Discussiones Mathematicae
Graph Theory 32 (2012) 91–108

THE PROJECTIVE PLANE CROSSING NUMBER OF THE CIRCULANT GRAPH $C(3k; \{1, k\})$

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Abstract

In this paper we prove that the projective plane crossing number of the circulant graph $C(3k; \{1, k\})$ is $k - 1$ for $k \geq 4$, and is 1 for $k = 3$.

Keywords: crossing number, circulant graph, projective plane.

2010 Mathematics Subject Classification: 05C10.

1. Introduction

The crossing number is an important measure of the non-planarity of a graph. Bhatt and Leighton [1] showed that the crossing number of a network (graph) is closely related to the minimum layout area required for the implementation of a VLSI circuit for that network. In general, determining the crossing number of a graph is hard. Garey and Johnson [3] showed that it is NP-complete. In fact, Hliněný [6] has proved that the problem remains NP-complete even when restricted to cubic graphs. Moreover, the exact crossing number is not known even for specific graph families, such as complete graphs [16], complete bipartite graphs [11, 22], and circulant graph [8, 12, 13, 14, 20, 23]. For more about crossing number, see [2, 21] and references therein.

Attention has been paid to the crossing number of graphs on surfaces [4, 5, 7, 9, 10, 17, 18, 19]. However, exact values are known only for very restricted classes of graphs. In this paper, we compute the projective plane crossing number of the circulant graph $C(3k; \{1, k\})$.

Theorem 1. *The projective plane crossing number of the circulant graph $C(3k; \{1, k\})$ is given by*

$$cr_1(C(3k; \{1, k\})) = \begin{cases} k - 1 & \text{for } k \geq 4, \\ 1 & \text{for } k = 3. \end{cases}$$

Note that there are only few infinite classes of graphs whose projective plane crossing number are known exactly. See [9, 19].

Here are some definitions. Let G be a simple graph with the vertex set $V = V(G)$ and the edge set $E = E(G)$. The *circulant graph* $C(n; S)$ is the graph with the vertex set $V(C(n; S)) = \{v_i \mid 1 \leq i \leq n\}$ and the edge set $E(C(n; S)) = \{v_i v_j \mid 1 \leq i, j \leq n, (i-j) \bmod n \in S\}$ where $S \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$.

The *projective plane crossing number* $cr_1(G)$ of G is the minimum number of crossings of all the drawings of G in the projective plane having the following properties: (i) no edge has a self-intersection; (ii) no two adjacent edges intersect; (iii) no two edges intersect each other more than once; (iv) each intersection of edges is a crossing rather than tangential; and (v) no three edges intersect in a common point. Similarly one can define the plane crossing number $cr(G)$ of the graph G. In a drawing D, if an edge (or a set of edges) does not cross other edges, we call it *clean*; otherwise, we call it *cross*. For a drawing D, the total number of crossings is denoted by $v(D)$.

Let A and B be two (not necessary disjoint) subsets of the edge set E. In a drawing D, the number of crossings crossed by an edge in A and another edge in B is denoted by $v_D(A, B)$. In particular, $v_D(A, A)$ is denoted by $v_D(A)$, and hence $v(D) = v_D(E)$. By counting the number of crossings in D, we have the following:

Lemma 2. *Let A, B, C be mutually disjoint subsets of E. Then,*

$$\begin{aligned} v_D(A, B \cup C) &= v_D(A, B) + v_D(A, C), \\ v_D(A \cup B) &= v_D(A) + v_D(B) + v_D(A, B). \end{aligned} \tag{1}$$

The plan of this paper is as follows. In Section 2 we prove the upper bound of the projective crossing number of $C(3k; \{1, k\})$. In Section 3, we prove the lower bound of the projective crossing number of $C(3k; \{1, k\})$ by assuming Lemma 7. In Section 4, we prove Lemma 7, which says that for any drawing of $C(3k; \{1, k\})$ with all of its cycles being clean, its number of crossing is at least $k - 1$.

2. Upper Bounds

From now on, we will denote the circulant graph $C(3k; \{1, k\})$ by $C(k)$ for simplicity. First we have the following:

Lemma 3. $cr_1(C(3)) \leq 1$.

Proof. One can refer to the drawing of $C(3)$ in the projective plane in Figure 1. ∎

Lemma 4. $cr_1(C(k)) \leq k - 1$ *for* $k \geq 4$.

Proof. For a non-planar graph G, the plane crossing number is strictly greater than the projective plane crossing number, i.e., $cr_1(G) \leq cr(G) - 1$. Lemma 4 follows from $cr(C(k)) = k$ for $k \geq 4$, which is proved in [12]. ∎

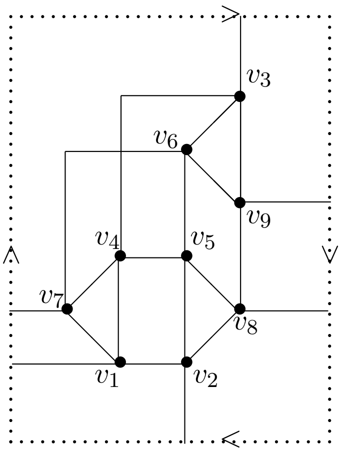

Figure 1. Drawing of $C(3)$.

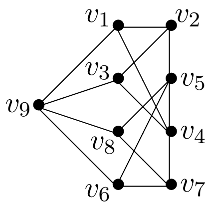

Figure 2. $F_1(9, 15)$.

3. Lower Bounds

Next, we have the following:

Lemma 5. $cr_1(C(3)) \geq 1$.

