = \{v_i \mid 1 \leq i \leq n\}$ and the edge set $E(C(n;S)) = \{v_iv_j \mid 1 \leq i,j \leq n, (i-j) \bmod n \in S\}$ where $S \subseteq \{1,2,\ldots,\lfloor n/2 \rfloor\}$.

The *projective plane crossing number* $cr_1(G)$ of $G$ is the minimum number of crossings of all the drawings of $G$ in the projective plane having the following properties: (i) no edge has a self-intersection; (ii) no two adjacent edges intersect; (iii) no two edges intersect each other more than once; (iv) each intersection of edges is a crossing rather than tangential; and (v) no three edges intersect in a common point. Similarly one can define the plane crossing number $cr(G)$ of the graph $G$. In a drawing $D$, if an edge (or a set of edges) does not cross other edges, we call it *clean*; otherwise, we call it *cross*. For a drawing $D$, the total number of crossings is denoted by $v(D)$.

Let $A$ and $B$ be two (not necessary disjoint) subsets of the edge set $E$. In a drawing $D$, the number of crossings crossed by an edge in $A$ and another edge in $B$ is denoted by $v_D(A,B)$. In particular, $v_D(A,A)$ is denoted by $v_D(A)$, and hence $v(D) = v_D(E)$. By counting the number of crossings in $D$, we have the following:

**Lemma 2.** *Let $A, B, C$ be mutually disjoint subsets of $E$. Then,*

$$\begin{aligned} v_D(A, B \cup C) &= v_D(A,B) + v_D(A,C), \\ v_D(A \cup B) &= v_D(A) + v_D(B) + v_D(A,B). \end{aligned} \tag{1}$$

The plan of this paper is as follows. In Section 2 we prove the upper bound of the projective crossing number of $C(3k;\{1,k\})$. In Section 3, we prove the lower bound of the projective crossing number of $C(3k;\{1,k\})$ by assuming Lemma 7. In Section 4, we prove Lemma 7, which says that for any drawing of $C(3k;\{1,k\})$ with all of its cycles being clean, its number of crossing is at least $k-1$.

## 2. Upper Bounds

From now on, we will denote the circulant graph $C(3k;\{1,k\})$ by $C(k)$ for simplicity. First we have the following:

**Lemma 3.** $cr_1(C(3)) \leq 1$.

***Proof.*** One can refer to the drawing of $C(3)$ in the projective plane in Figure 1. ∎

**Lemma 4.** $cr_1(C(k)) \leq k-1$ *for* $k \geq 4$.

***Proof.*** For a non-planar graph $G$, the plane crossing number is strictly greater than the projective plane crossing number, i.e., $cr_1(G) \le cr(G) - 1$. Lemma 4 follows from $cr(C(k)) = k$ for $k \ge 4$, which is proved in [12]. ■

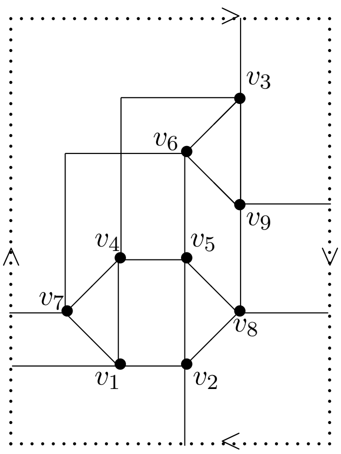


Figure 1. Drawing of $C(3)$.

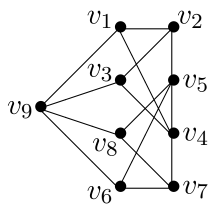


Figure 2. $F_1(9, 15)$.

## 3. LOWER BOUNDS

Next, we have the following:

**Lemma 5.** $cr_1(C(3)) \ge 1$.

***Proof.*** It suffices to show that $C(3)$ cannot be embedded in the projective plane. Note that $C(3) - \{v_1v_7, v_2v_8, v_3v_6\}$ is isomorphic to $F_1(9, 15)$ (see Figure 2) in the list of the minimal forbidden subgraphs for the projective plane (see Appendix A in [15]). This shows that $C(3)$ cannot be embedded in the projective plane. ■

In fact, we have shown the following:

**Corollary 6.** *If $e$ is an edge in the cycle $C_i$ (see the definition below) in $C(3)$, then $cr_1(C(3) - e) \ge 1$.*

In $C(k)$, we define

$$C_i = \{v_iv_{k+i}, v_iv_{2k+i}, v_{k+i}v_{2k+i}\},$$

where $1 \le i \le k$. We have the following:

**Lemma 7.** *For $k \ge 4$, let $D$ be a drawing of $C(k)$ such that $C_i$ is clean for all $1 \le i \le k$. Then $v(D) \ge k - 1$.*

We postpone its proof to Section 4. By assuming Lemma 7, we are in a position to prove the lower bound of $cr_1(C(k))$.

**Lemma 8.**

$$cr_1(C(k)) \geq k-1 \textit{ for } k \geq 4. \tag{2}$$

***Proof.*** We will prove (2) by induction on $k$. First consider $k=4$. Suppose $D$ is a drawing of $C(4)$. We will prove $v(D) \geq 3$ by contradiction. Suppose that $v(D) \leq 2$. Then there exists $C_i$ which crosses; otherwise, if all $C_i$ are clean, $v(D) \geq 3$ by Lemma 7.