Proof. It suffices to show that $C(3)$ cannot be embedded in the projective plane. Note that $C(3) - \{v_1v_7, v_2v_8, v_3v_6\}$ is isomorphic to $F_1(9, 15)$ (see Figure 2) in the list of the minimal forbidden subgraphs for the projective plane (see Appendix A in [15]). This shows that $C(3)$ cannot be embedded in the projective plane. ∎

In fact, we have shown the following:

Corollary 6. *If e is an edge in the cycle C_i (see the definition below) in $C(3)$, then $cr_1(C(3) - e) \geq 1$.*

In $C(k)$, we define
$$C_i = \{v_iv_{k+i}, v_iv_{2k+i}, v_{k+i}v_{2k+i}\},$$
where $1 \leq i \leq k$. We have the following:

Lemma 7. *For $k \geq 4$, let D be a drawing of $C(k)$ such that C_i is clean for all $1 \leq i \leq k$. Then $v(D) \geq k - 1$.*

We postpone its proof to Section 4. By assuming Lemma 7, we are in a position to prove the lower bound of $cr_1(C(k))$.

Lemma 8.

$$cr_1(C(k)) \geq k-1 \text{ for } k \geq 4. \tag{2}$$

Proof. We will prove (2) by induction on k. First consider $k = 4$. Suppose D is a drawing of $C(4)$. We will prove $v(D) \geq 3$ by contradiction. Suppose that $v(D) \leq 2$. Then there exists C_i which crosses; otherwise, if all C_i are clean, $v(D) \geq 3$ by Lemma 7.

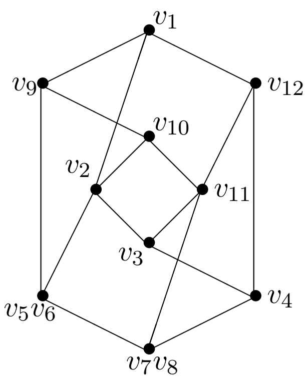

Figure 3(a)

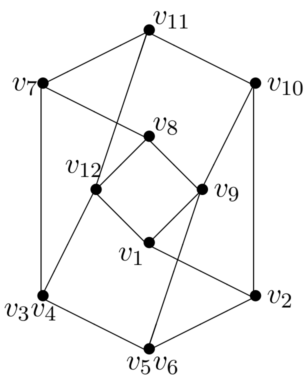

Figure 3(b)

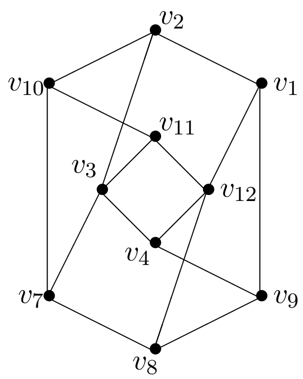

Figure 3(c)

Without loss of generality, we may assume that the edge v_1v_5 in C_1 crosses. Then there exists an edge e in $D - v_1v_5$ such that $D - v_1v_5 - e$ is an embedding in the projective plane. Note that e cannot be the edge in any cycle C_1: If e is an edge in C_1 other than v_1v_5, then $D - C_1$, which is a subdivision of $C(3)$, is an embedding in the projective plane, which is impossible by Lemma 5. If e is an edge in C_i with $i \neq 1$, then $D - C_1 - e$, which is a subdivision of $C(3)$ minus an edge in the cycle C^i is an embedding in the projective plane, which contradicts Corollary 6.

Therefore, by symmetry, we have the following possibilities: $e = v_2v_3$, $e = v_4v_5$, $e = v_5v_6$, $e = v_6v_7$, $e = v_7v_8$, $e = v_8v_9$. We will show that it is impossible for $C(4) - v_1v_5 - e$ to be embedded in the projective plane for each of these cases, which will give the required contradiction.

First, by contracting the edges v_5v_6 and v_7v_8 in $C(4) - \{v_1v_5, v_4v_5, v_8v_9\}$, we get a graph which contains a subgraph isomorphic to $F_4(10, 16)$ (see Figure 3(a)) in the list of the minimal forbidden subgraphs for the projective plane (see Appendix A in [15]). Moreover, by contracting the edges v_3v_4 and v_5v_6 in $C(4) - \{v_1v_5, v_2v_3, v_6v_7\}$, we get a graph which contains a subgraph isomorphic to $F_4(10, 16)$ (see Figure 3(b)).

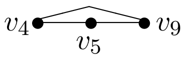

Figure 4(a)

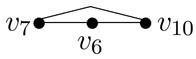

Figure 4(b)

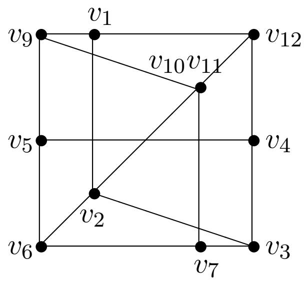

Figure 5

Next we are going to show that $C(4) - \{v_1v_5, v_5v_6\}$ cannot be embedded in the projective plane. Suppose it is not true and let D be an embedding of $C(4) - \{v_1v_5, v_5v_6\}$ in the projective plane. Delete the edge v_2v_6 in the drawing. Since v_1v_5 and v_5v_6 are absent, we can always draw an edge connecting v_4 and v_9 which is close to the edges v_4v_5 and v_5v_9 without producing new crossings (see Figure 4(a)). Similarly, since v_2v_6 and v_5v_6 are absent, we can draw an edge connecting v_7 and v_{10} which is close to the edges v_6v_7 and v_6v_{10} without producing new crossings (see Figure 4(b)). Therefore, we obtain an embedding of $C(12, \{1, 4\}) - \{v_1v_5, v_5v_6, v_2v_6\} + \{v_4v_9, v_7v_{10}\}$ in the projective plane, which is impossible since it contains a minor isomorphic to $F_4(10, 16)$ (see Figure 3(c)).

Finally, one can see that $C(12, \{1, 4\}) - \{v_1v_5, v_7v_8\}$ contains a minor isomorphic to $F_5(10, 16)$ (see Figure 5) in the list of the minimal forbidden subgraphs for the projective plane (see Appendix A in [15]).

Therefore, (2) is true for $k = 4$. Now suppose that (2) is true for all values less than $k \geq 5$. Let D be a drawing of $C(k)$ in the projective plane and we are going to show that $v(D) \geq k - 1$.

If there exists $1 \leq i \leq 3k$ such that v_iv_{k+i} crosses, then by deleting v_iv_{k+i}, $v_{k+i}v_{2k+i}$, $v_{2k+i}v_i$, we obtain a drawing of a subdivision of $C(k-1)$, denote it by D'. By our construction, $v(D') \leq v(D) - 1$. On the other hand, $v(D') \geq k - 2$ by induction assumption. This implies $v(D) \geq k - 1$. Therefore, we may assume that v_iv_{k+i} is clean in D for all $1 \leq i \leq 3k$, i.e., C_i is clean for all $1 \leq i \leq k$. Then by Lemma 7, we have $v(D) \geq k - 1$. ■

***Proof of Theorem* 1.** It follows from Lemma 3, 4, 5 and 8. ■

4. PROOF OF LEMMA 7

This section is devoted to proving Lemma 7. Throughout this section, we assume that C_i is clean for $1 \leq i \leq k$, as we have assumed in Lemma 7.

For $1 \leq i \leq k$, let

$$F_i = \{v_i v_{k+i}, v_i v_{2k+i}, v_{k+i} v_{2k+i}, v_i v_{i+1}, v_{k+i} v_{k+i+1}, v_{2k+i} v_{2k+i+1}\}.$$

Note that the set of all F_i is a partition of the edge set E of $C(k)$, i.e.,

$$E = \bigcup_{i=1}^{k} F_i \text{ and } F_i \cap F_j = \emptyset \text{ for } i \neq j. \tag{3}$$

For $1 \leq i \leq k$, define

$$f_D(F_i) = v_D(F_i) + \frac{1}{2} \sum_{j \neq i} v_D(F_i, F_j). \tag{4}$$

Since we have assumed that each C_i is clean, there are only two possible ways of drawing C_i, depending on whether it is contractible or not, which are shown in Figure 6(a) and 6(b).

If C_i and C_{i+1} are both contractible, there are three possible ways of drawing $C_i \cup C_{i+1}$ for each i, which are shown in Figure 7(a), 7(b) and 7(c).

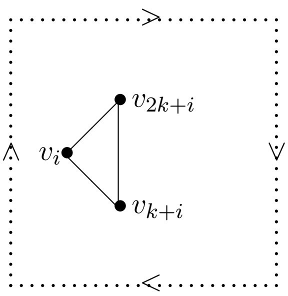

Figure 6(a). C_i is contractible.

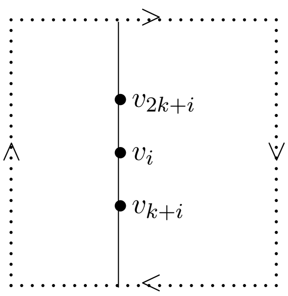

Figure 6(b). C_i is non-contractible.

We have the following:

Proposition 9. *If C_i and C_{i+1} are drawn as in Figure 7(a) or 7(b), then* $f_D(F_i) \geq 1$.

Proof. Suppose $f_D(F_i) < 1$. By (4), $v_i v_{i+1}, v_{k+i} v_{k+i+1}, v_{2k+i} v_{2k+i+1}$ do not cross each other. If $C_i \cup C_{i+1}$ is drawn as in Figure 7(a), $F_i \cup C_{i+1}$ must be drawn as in Figure 8 since C_i, C_{i+1} are clean and $v_i v_{i+1}$, $v_{k+i} v_{k+i+1}$, $v_{2k+i} v_{2k+i+1}$ do not cross each other. Since C_{i-1} is clean, C_{i-1} must lies entirely in one of the regions f_1, f_2 or f_3. We may assume that C_{i-1} lies in the region f_1, for other cases the proof is the same. If C_{i-1} lies in f_1, then $v_{k+i-1} v_{k+i}$ must cross $v_i v_{i+1}$ or $v_{2k+i} v_{2k+i+1}$. On the other hand, the path $v_{k+i+1} v_{k+i+2} \cdots v_{2k-i-1}$ must cross $v_i v_{i+1}$ or $v_{2k+i} v_{2k+i+1}$. Hence, by (4), $f_D(F_i) \geq 1$. Similarly, one can show that $f_D(F_i) \geq 1$ if $C_i \cup C_{i+1}$ is drawn as in Figure 7(b). ■

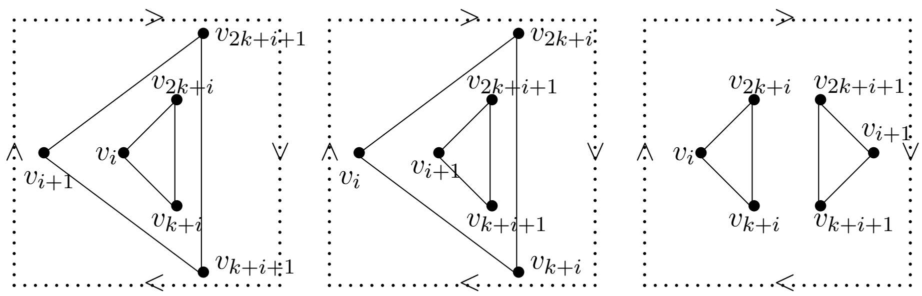

Figure 7(a) Figure 7(b) Figure 7(c)

Proposition 10. *If $C_i \cup C_{i+1}$ is drawn as in Figure 7(c) and $f_D(F_i) < 1$, then $F_i \cup C_{i+1}$ must be drawn as in Figure 9(b).*

Proof. Since $f_D(F_i) < 1$, by (4), $v_{k+i}v_{k+i+1}, v_{2k+i}v_{2k+i+1}$ do not cross each other. Then $F_i \cup C_{i+1}$ must be drawn as in Figure 9(a) or 9(b). If $F_i \cup C_{i+1}$ is drawn as in Figure 9(a), then C_{i-1} must lie entirely in one of the regions f_1, f_2 or f_3 since C_{i-1} is clean. We may assume that C_{i-1} lies in the region f_1, for other cases the proof is the same. If C_{i-1} lies in f_1, then $v_{i-1}v_i$ must cross $v_{k+i}v_{k+i+1}$ or $v_{2k+i}v_{2k+i+1}$ since C_i and C_{i+1} are clean. On the other hand, the path $v_{i+1}v_{i+2}\cdots v_{k-i-1}$ must cross F_i. Hence, by (4), we have $f_D(F_i) \geq 1$, which contradicts that $f_D(F_i) < 1$. ■