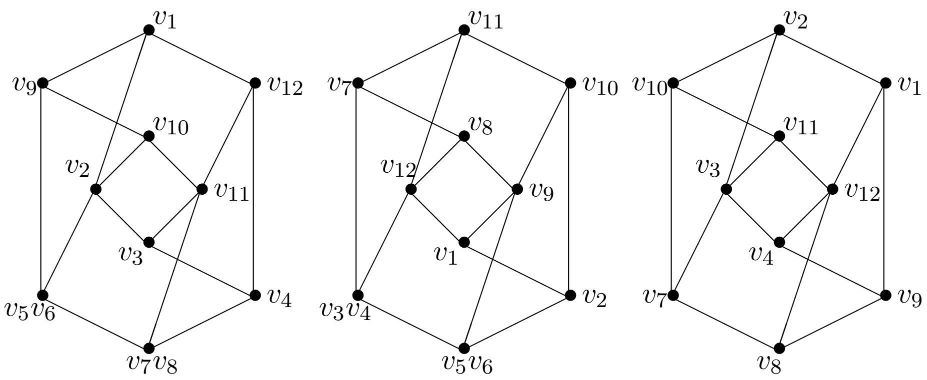


Figure 3(a) Figure 3(b) Figure 3(c)

Without loss of generality, we may assume that the edge $v_1v_5$ in $C_1$ crosses. Then there exists an edge $e$ in $D - v_1v_5$ such that $D - v_1v_5 - e$ is an embedding in the projective plane. Note that $e$ cannot be the edge in any cycle $C_1$: If $e$ is an edge in $C_1$ other than $v_1v_5$, then $D - C_1$, which is a subdivision of $C(3)$, is an embedding in the projective plane, which is impossible by Lemma 5. If $e$ is an edge in $C_i$ with $i \neq 1$, then $D - C_1 - e$, which is a subdivision of $C(3)$ minus an edge in the cycle $C^i$ is an embedding in the projective plane, which contradicts Corollary 6.

Therefore, by symmetry, we have the following possibilities: $e = v_2v_3$, $e = v_4v_5$, $e = v_5v_6$, $e = v_6v_7$, $e = v_7v_8$, $e = v_8v_9$. We will show that it is impossible for $C(4) - v_1v_5 - e$ to be embedded in the projective plane for each of these cases, which will give the required contradiction.

First, by contracting the edges $v_5v_6$ and $v_7v_8$ in $C(4) - \{v_1v_5, v_4v_5, v_8v_9\}$, we get a graph which contains a subgraph isomorphic to $F_4(10, 16)$ (see Figure 3(a)) in the list of the minimal forbidden subgraphs for the projective plane (see Appendix A in [15]). Moreover, by contracting the edges $v_3v_4$ and $v_5v_6$ in $C(4) - \{v_1v_5, v_2v_3, v_6v_7\}$, we get a graph which contains a subgraph isomorphic to $F_4(10, 16)$ (see Figure 3(b)).

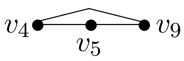


Figure 4(a)

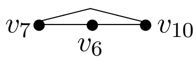


Figure 4(b)

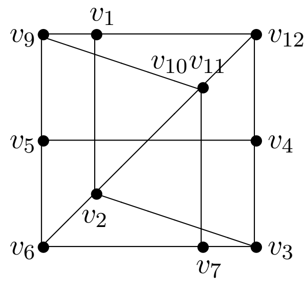


Figure 5

Next we are going to show that $C(4) - \{v_1v_5, v_5v_6\}$ cannot be embedded in the projective plane. Suppose it is not true and let $D$ be an embedding of $C(4) - \{v_1v_5, v_5v_6\}$ in the projective plane. Delete the edge $v_2v_6$ in the drawing. Since $v_1v_5$ and $v_5v_6$ are absent, we can always draw an edge connecting $v_4$ and $v_9$ which is close to the edges $v_4v_5$ and $v_5v_9$ without producing new crossings (see Figure 4(a)). Similarly, since $v_2v_6$ and $v_5v_6$ are absent, we can draw an edge connecting $v_7$ and $v_{10}$ which is close to the edges $v_6v_7$ and $v_6v_{10}$ without producing new crossings (see Figure 4(b)). Therefore, we obtain an embedding of $C(12, \{1, 4\}) - \{v_1v_5, v_5v_6, v_2v_6\} + \{v_4v_9, v_7v_{10}\}$ in the projective plane, which is impossible since it contains a minor isomorphic to $F_4(10, 16)$ (see Figure 3(c)).

Finally, one can see that $C(12, \{1, 4\}) - \{v_1v_5, v_7v_8\}$ contains a minor isomorphic to $F_5(10, 16)$ (see Figure 5) in the list of the minimal forbidden subgraphs for the projective plane (see Appendix A in [15]).

Therefore, (2) is true for $k = 4$. Now suppose that (2) is true for all values less than $k \geq 5$. Let $D$ be a drawing of $C(k)$ in the projective plane and we are going to show that $v(D) \geq k - 1$.

If there exists $1 \leq i \leq 3k$ such that $v_iv_{k+i}$ crosses, then by deleting $v_iv_{k+i}$, $v_{k+i}v_{2k+i}$, $v_{2k+i}v_i$, we obtain a drawing of a subdivision of $C(k-1)$, denote it by $D^{'}$. By our construction, $v(D^{'}) \leq v(D) - 1$. On the other hand, $v(D^{'}) \geq k - 2$ by induction assumption. This implies $v(D) \geq k - 1$. Therefore, we may assume that $v_iv_{k+i}$ is clean in $D$ for all $1 \leq i \leq 3k$, i.e., $C_i$ is clean for all $1 \leq i \leq k$. Then by Lemma 7, we have $v(D) \geq k - 1$. ■

***Proof of Theorem* 1.** It follows from Lemma 3, 4, 5 and 8. ■

## 4. PROOF OF LEMMA 7

This section is devoted to proving Lemma 7. Throughout this section, we assume that $C_i$ is clean for $1 \leq i \leq k$, as we have assumed in Lemma 7.

For $1 \leq i \leq k$, let

$$F_i = \{v_i v_{k+i}, v_i v_{2k+i}, v_{k+i} v_{2k+i}, v_i v_{i+1}, v_{k+i} v_{k+i+1}, v_{2k+i} v_{2k+i+1}\}.$$

Note that the set of all $F_i$ is a partition of the edge set $E$ of $C(k)$, i.e.,

$$E = \bigcup_{i=1}^{k} F_i \text{ and } F_i \cap F_j = \emptyset \text{ for } i \neq j. \tag{3}$$

For $1 \leq i \leq k$, define

$$f_D(F_i) = v_D(F_i) + \frac{1}{2} \sum_{j \neq i} v_D(F_i, F_j). \tag{4}$$

Since we have assumed that each $C_i$ is clean, there are only two possible ways of drawing $C_i$, depending on whether it is contractible or not, which are shown in Figure 6(a) and 6(b).

If $C_i$ and $C_{i+1}$ are both contractible, there are three possible ways of drawing $C_i \cup C_{i+1}$ for each $i$, which are shown in Figure 7(a), 7(b) and 7(c).

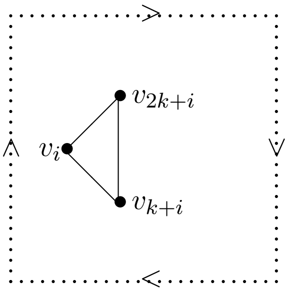


Figure 6(a). $C_i$ is contractible.

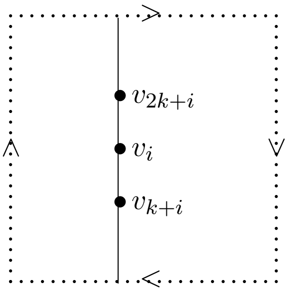


Figure 6(b). $C_i$ is non-contractible.

We have the following:

**Proposition 9.** *If $C_i$ and $C_{i+1}$ are drawn as in Figure 7(a) or 7(b), then* $f_D(F_i) \geq 1$.

***Proof.*** Suppose $f_D(F_i) < 1$. By (4), $v_i v_{i+1}, v_{k+i} v_{k+i+1}, v_{2k+i} v_{2k+i+1}$ do not cross each other. If $C_i \cup C_{i+1}$ is drawn as in Figure 7(a), $F_i \cup C_{i+1}$ must be drawn as in Figure 8 since $C_i, C_{i+1}$ are clean and $v_i v_{i+1}$, $v_{k+i} v_{k+i+1}$, $v_{2k+i} v_{2k+i+1}$ do not cross each other. Since $C_{i-1}$ is clean, $C_{i-1}$ must lies entirely in one of the regions $f_1$, $f_2$ or $f_3$. We may assume that $C_{i-1}$ lies in the region $f_1$, for other cases the proof is the same. If $C_{i-1}$ lies in $f_1$, then $v_{k+i-1} v_{k+i}$ must cross $v_i v_{i+1}$ or $v_{2k+i} v_{2k+i+1}$. On the other hand, the path $v_{k+i+1} v_{k+i+2} \cdots v_{2k-i-1}$ must cross $v_i v_{i+1}$ or $v_{2k+i} v_{2k+i+1}$. Hence, by (4), $f_D(F_i) \geq 1$. Similarly, one can show that $f_D(F_i) \geq 1$ if $C_i \cup C_{i+1}$ is drawn as in Figure 7(b). ∎

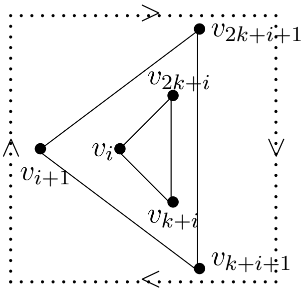


Figure 7(a)

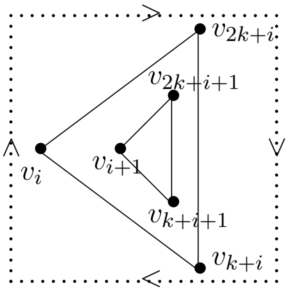


Figure 7(b)

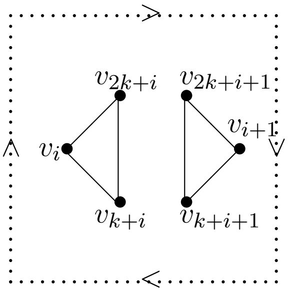


Figure 7(c)

**Proposition 10.** *If $C_i \cup C_{i+1}$ is drawn as in Figure 7(c) and $f_D(F_i) < 1$, then $F_i \cup C_{i+1}$ must be drawn as in Figure 9(b).*

***Proof.*** Since $f_D(F_i) < 1$, by (4), $v_{k+i}v_{k+i+1}, v_{2k+i}v_{2k+i+1}$ do not cross each other. Then $F_i \cup C_{i+1}$ must be drawn as in Figure 9(a) or 9(b). If $F_i \cup C_{i+1}$ is drawn as in Figure 9(a), then $C_{i-1}$ must lie entirely in one of the regions $f_1$, $f_2$ or $f_3$ since $C_{i-1}$ is clean. We may assume that $C_{i-1}$ lies in the region $f_1$, for other cases the proof is the same. If $C_{i-1}$ lies in $f_1$, then $v_{i-1}v_i$ must cross $v_{k+i}v_{k+i+1}$ or $v_{2k+i}v_{2k+i+1}$ since $C_i$ and $C_{i+1}$ are clean. On the other hand, the path $v_{i+1}v_{i+2}\cdots v_{k-i-1}$ must cross $F_i$. Hence, by (4), we have $f_D(F_i) \geq 1$, which contradicts that $f_D(F_i) < 1$. ■

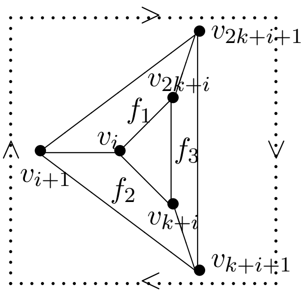


Figure 8

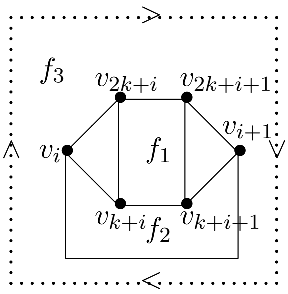


Figure 9(a)

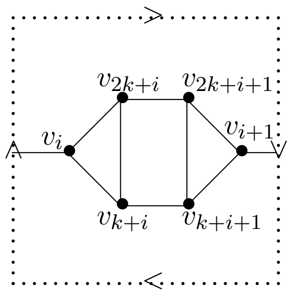


Figure 9(b)

Combining Proposition 9 and 10, we have the following:

**Corollary 11.** *If $F_i \cup C_{i+1}$ is not drawn as in Figure 9(b), then $f_D(F_i) \geq 1$.*

***Proof.*** By Proposition 10, either $f_D(F_i) \geq 1$ or $C_i \cup C_{i+1}$ is not drawn as in Figure 7(c). In the latter case, $C_i \cup C_{i+1}$ must be drawn as in Figure 7(a) or 7(b). By Proposition 9, again we have $f_D(F_i) \geq 1$. ■

**Remark 12.** Hereafter, we say that $F_j \cup C_{j+1}$ is drawn as in Figure 9(b) if it is drawn as in Figure 9(c), i.e., replacing all the indices $i$ by $j$.