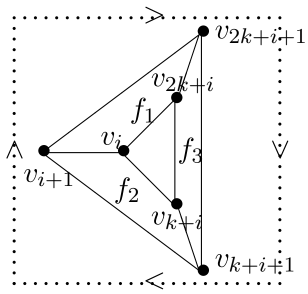

Figure 8

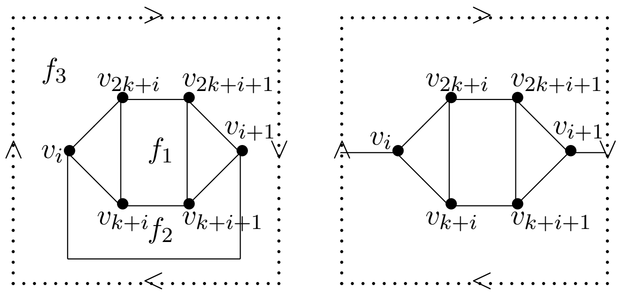

Figure 9(a) Figure 9(b)

Combining Proposition 9 and 10, we have the following:

Corollary 11. *If $F_i \cup C_{i+1}$ is not drawn as in Figure 9(b), then $f_D(F_i) \geq 1$.*

Proof. By Proposition 10, either $f_D(F_i) \geq 1$ or $C_i \cup C_{i+1}$ is not drawn as in Figure 7(c). In the latter case, $C_i \cup C_{i+1}$ must be drawn as in Figure 7(a) or 7(b). By Proposition 9, again we have $f_D(F_i) \geq 1$. ■

Remark 12. Hereafter, we say that $F_j \cup C_{j+1}$ is drawn as in Figure 9(b) if it is drawn as in Figure 9(c), i.e., replacing all the indices i by j.

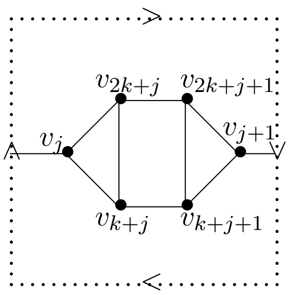

Figure 9(c)

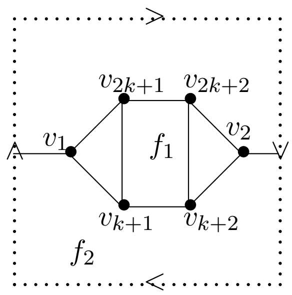

Figure 9(d)

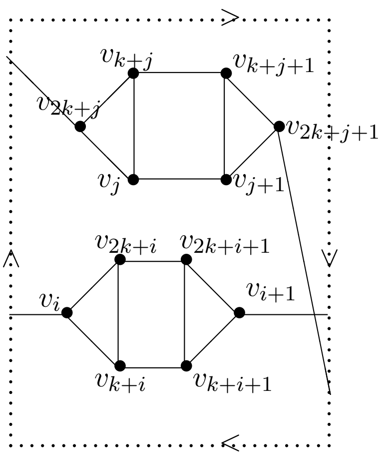

Figure 10. $F_i \cup C_{i+1} \cup F_j \cup C_{j+1}$.

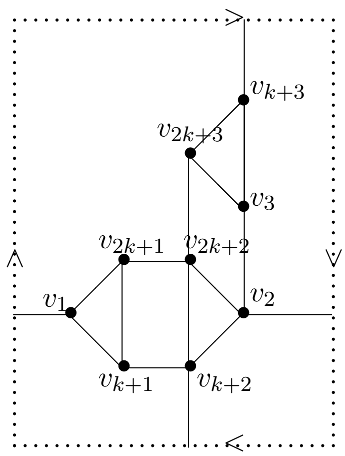

Figure 11. $F_1 \cup F_2 \cup C_3$.

Proposition 13. *Suppose that $F_i \cup C_{i+1}$ is drawn as in Figure* 9(b)*. If $j \neq i-1, i, i+1$ such that $F_j \cup C_{j+1}$ is drawn as in Figure* 9(b)*, then F_i and F_j must cross each other. In particular, we have $f_D(F_i) \geq 1/2$ and $f_D(F_j) \geq 1/2$.*

Proof. Note that two non-contractible curves in the projective plane must cross each other. Since $F_i \cup C_{i+1}$ and $F_j \cup C_{j+1}$ are drawn as in Figure 9(b) where $j \neq i-1, i+1$, F_i and F_j must cross each other since $C_i, C_{i+1}, C_j, C_{j+1}$ are clean. See Figure 10 for a possible drawing of $F_i \cup C_{i+1} \cup F_j \cup C_{j+1}$. Since F_i and F_j cross each other, we have $v_D(F_i, F_j) \geq 1$, which implies that $f_D(F_i) \geq 1/2$ and $f_D(F_j) \geq 1/2$ by (4). ∎

Here is the outline of the proof of Lemma 7. We will consider two cases:

Case 1. C_i is contractible for all $1 \leq i \leq k$.

Case 2. C_i is non-contractible for some $1 \leq i \leq k$.

For Case 1, by simple arguments, we can show that $F_1 \cup C_2$ is drawn as in Figure 9(b). Moreover, we can show that $f_D(F_{i_0}) < 1$ for some $i_0 \neq 1$. Then we will consider two cases:

Case 1.1. $i_0 \neq 2, k$.

Case 1.2. $i_0 = 2$ or k.

Case 1.1 can be solved easily. For Case 1.2, we will assume that $i_0 = 2$ since the proof for $i_0 = k$ is the same. Then we will consider two cases:

Case 1.2.1. $f_D(F_j) \geq 1$ for all $j \neq 1, 2$.

Case 1.2.2. $f_D(F_j) < 1$ for some $j \neq 1, 2$.