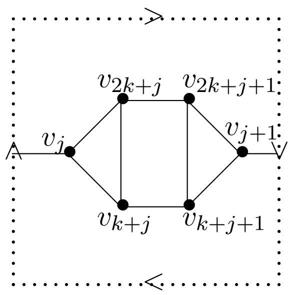


Figure 9(c)

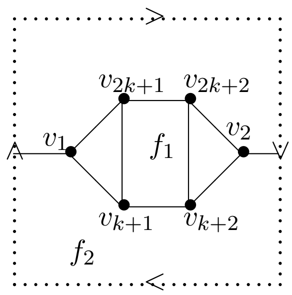


Figure 9(d)

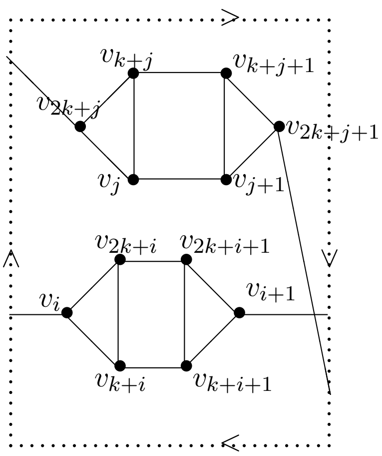


Figure 10. $F_i \cup C_{i+1} \cup F_j \cup C_{j+1}$.

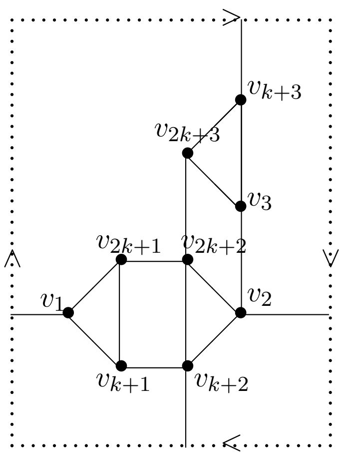


Figure 11. $F_1 \cup F_2 \cup C_3$.

**Proposition 13.** *Suppose that $F_i \cup C_{i+1}$ is drawn as in Figure* 9(b)*. If $j \neq i-1, i, i+1$ such that $F_j \cup C_{j+1}$ is drawn as in Figure* 9(b)*, then $F_i$ and $F_j$ must cross each other. In particular, we have $f_D(F_i) \geq 1/2$ and $f_D(F_j) \geq 1/2$.*

***Proof.*** Note that two non-contractible curves in the projective plane must cross each other. Since $F_i \cup C_{i+1}$ and $F_j \cup C_{j+1}$ are drawn as in Figure 9(b) where $j \neq i-1, i+1$, $F_i$ and $F_j$ must cross each other since $C_i, C_{i+1}, C_j, C_{j+1}$ are clean. See Figure 10 for a possible drawing of $F_i \cup C_{i+1} \cup F_j \cup C_{j+1}$. Since $F_i$ and $F_j$ cross each other, we have $v_D(F_i, F_j) \geq 1$, which implies that $f_D(F_i) \geq 1/2$ and $f_D(F_j) \geq 1/2$ by (4). ■

Here is the outline of the proof of Lemma 7. We will consider two cases:

*Case* 1. $C_i$ is contractible for all $1 \leq i \leq k$.

*Case* 2. $C_i$ is non-contractible for some $1 \leq i \leq k$.

For Case 1, by simple arguments, we can show that $F_1 \cup C_2$ is drawn as in Figure 9(b). Moreover, we can show that $f_D(F_{i_0}) < 1$ for some $i_0 \neq 1$. Then we will consider two cases:

*Case* 1.1. $i_0 \neq 2, k$.

*Case* 1.2. $i_0 = 2$ or $k$.

Case 1.1 can be solved easily. For Case 1.2, we will assume that $i_0 = 2$ since the proof for $i_0 = k$ is the same. Then we will consider two cases:

*Case* 1.2.1. $f_D(F_j) \geq 1$ for all $j \neq 1, 2$.

*Case* 1.2.2. $f_D(F_j) < 1$ for some $j \neq 1, 2$.

For Case 1.2.1, by assumption, $f_D(F_j) \geq 1$ for all $j \neq 1, 2$. We just need to show that $f_D(F_1) + f_D(F_2) > 0$, which implies that $v(D) = \sum_{j=1}^{k} f_D(F_j) = f_D(F_1) + f_D(F_2) + \sum_{j \neq 1,2} f_D(F_j) > k - 2$, and hence $v(D) \geq k - 1$ since $v(D)$ is an integer. For Case 1.2.2, by assumption, $f_D(F_j) < 1$ for some $j \neq 1, 2$. Then we will consider two cases:

*Case* 1.2.2.1. $j \neq 3$, $k$.

*Case* 1.2.2.2. $j = 3$ or $k$.

Case 1.2.2.1 can be solved easily.
For Case 1.2.2.2, we can assume that

$$f_D(F_l) \geq 1 \text{ for } l \neq 1, 2, 3, k. \tag{5}$$

Otherwise, if $f_D(F_l) < 1$ for some $l \neq 1, 2, 3, k$, then it can be reduces to Case 1.2.2.1 by taking $j = l$. By simple arguments, we can reduced it to the case when both $F_3 \cup C_4$ and $F_k \cup C_1$ are drawn as in Figure 9(b). That is to say, $F_i \cup C_{i+1}$ is drawn as in Figure 9(b) for $i = 1, 2, 3, k$. Then by Proposition 13, $F_1$ crosses $F_3$ and $F_2$ crosses $F_k$. Moreover, if $k \geq 5$, then $F_1$ also crosses $F_k$. All these implies

$$f_D(F_1) \geq 1, f_D(F_k) \geq 1, f_D(F_2) \geq 1/2, \text{ and } f_D(F_3) \geq 1/2. \tag{6}$$

Combining (5) and (6), we get $v(D) \geq k - 1$. For $k = 4$, we will use different arguments by making use the fact that $F_i \cup C_{i+1}$ is drawn as in Figure 9(b) for $i = 1, 2, 3, 4$.

Now we are ready to prove Lemma 7.

***Proof of Lemma 7.*** By (1), (3) and (4), the total number of crossing of the drawing $D$ is $v(D) = v_D(E) = \sum_{i=1}^{k} f_D(F_i)$. Therefore, it suffices to prove that $\sum_{i=1}^{k} f_D(F_i) \geq k-1$. To prove by contradiction, we assume that

$$\sum_{i=1}^{k} f_D(F_i) < k-1. \tag{7}$$

We will consider two cases: Case 1. $C_i$ is contractible for all $1 \leq i \leq k$ and Case 2. $C_i$ is non-contractible for some $1 \leq i \leq k$.

*Case* 1. Since we have assumed that $C_i$ is clean for $1 \leq i \leq k$, as we have said at the beginning of this section, there are three possible ways of drawing $C_i \cup C_{i+1}$ for each $i$, which are shown in Figure 7(a), 7(b) or 7(c).

Note that (7) implies that $f_D(F_i) < 1$ for some $i$. Without loss of generality, we may assume $i = 1$, i.e.,

$$f_D(F_1) < 1. \tag{8}$$

By Proposition 9, $C_1 \cup C_2$ must be drawn as in Figure 7(c). Hence, by (8) and Proposition 10, $F_1 \cup C_2$ is drawn as in Figure 9(b) (see Figure 9(d)).

There exists $i_0 \neq 1$ such that $F_{i_0} \cup C_{i_0+1}$ is drawn as in Figure 9(b). (Otherwise, if $F_j \cup C_{j+1}$ is not drawn as in Figure 9(b) for all $j \neq 1$, $f_D(F_j) \geq 1$ for all $j \neq 1$ by Corollary 11, which implies $\sum_{j=1}^{k} f_D(F_j) \geq \sum_{j \neq 1} f_D(F_j) \geq k-1$.) We will consider two cases: Case 1.1. $i_0 \neq 2, k$ and Case 1.2. $i_0 = 2$ or $k$.

*Case* 1.1. If $i_0 \neq 2, k$, i.e., $C_{i_0} \cup C_{i_0+1}$ is drawn as in Figure 9(b) for some $i_0 \neq 1, 2, k$, then by Proposition 13, $F_1$ and $F_{i_0}$ cross each others,

$$f_D(F_1) \geq 1/2 \text{ and } f_D(F_{i_0}) \geq 1/2. \tag{9}$$

Moreover, if there exists $j \neq 1, 2, i_0, k$ such that $f_D(F_j) < 1$, then $F_j \cup C_{j+1}$ must be drawn as in Figure 9(b) by Proposition 10. By Proposition 13, $F_j$ and $F_1$ must also cross each other. Hence, $f_D(F_1) \geq 1$ since $F_1$ crosses both $F_{i_0}$ and $F_j$, which contradicts (8). Therefore,

$$f_D(F_j) \geq 1 \text{ for all } j \neq 1, 2, i_0, k. \tag{10}$$

Moreover, we can assume that

$$f_D(F_2) \geq 1 \text{ and } f_D(F_k) \geq 1. \tag{11}$$

(Otherwise, $f_D(F_2) < 1$ or $f_D(F_k) < 1$ implies that $F_2 \cup C_3$ or $F_k \cup C_1$ is drawn as in Figure 9(b) by Proposition 10. Replacing $i_0$ by 2 or $k$, one can reduce this to Case 1.2.) Combining (9), (10) and (11), we have $\sum_{j=1}^{k} f_D(F_j) \geq f_D(F_1) + f_D(F_{i_0}) + \sum_{j \neq 1, i_0} f_D(F_j) \geq k-1$.

*Case* 1.2. If $i_0 = 2$ or $k$, then we may assume that $i_0 = 2$ since the proof for $i_0 = k$ is the same. Then $F_2 \cup C_3$ is drawn as in Figure 9(b). We will consider two cases: Case 1.2.1. $f_D(F_j) \geq 1$ for all $j \neq 1, 2$ and Case 1.2.2. $f_D(F_j) < 1$ for some $j \neq 1, 2$.

*Case* 1.2.1. By assumption,

$$f_D(F_j) \geq 1 \text{ for all } j \neq 1, 2. \tag{12}$$

If we can show that

$$f_D(F_1) + f_D(F_2) > 0, \tag{13}$$

then by (12) and (13),
$v(D) = \sum_{j=1}^{k} f_D(F_j) = f_D(F_1) + f_D(F_2) + \sum_{j \neq 1,2} f_D(F_j) > k - 2$, which implies that $v(D) \geq k - 1$ since the total number of crossing $v(D)$ is an integer.