For Case 1.2.1, by assumption, $f_D(F_j) \geq 1$ for all $j \neq 1, 2$. We just need to show that $f_D(F_1) + f_D(F_2) > 0$, which implies that $v(D) = \sum_{j=1}^{k} f_D(F_j) = f_D(F_1) + f_D(F_2) + \sum_{j \neq 1,2} f_D(F_j) > k - 2$, and hence $v(D) \geq k - 1$ since $v(D)$ is an integer. For Case 1.2.2, by assumption, $f_D(F_j) < 1$ for some $j \neq 1, 2$. Then we will consider two cases:

Case 1.2.2.1. $j \neq 3$, k.

Case 1.2.2.2. $j = 3$ or k.

Case 1.2.2.1 can be solved easily.
For Case 1.2.2.2, we can assume that

$$f_D(F_l) \geq 1 \text{ for } l \neq 1, 2, 3, k. \tag{5}$$

Otherwise, if $f_D(F_l) < 1$ for some $l \neq 1, 2, 3, k$, then it can be reduces to Case 1.2.2.1 by taking $j = l$. By simple arguments, we can reduced it to the case when both $F_3 \cup C_4$ and $F_k \cup C_1$ are drawn as in Figure 9(b). That is to say, $F_i \cup C_{i+1}$ is drawn as in Figure 9(b) for $i = 1, 2, 3, k$. Then by Proposition 13, F_1 crosses F_3 and F_2 crosses F_k. Moreover, if $k \geq 5$, then F_1 also crosses F_k. All these implies

$$f_D(F_1) \geq 1, f_D(F_k) \geq 1, f_D(F_2) \geq 1/2, \text{ and } f_D(F_3) \geq 1/2. \tag{6}$$

Combining (5) and (6), we get $v(D) \geq k - 1$. For $k = 4$, we will use different arguments by making use the fact that $F_i \cup C_{i+1}$ is drawn as in Figure 9(b) for $i = 1, 2, 3, 4$.

Now we are ready to prove Lemma 7.

Proof of Lemma 7. By (1), (3) and (4), the total number of crossing of the drawing D is $v(D) = v_D(E) = \sum_{i=1}^{k} f_D(F_i)$. Therefore, it suffices to prove that $\sum_{i=1}^{k} f_D(F_i) \geq k-1$. To prove by contradiction, we assume that

$$\sum_{i=1}^{k} f_D(F_i) < k-1. \tag{7}$$

We will consider two cases: Case 1. C_i is contractible for all $1 \leq i \leq k$ and Case 2. C_i is non-contractible for some $1 \leq i \leq k$.

Case 1. Since we have assumed that C_i is clean for $1 \leq i \leq k$, as we have said at the beginning of this section, there are three possible ways of drawing $C_i \cup C_{i+1}$ for each i, which are shown in Figure 7(a), 7(b) or 7(c).

Note that (7) implies that $f_D(F_i) < 1$ for some i. Without loss of generality, we may assume $i = 1$, i.e.,

$$f_D(F_1) < 1. \tag{8}$$

By Proposition 9, $C_1 \cup C_2$ must be drawn as in Figure 7(c). Hence, by (8) and Proposition 10, $F_1 \cup C_2$ is drawn as in Figure 9(b) (see Figure 9(d)).

There exists $i_0 \neq 1$ such that $F_{i_0} \cup C_{i_0+1}$ is drawn as in Figure 9(b). (Otherwise, if $F_j \cup C_{j+1}$ is not drawn as in Figure 9(b) for all $j \neq 1$, $f_D(F_j) \geq 1$ for all $j \neq 1$ by Corollary 11, which implies $\sum_{j=1}^{k} f_D(F_j) \geq \sum_{j\neq 1} f_D(F_j) \geq k-1$.) We will consider two cases: Case 1.1. $i_0 \neq 2, k$ and Case 1.2. $i_0 = 2$ or k.

Case 1.1. If $i_0 \neq 2, k$, i.e., $C_{i_0} \cup C_{i_0+1}$ is drawn as in Figure 9(b) for some $i_0 \neq 1, 2, k$, then by Proposition 13, F_1 and F_{i_0} cross each others,

$$f_D(F_1) \geq 1/2 \text{ and } f_D(F_{i_0}) \geq 1/2. \tag{9}$$

Moreover, if there exists $j \neq 1, 2, i_0, k$ such that $f_D(F_j) < 1$, then $F_j \cup C_{j+1}$ must be drawn as in Figure 9(b) by Proposition 10. By Proposition 13, F_j and F_1 must also cross each other. Hence, $f_D(F_1) \geq 1$ since F_1 crosses both F_{i_0} and F_j, which contradicts (8). Therefore,

$$f_D(F_j) \geq 1 \text{ for all } j \neq 1, 2, i_0, k. \tag{10}$$

Moreover, we can assume that

$$f_D(F_2) \geq 1 \text{ and } f_D(F_k) \geq 1. \tag{11}$$

(Otherwise, $f_D(F_2) < 1$ or $f_D(F_k) < 1$ implies that $F_2 \cup C_3$ or $F_k \cup C_1$ is drawn as in Figure 9(b) by Proposition 10. Replacing i_0 by 2 or k, one can reduce this to Case 1.2.) Combining (9), (10) and (11), we have $\sum_{j=1}^{k} f_D(F_j) \geq f_D(F_1) + f_D(F_{i_0}) + \sum_{j\neq 1, i_0} f_D(F_j) \geq k-1$.

Case 1.2. If $i_0 = 2$ or k, then we may assume that $i_0 = 2$ since the proof for $i_0 = k$ is the same. Then $F_2 \cup C_3$ is drawn as in Figure 9(b). We will consider two cases: Case 1.2.1. $f_D(F_j) \geq 1$ for all $j \neq 1, 2$ and Case 1.2.2. $f_D(F_j) < 1$ for some $j \neq 1, 2$.

Case 1.2.1. By assumption,

$$f_D(F_j) \geq 1 \text{ for all } j \neq 1, 2. \tag{12}$$

If we can show that

$$f_D(F_1) + f_D(F_2) > 0, \tag{13}$$

then by (12) and (13),
$v(D) = \sum_{j=1}^{k} f_D(F_j) = f_D(F_1) + f_D(F_2) + \sum_{j \neq 1,2} f_D(F_j) > k - 2$, which implies that $v(D) \geq k - 1$ since the total number of crossing $v(D)$ is an integer.