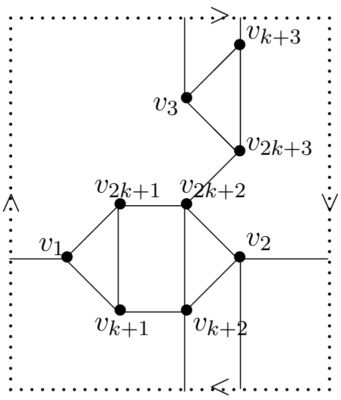


Figure 12

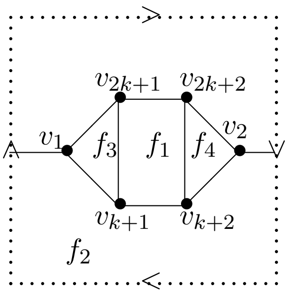


Figure 13

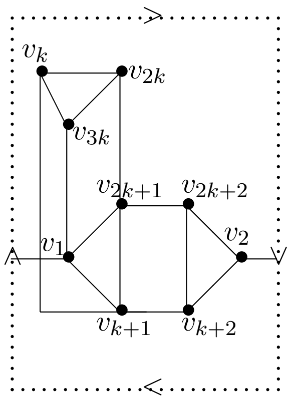


Figure 14

Suppose (13) is not true, i.e.,

$$f_D(F_1) = f_D(F_2) = 0. \tag{14}$$

Recall that $F_1 \cup C_2$ is drawn as in Figure 9(d). Since $C_3$ is clean, $C_3$ must lie entirely in regions $f_1$ or $f_2$ in Figure 9(d). If $C_3$ lies in $f_1$, then $v_2v_3$ must cross $v_{k+1}v_{k+2}$ or $v_{2k+1}v_{2k+2}$. By (4), $f_D(F_2) \geq 1/2$, which contradicts (14). Therefore, $C_3$ lies in $f_2$. By (4) and (14), $v_2v_3$, $v_{k+2}v_{k+3}$, $v_{2k+2}v_{2k+3}$ are clean. Then the only possible drawing of $F_1 \cup F_2 \cup C_3$ is shown as in Figure 11. (It is true up to renaming the vertices. For example, it is possible for $F_1 \cup F_2 \cup C_3$ to be drawn as in Figure 12. But one can reduce it to Figure 11 by the transformation $v_j \mapsto v_{j-k}$.)

Since $C_4$ is clean, it must lie entirely in one of the regions in Figure 11. Note that $v_3$, $v_{k+3}$ and $v_{2k+3}$ do not lie in the the same region in Figure 11. No matter which region $C_4$ lies in Figure 11, one of the edges $v_3v_4$, $v_{k+3}v_{k+4}$ and $v_{2k+3}v_{2k+4}$ must cross the $F_1$ or $F_2$ (Note that $k \geq 4$ is crucial here for $C_4$ being not equal to $C_1$). Hence, $f_D(F_1) + f_D(F_2) > 0$ which gives (13).

*Case* 1.2.2. If $f_D(F_j) < 1$ for some $j \neq 1, 2$, then $F_j \cup C_{j+1}$ must be drawn as in Figure 9(b) by Proposition 10. We will consider two cases: Case 1.2.2.1. $j \neq 3$, $k$ and Case 1.2.2.2. $j = 3$ or $k$.

*Case* 1.2.2.1. Since $F_j \cup C_{j+1}$ is drawn as in Figure 9(b) where $j \neq 1, 2, 3, k$, $F_j$ must cross $F_1$ and $F_2$ by Proposition 13, since $F_1 \cup C_2$ and $F_2 \cup C_3$ are drawn as in Figure 9(b). This implies that, by (4),

$$f_D(F_1) \geq 1/2, f_D(F_2) \geq 1/2, \text{ and } f_D(F_j) \geq 1. \tag{15}$$

Note that

$$f_D(F_r) \geq 1 \text{ for all } r \neq 1, 2, 3, j, k. \tag{16}$$

Otherwise, if $f_D(F_r) < 1$ for some $r \neq 1, 2, 3, j, k$ , then by Proposition 10, $F_r \cup C_{r+1}$ is drawn as in Figure 9(b). By Proposition 13, $F_r$ also crosses $F_1$. This implies $f_D(F_1) \geq 1$ since $F_1$ cross $F_j$ and $F_r$, which contradicts (8).

We claim that

$$f_D(F_3) \geq 1 \text{ and } f_D(F_k) \geq 1. \tag{17}$$

To see this, suppose that $f_D(F_3) < 1$. Then $F_3 \cup C_4$ is drawn as in Figure 9(b) by Proposition 10. Hence $F_1$ must cross $F_3$ and $F_j$ by Proposition 13, which implies that $f_D(F_1) \geq 1$ and contradicts (8). On the other hand, if $f_D(F_k) < 1$, then $F_k \cup C_1$ must be drawn as in Figure 9(b) by Proposition 10. Hence $F_2$ must cross $F_k$ and $F_j$ by Proposition 13, which implies that $f_D(F_2) \geq 1$ and contradicts (8). This proves (17).

Combining (15), (16) and (17), we get $\sum_{r=1}^{k} f_D(F_r) = f_D(F_1) + f_D(F_2) + \sum_{r \neq 1,2} f_D(F_r) \geq k - 1$.

*Case* 1.2.2.2. If $j = 3$ or $k$, then $F_k \cup C_1$ or $F_3 \cup C_4$ is drawn as in Figure 9(b). We may assume that

$$f_D(F_l) \geq 1 \text{ for } l \neq 1, 2, 3, k. \tag{18}$$

(Otherwise, if $f_D(F_l) < 1$ for some $l \neq 1, 2, 3, k$, then it can be reduces to Case 1.2.2.1 by taking $j = l$.) It can be reduced to the case when both $F_3 \cup C_4$ and $F_k \cup C_1$ are drawn as in Figure 9(b).