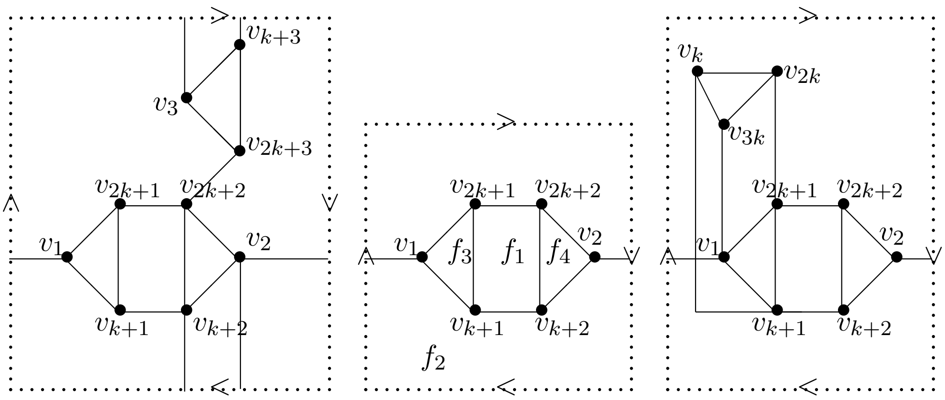

Figure 12 Figure 13 Figure 14

Suppose (13) is not true, i.e.,

$$f_D(F_1) = f_D(F_2) = 0. \tag{14}$$

Recall that $F_1 \cup C_2$ is drawn as in Figure 9(d). Since C_3 is clean, C_3 must lie entirely in regions f_1 or f_2 in Figure 9(d). If C_3 lies in f_1, then v_2v_3 must cross $v_{k+1}v_{k+2}$ or $v_{2k+1}v_{2k+2}$. By (4), $f_D(F_2) \geq 1/2$, which contradicts (14). Therefore, C_3 lies in f_2. By (4) and (14), v_2v_3, $v_{k+2}v_{k+3}$, $v_{2k+2}v_{2k+3}$ are clean. Then the only possible drawing of $F_1 \cup F_2 \cup C_3$ is shown as in Figure 11. (It is true up to renaming the vertices. For example, it is possible for $F_1 \cup F_2 \cup C_3$ to be drawn as in Figure 12. But one can reduce it to Figure 11 by the transformation $v_j \mapsto v_{j-k}$.)

Since C_4 is clean, it must lie entirely in one of the regions in Figure 11. Note that v_3, v_{k+3} and v_{2k+3} do not lie in the the same region in Figure 11. No matter which region C_4 lies in Figure 11, one of the edges v_3v_4, $v_{k+3}v_{k+4}$ and $v_{2k+3}v_{2k+4}$ must cross the F_1 or F_2 (Note that $k \geq 4$ is crucial here for C_4 being not equal to C_1). Hence, $f_D(F_1) + f_D(F_2) > 0$ which gives (13).

Case 1.2.2. If $f_D(F_j) < 1$ for some $j \neq 1, 2$, then $F_j \cup C_{j+1}$ must be drawn as in Figure 9(b) by Proposition 10. We will consider two cases: Case 1.2.2.1. $j \neq 3$, k and Case 1.2.2.2. $j = 3$ or k.

Case 1.2.2.1. Since $F_j \cup C_{j+1}$ is drawn as in Figure 9(b) where $j \neq 1, 2, 3, k$, F_j must cross F_1 and F_2 by Proposition 13, since $F_1 \cup C_2$ and $F_2 \cup C_3$ are drawn as in Figure 9(b). This implies that, by (4),

$$f_D(F_1) \geq 1/2, f_D(F_2) \geq 1/2, \text{ and } f_D(F_j) \geq 1. \tag{15}$$

Note that

$$f_D(F_r) \geq 1 \text{ for all } r \neq 1, 2, 3, j, k. \tag{16}$$

Otherwise, if $f_D(F_r) < 1$ for some $r \neq 1, 2, 3, j, k$, then by Proposition 10, $F_r \cup C_{r+1}$ is drawn as in Figure 9(b). By Proposition 13, F_r also crosses F_1. This implies $f_D(F_1) \geq 1$ since F_1 cross F_j and F_r, which contradicts (8).

We claim that

$$f_D(F_3) \geq 1 \text{ and } f_D(F_k) \geq 1. \tag{17}$$

To see this, suppose that $f_D(F_3) < 1$. Then $F_3 \cup C_4$ is drawn as in Figure 9(b) by Proposition 10. Hence F_1 must cross F_3 and F_j by Proposition 13, which implies that $f_D(F_1) \geq 1$ and contradicts (8). On the other hand, if $f_D(F_k) < 1$, then $F_k \cup C_1$ must be drawn as in Figure 9(b) by Proposition 10. Hence F_2 must cross F_k and F_j by Proposition 13, which implies that $f_D(F_2) \geq 1$ and contradicts (8). This proves (17).

Combining (15), (16) and (17), we get $\sum_{r=1}^{k} f_D(F_r) = f_D(F_1) + f_D(F_2) + \sum_{r \neq 1,2} f_D(F_r) \geq k - 1$.

Case 1.2.2.2. If $j = 3$ or k, then $F_k \cup C_1$ or $F_3 \cup C_4$ is drawn as in Figure 9(b). We may assume that

$$f_D(F_l) \geq 1 \text{ for } l \neq 1, 2, 3, k. \tag{18}$$

(Otherwise, if $f_D(F_l) < 1$ for some $l \neq 1, 2, 3, k$, then it can be reduces to Case 1.2.2.1 by taking $j = l$.) It can be reduced to the case when both $F_3 \cup C_4$ and $F_k \cup C_1$ are drawn as in Figure 9(b).

To see this, suppose that $F_3 \cup C_4$ is drawn as in Figure 9(b) and $F_k \cup C_1$ is not. Then by Corollary 11

$$f_D(F_k) \geq 1, \tag{19}$$

and F_3 must cross F_1 by Proposition 13 since $F_1 \cup C_2$ is drawn as in Figure 9(b). We claim that F_1 must cross F_k. Assuming the claim, we have

$$f_D(F_1) \geq 1 \text{ and } f_D(F_3) \geq 1/2. \tag{20}$$

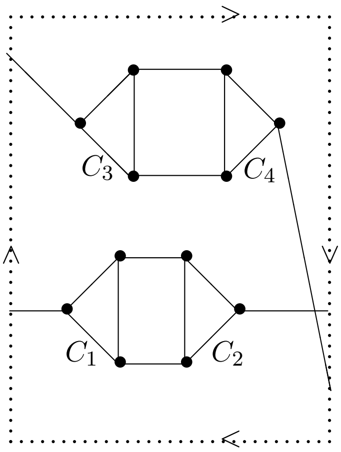

Figure 15(a)

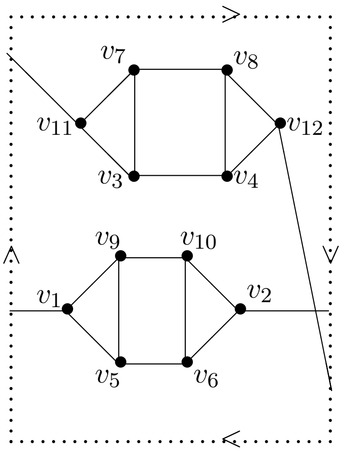

Figure 15(b)

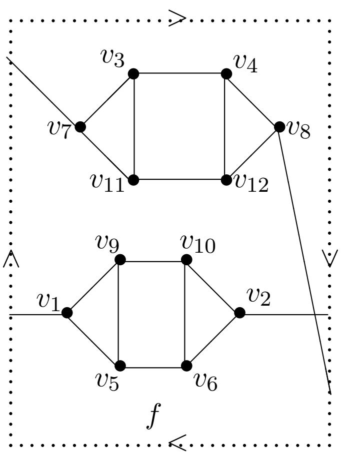

Figure 15(c)

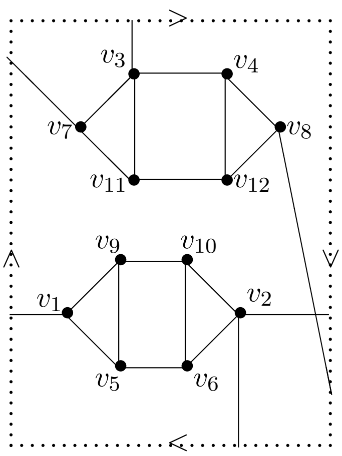

Figure 15(d)

Combining (18), (19) and (20), we get $\sum_{r=1}^{k} f_D(F_r) > k - 2$, which implies that $v(D) = \sum_{i=1}^{k} f_D(F_i) \geq k - 1$ since $v(D)$ is an integer.

To show the claim, i.e., F_1 crosses F_k, we note that $F_1 \cup C_2$ is drawn as in Figure 9(b). See Figure 13. Since C_k is clean, it must lie entirely in one of the regions in Figure 13. It is impossible for C_k to lie in f_3, otherwise, the path $v_2v_3 \cdots v_k$ crosses C_1. It is also impossible for C_{i-1} to lie in f_4, otherwise, v_kv_{k+1} crosses C_2. If C_k lies in f_1, $v_{3k}v_1$ must cross with $v_{k+1}v_{k+2}$ or $v_{2k+1}v_{2k+2,}$ which implies that F_k crosses F_1. If C_k lies in f_2, then F_k must cross F_1 since $F_k \cup C_1$ is not drawn as in Figure 9(b) by our assumption (See Figure 14 for example). Therefore, F_1 must cross F_k, as we claimed.

Similarly, if $F_k \cup C_1$ is drawn as in Figure 9(b) and $F_3 \cup C_4$ is not, then $\sum_{r=1}^{k} f_D(F_r) \geq k-1$.

Therefore, we can assume that both $F_3 \cup C_4$ and $F_k \cup C_1$ are drawn as in Figure 9(b). Then F_k must cross F_2, and F_1 must cross with F_3 by Proposition 13. Moreover, if $k \geq 5$, then F_3 and F_k must also cross each other by Proposition 13. All these imply that

$$f_D(F_1) \geq 1/2, f_D(F_2) \geq 1/2, f_D(F_3) \geq 1, \text{ and } f_D(F_k) \geq 1. \tag{21}$$

Combining (18) and (21), we infer $\sum_{r=1}^{k} f_D(F_r) \geq k-1$ if $k \geq 5$.

On the other hand, if $k = 4$, then $F_k \cup C_1 = F_4 \cup C_1$, $F_1 \cup C_2$, $F_2 \cup C_3$ and $F_3 \cup C_4$ are drawn as in Figure 9(b) by assumptions. By Proposition 13, F_1 must cross F_3, and F_2 must cross F_4. This implies that

$$f_D(F_i) \geq 1/2 \text{ for } 1 \leq i \leq 4. \tag{22}$$

We will show that $v(D) \geq 3$. By contradiction, suppose that $v(D) \leq 2$. By (1) and (22), we have

$$f_D(F_1) = f_D(F_2) = f_D(F_3) = f_D(F_4) = 1/2. \tag{23}$$

Since F_1 crosses F_3, by (4) and (23) we get

$$v_D(F_1, F_3) = 1, v_D(F_1, F_j) = 0 \text{ for } j \neq 3, v_D(F_3, F_j) = 0 \text{ for } j \neq 1. \tag{24}$$

Similarly, since F_2 crosses F_4, by (4) and (23) we get

$$v_D(F_2, F_4) = 1, v_D(F_2, F_j) = 0 \text{ for } j \neq 4, v_D(F_4, F_j) = 0 \text{ for } j \neq 2. \tag{25}$$

Since $F_1 \cup C_2$ and $F_3 \cup C_4$ are drawn as in Figure 9(b), the only possible drawing of $F_1 \cup C_2 \cup F_3 \cup C_4$ is shown in Figure 15(a) in view of (24) and (25). However, one can show that it is impossible for (24), (25) to hold. For example, if $F_1 \cup C_2 \cup F_3 \cup C_4$ is drawn in Figure 15(b), then the edge v_8v_9 must cross with F_1 or F_3, which contradicts (24); and if $F_1 \cup C_2 \cup F_3 \cup C_4$ is drawn in Figure 15(c), then the edge v_2v_3 must lie entirely in the region f, as in Figure 15(d), since $v_D(F_2, F_j) = 0$ for $j \neq 4$ by (25). However, in Figure 15(d), no matter how v_6v_7

is drawn, v_6v_7 must either (i) cross v_2v_3 which contradicts (25), or (ii) cross C_i which contradicts that C_i are all clean, or (iii) cross F_1 or F_3 which contradicts (25). We leave other cases to the reader.