To see this, suppose that $F_3 \cup C_4$ is drawn as in Figure 9(b) and $F_k \cup C_1$ is not. Then by Corollary 11

$$f_D(F_k) \geq 1, \tag{19}$$

and $F_3$ must cross $F_1$ by Proposition 13 since $F_1 \cup C_2$ is drawn as in Figure 9(b). We claim that $F_1$ must cross $F_k$. Assuming the claim, we have

$$f_D(F_1) \geq 1 \text{ and } f_D(F_3) \geq 1/2. \tag{20}$$

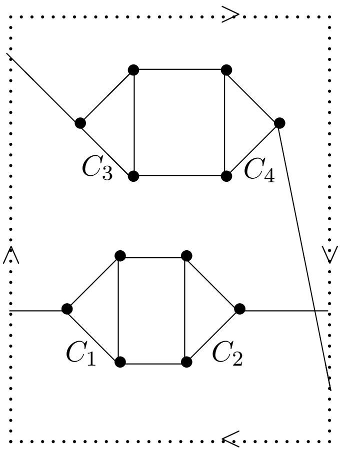


Figure 15(a)

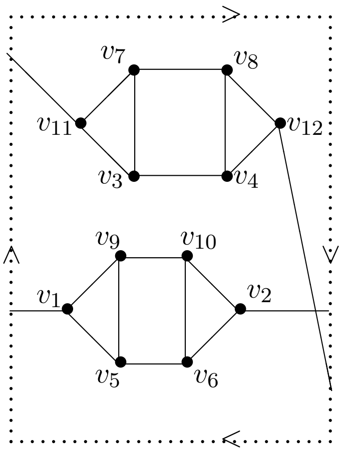


Figure 15(b)

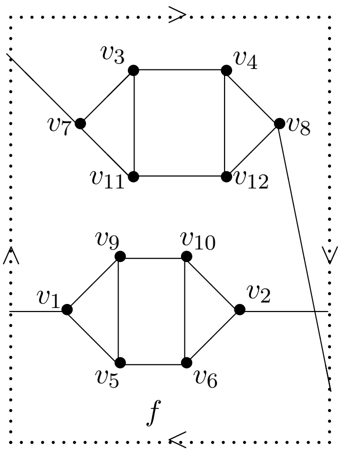


Figure 15(c)

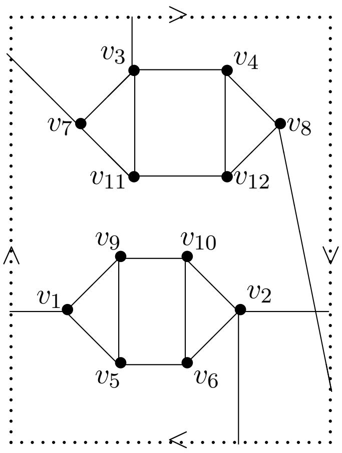


Figure 15(d)

Combining (18), (19) and (20), we get $\sum_{r=1}^{k} f_D(F_r) > k - 2$, which implies that $v(D) = \sum_{i=1}^{k} f_D(F_i) \geq k - 1$ since $v(D)$ is an integer.

To show the claim, i.e., $F_1$ crosses $F_k$, we note that $F_1 \cup C_2$ is drawn as in Figure 9(b). See Figure 13. Since $C_k$ is clean, it must lie entirely in one of the regions in Figure 13. It is impossible for $C_k$ to lie in $f_3$, otherwise, the path $v_2 v_3 \cdots v_k$ crosses $C_1$. It is also impossible for $C_{i-1}$ to lie in $f_4$, otherwise, $v_k v_{k+1}$ crosses $C_2$. If $C_k$ lies in $f_1$, $v_{3k} v_1$ must cross with $v_{k+1} v_{k+2}$ or $v_{2k+1} v_{2k+2}$, which implies that $F_k$ crosses $F_1$. If $C_k$ lies in $f_2$, then $F_k$ must cross $F_1$ since $F_k \cup C_1$ is not drawn as in Figure 9(b) by our assumption (See Figure 14 for example). Therefore, $F_1$ must cross $F_k$, as we claimed.

Similarly, if $F_k \cup C_1$ is drawn as in Figure 9(b) and $F_3 \cup C_4$ is not, then $\sum_{r=1}^{k} f_D(F_r) \geq k - 1$.

Therefore, we can assume that both $F_3 \cup C_4$ and $F_k \cup C_1$ are drawn as in Figure 9(b). Then $F_k$ must cross $F_2$, and $F_1$ must cross with $F_3$ by Proposition 13. Moreover, if $k \geq 5$, then $F_3$ and $F_k$ must also cross each other by Proposition 13. All these imply that

$$f_D(F_1) \geq 1/2, f_D(F_2) \geq 1/2, f_D(F_3) \geq 1, \text{ and } f_D(F_k) \geq 1. \tag{21}$$

Combining (18) and (21), we infer $\sum_{r=1}^{k} f_D(F_r) \geq k - 1$ if $k \geq 5$.

On the other hand, if $k = 4$, then $F_k \cup C_1 = F_4 \cup C_1$, $F_1 \cup C_2$, $F_2 \cup C_3$ and $F_3 \cup C_4$ are drawn as in Figure 9(b) by assumptions. By Proposition 13, $F_1$ must cross $F_3$, and $F_2$ must cross $F_4$. This implies that

$$f_D(F_i) \geq 1/2 \text{ for } 1 \leq i \leq 4. \tag{22}$$

We will show that $v(D) \geq 3$. By contradiction, suppose that $v(D) \leq 2$. By (1) and (22), we have

$$f_D(F_1) = f_D(F_2) = f_D(F_3) = f_D(F_4) = 1/2. \tag{23}$$

Since $F_1$ crosses $F_3$, by (4) and (23) we get

$$v_D(F_1, F_3) = 1, v_D(F_1, F_j) = 0 \text{ for } j \neq 3, v_D(F_3, F_j) = 0 \text{ for } j \neq 1. \tag{24}$$

Similarly, since $F_2$ crosses $F_4$, by (4) and (23) we get

$$v_D(F_2, F_4) = 1, v_D(F_2, F_j) = 0 \text{ for } j \neq 4, v_D(F_4, F_j) = 0 \text{ for } j \neq 2. \tag{25}$$

Since $F_1 \cup C_2$ and $F_3 \cup C_4$ are drawn as in Figure 9(b), the only possible drawing of $F_1 \cup C_2 \cup F_3 \cup C_4$ is shown in Figure 15(a) in view of (24) and (25). However, one can show that it is impossible for (24), (25) to hold. For example, if $F_1 \cup C_2 \cup F_3 \cup C_4$ is drawn in Figure 15(b), then the edge $v_8 v_9$ must cross with $F_1$ or $F_3$, which contradicts (24); and if $F_1 \cup C_2 \cup F_3 \cup C_4$ is drawn in Figure 15(c), then the edge $v_2 v_3$ must lie entirely in the region $f$, as in Figure 15(d), since $v_D(F_2, F_j) = 0$ for $j \neq 4$ by (25). However, in Figure 15(d), no matter how $v_6 v_7$

is drawn, $v_6v_7$ must either (i) cross $v_2v_3$ which contradicts (25), or (ii) cross $C_i$ which contradicts that $C_i$ are all clean, or (iii) cross $F_1$ or $F_3$ which contradicts (25). We leave other cases to the reader.