Case 2. If there exists $1 \leq i \leq k$ such that C_i is non-contractible, then we may assume that C_1 is non-contractible. Then C_i is contractible for all $i \neq 1$. (Otherwise, C_i crosses C_1 since two non-contractible curves in the projective plane must cross each other. This contradicts the assumption that all C_i are clean.) Since C_i and C_{i+1} are clean and contractible for $i \neq 1, k$, there are three possible ways of drawing $C_i \cup C_{i+1}$, which are shown in Figure 7(a), 7(b) or 7(c).

We claim that

$$f_D(F_i) \geq 1 \text{ for } i \neq 1, k. \tag{26}$$

To prove this, suppose that $f_D(F_i) < 1$ for some $i \neq 1, k$. By Corollary 11, $F_i \cup C_{i+1}$ must be drawn as in Figure 9(b), which crosses the non-contractible C_1. This contradicts that C_1 is clean. This proves (26).

Now we are going to show that

$$f_D(F_1) + f_D(F_k) > 0. \tag{27}$$

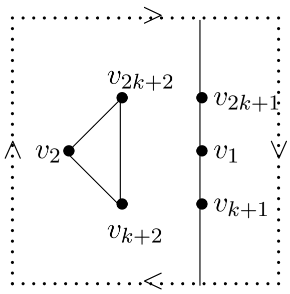

Figure 16. $C_1 \cup C_2$.

Combining this with (26), we will get $\sum_{r=1}^{k} f_D(F_r) > k-2$, which gives $v(D) = \sum_{i=1}^{k} f_D(F_i) \geq k-1$ since $v(D)$ is an integer. Suppose that (27) is not true, i.e.,

$$f_D(F_1) = f_D(F_k) = 0. \tag{28}$$

Since C_1 is non-contractile and C_2 is contractible, $C_1 \cup C_2$ must be drawn as in Figure 16. On the other hand, by the same reasons, $C_1 \cup C_k$ must be drawn as in Figure 16 by replacing C_2 by C_k.

By (4) and (28), v_1v_2, $v_{k+1}v_{k+2}$, $v_{2k+1}v_{2k+2}$ do not cross. From Figure 16, one can see that there are three possible ways of drawing $F_1 \cup C_2$, which are shown in Figure 17(a), 17(b) and 17(c).

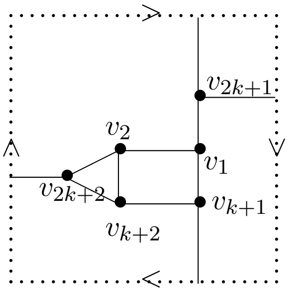

Figure 17(a)

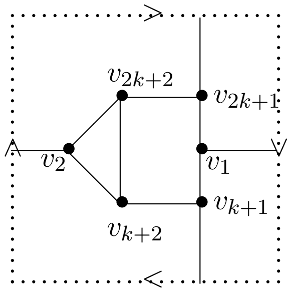

Figure 17(b)

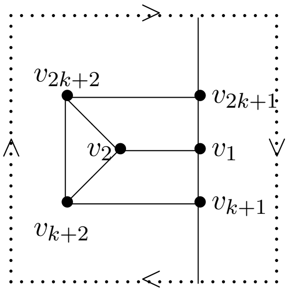

Figure 17(c)

If $F_1 \cup C_2$ is drawn as in Figure 17(b) and 17(c), then C_3 must lie entirely in one of the regions since C_3 is clean. Then F_2 must cross with F_1 since there is no region in Figure 17(b) or 17(c) containing all of the vertices v_2, v_{k+2} and v_{2k+2}. This implies $f_D(F_1) > 0$, which contradicts (28).

Therefore, $F_1 \cup C_2$ must be drawn as in Figure 17(a). By the same argument, $F_k \cup C_1$ must be drawn as in Figure 17(a) by replacing C_2 by C_k. Hence, $F_k \cup F_1 \cup C_2$ must be drawn as in Figure 18(a) or 18(b) since F_1 does not cross F_k by (28).

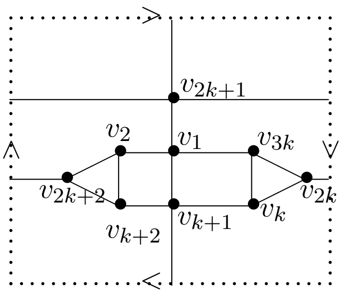

Figure 18(a)

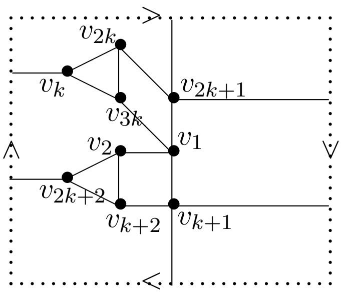

Figure 18(b)

Note that C_3 must lie in one of the regions in Figure 18(a) or 18(b). Since there exists no region in Figure 18(a) or 18(b) which contains all of the vertices v_2, v_{k+2} and v_{2k+2}, F_3 must cross either F_k or F_1 ($k \geq 4$ is needed here for F_3 being not equal to F_k). This implies that $f_D(F_1) > 0$ or $f_D(F_k) > 0$, which gives (27).

This finishes the the proof of Lemma 7. ∎

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Received 2 September 2010
Revised 26 January 2011
Accepted 26 January 2011