*Case* 2. If there exists $1 \le i \le k$ such that $C_i$ is non-contractible, then we may assume that $C_1$ is non-contractible. Then $C_i$ is contractible for all $i \ne 1$. (Otherwise, $C_i$ crosses $C_1$ since two non-contractible curves in the projective plane must cross each other. This contradicts the assumption that all $C_i$ are clean.) Since $C_i$ and $C_{i+1}$ are clean and contractible for $i \ne 1, k$, there are three possible ways of drawing $C_i \cup C_{i+1}$, which are shown in Figure 7(a), 7(b) or 7(c).

We claim that

$$f_D(F_i) \ge 1 \text{ for } i \ne 1, k. \tag{26}$$

To prove this, suppose that $f_D(F_i) < 1$ for some $i \ne 1, k$. By Corollary 11, $F_i \cup C_{i+1}$ must be drawn as in Figure 9(b), which crosses the non-contractible $C_1$. This contradicts that $C_1$ is clean. This proves (26).

Now we are going to show that

$$f_D(F_1) + f_D(F_k) > 0. \tag{27}$$

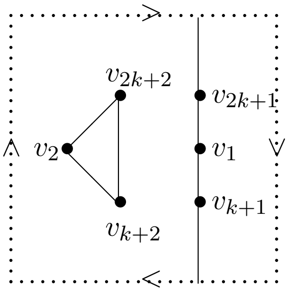


Figure 16. $C_1 \cup C_2$.

Combining this with (26), we will get $\sum_{r=1}^{k} f_D(F_r) > k-2$, which gives $v(D) = \sum_{i=1}^{k} f_D(F_i) \ge k-1$ since $v(D)$ is an integer. Suppose that (27) is not true, i.e.,

$$f_D(F_1) = f_D(F_k) = 0. \tag{28}$$

Since $C_1$ is non-contractile and $C_2$ is contractible, $C_1 \cup C_2$ must be drawn as in Figure 16. On the other hand, by the same reasons, $C_1 \cup C_k$ must be drawn as in Figure 16 by replacing $C_2$ by $C_k$.

By (4) and (28), $v_1v_2$, $v_{k+1}v_{k+2}$, $v_{2k+1}v_{2k+2}$ do not cross. From Figure 16, one can see that there are three possible ways of drawing $F_1 \cup C_2$, which are shown in Figure 17(a), 17(b) and 17(c).

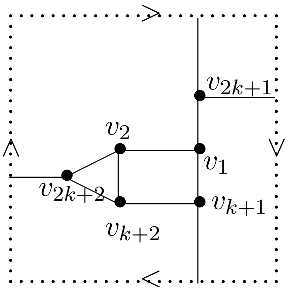


Figure 17(a)

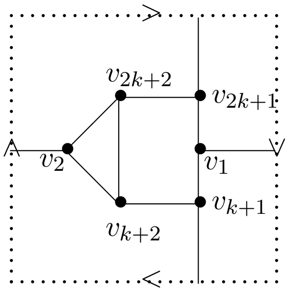


Figure 17(b)

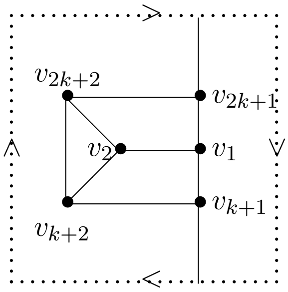


Figure 17(c)

If $F_1 \cup C_2$ is drawn as in Figure 17(b) and 17(c), then $C_3$ must lie entirely in one of the regions since $C_3$ is clean. Then $F_2$ must cross with $F_1$ since there is no region in Figure 17(b) or 17(c) containing all of the vertices $v_2$, $v_{k+2}$ and $v_{2k+2}$. This implies $f_D(F_1) > 0$, which contradicts (28).

Therefore, $F_1 \cup C_2$ must be drawn as in Figure 17(a). By the same argument, $F_k \cup C_1$ must be drawn as in Figure 17(a) by replacing $C_2$ by $C_k$. Hence, $F_k \cup F_1 \cup C_2$ must be drawn as in Figure 18(a) or 18(b) since $F_1$ does not cross $F_k$ by (28).

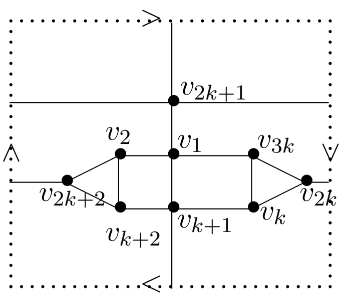


Figure 18(a)

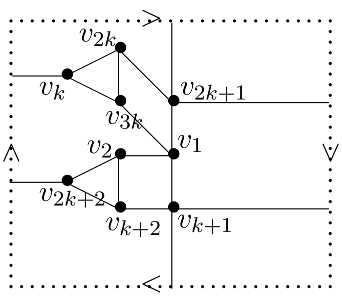


Figure 18(b)

Note that $C_3$ must lie in one of the regions in Figure 18(a) or 18(b). Since there exists no region in Figure 18(a) or 18(b) which contains all of the vertices $v_2$, $v_{k+2}$ and $v_{2k+2}$, $F_3$ must cross either $F_k$ or $F_1$ ($k \geq 4$ is needed here for $F_3$ being not equal to $F_k$). This implies that $f_D(F_1) > 0$ or $f_D(F_k) > 0$, which gives (27).

This finishes the the proof of Lemma 7. ■

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