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RECOGNIZABLE COLORINGS OF CYCLES AND TREES

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Abstract

For a graph G and a vertex-coloring $c : V(G) \to \{1, 2, \ldots, k\}$, the color code of a vertex v is the $(k+1)$-tuple $(a_0, a_1, \ldots, a_k)$, where $a_0 = c(v)$, and for $1 \leq i \leq k$, a_i is the number of neighbors of v colored i. A recognizable coloring is a coloring such that distinct vertices have distinct color codes. The recognition number of a graph is the minimum k for which G has a recognizable k-coloring. In this paper we prove three conjectures of Chartrand *et al.* in [8] regarding the recognition number of cycles and trees.

Keywords: recognizable coloring, recognition number.

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1. Introduction

Distinguishing the vertices of a graph G by means of a coloring is a topic that has received much attention in the literature. Typically, the edges of G are colored and the vertices are distinguished based on the coloring of their incident edges. For example, in [9], given an edge-coloring of G, two vertices of G are distinguished if the sets of colors assigned to their incident edges are different.

Another example is that of *irregular* edge-colorings, where two vertices are distinguished if, for some color k, they are incident with different numbers of edges colored k. Irregular colorings were studied in [1, 2, 3, 4] and [5]. Another way is to distinguish vertices according to the sum of the colors of their incident edges. See [6].

In [8], a new method of distinguishing the vertices of a graph G was introduced. This method involves coloring vertices rather than edges, and combines a number of the features of the various previous methods.

For a graph G and a (not necessarily proper) vertex coloring $c : V(G) \to \{1, 2, \ldots, k\}$, the color code of a vertex v is the $(k+1)$-tuple $(a_0, a_1, \ldots, a_k)$, where $a_0 = c(v)$, and for $1 \le i \le k$, a_i is the number of neighbors of v colored i. A *recognizable coloring* is a coloring such that distinct vertices have distinct color codes. The *recognition number* of a graph G, denoted by $\mathrm{rn}(G)$, is the minimum k for which G has a recognizable k-coloring. Such a coloring is called a *minimum recognizable coloring.*

Since every coloring that assigns distinct colors to the vertices of a connected graph is recognizable, the recognition number is always defined.

In this paper we will study the recognition number of cycles, paths and trees and prove three results conjectured in [8] about these three classes of graphs.

For graph-theoretical notation or terminology not defined in this paper we refer the reader to [7].

2. Cycles

The following observation from [8], which follows easily by standard counting methods, will be useful.

Observation 1. *The number of distinct color codes on k colors for vertices of degree r is $k\binom{k+r-1}{r}$. In particular, for vertices of degree 2, there are $(k^3+k^2)/2$ distinct color codes on k colors.*

Our main result is the following theorem, which was conjectured in [8]. It is also used to prove the other two conjectures regarding paths and trees in [8].

Theorem 2. *Let $k \ge 3$ be an integer. Then $\mathrm{rn}(C_n) = k$ for all integers n such that*

$$\begin{aligned} \tfrac{(k-1)^3+(k-1)^2-2(k-1)+2}{2} \quad &\le \quad n \quad \le \quad \tfrac{k^3+k^2}{2} \qquad \textit{if } k \textit{ is odd,} \\ \tfrac{(k-1)^3+(k-1)^2+2}{2} \quad &\le \quad n \quad \le \quad \tfrac{k^3+k^2-2k}{2} \qquad \textit{if } k \textit{ is even.} \end{aligned}$$

It is interesting to note that the monotonicity of $\mathrm{rn}(C_n)$ (i.e., for any integers n_1, n_2, if $n_1 \le n_2$, then $\mathrm{rn}(C_{n_1}) \le \mathrm{rn}C(n_2)$) follows immediately from Theorem 2.

However, this is surprisingly difficult to prove directly. The situation with paths is different. There we can prove monotonicity, and the full result then follows from extreme cases. See Section 3.

It will be convenient to use a simpler notation for color codes of vertices of degree two: If a vertex v of degree two has color a and its neighbors are colored b and c, (where possibly $b = c$) we will also denote the color code of v by $(a; b, c)$. Note that $(a; b, c)$ and $(a; c, b)$ are equal.

That $\mathrm{rn}(C_n) \geq k$ if n is as in the statement of Theorem 2 is proved in [8]. When k is even, this follows immediately from Observation 1 applied to $k-1$ colors. For k odd, this follows from the fact that on a cycle, codes of the form $(a; a, b)$ with $a \neq b$ occur in pairs, so at least $k-1$ codes cannot occur in a recognizable $k-1$ coloring of C_n if $k-1$ is even. See [8] for details.

Hence, to prove Theorem 2, we must show that $\mathrm{rn}(C_n) \leq k$ if $n \leq (k^3+k^2)/2$ and k is odd, or $n \leq (k^3 + k^2 - 2k)/2$ and k is even. In [8], the authors construct a recognizable 5-coloring of C_{75} by finding an appropriate Eulerian subdigraph of a de Bruijn digraph. This is the approach we will use.

Consider the de Bruijn digraph D_k where $V(D_k) = \{(a, b) : 1 \leq a, b \leq k\}$ and $E(D_k) = \{(a, b)(b, c) : 1 \leq a, b, c \leq k\}$. Any circuit C in D_k, as a sequence of vertices, has the form $(a_1, a_2), (a_2, a_3), \ldots, (a_m, a_1)$. Such a circuit corresponds to a k-coloring g of C_m by setting $g(x) = a_x$, where we take $V(C_m)$ to be $\{1, 2, \ldots, m\}$. Each arc $e = (a, b)(b, c)$ of C corresponds to a vertex of C_m with code $(b; a, c)$. The only way an arc $f \neq e$ can correspond to a vertex with the same code is if $f = (c, b)(b, a)$. Therefore, if for all a, b and c, at most one element of $\{(a, b)(b, c), (c, b)(b, a)\}$ is on C, g is a recognizable k-coloring of C_m. (Note that $\{(a, b)(b, c), (c, b)(b, a)\}$ can be a singleton set.)

To show that $\mathrm{rn}(C_n) \leq k$ for some n and k, it suffices then to find a subdigraph G of D_k such that G contains a circuit of length n and at most one element of $\{(a, b)(b, c), (c, b)(b, a)\}$, for all a, b and c. To this end, we make the following definitions:

For three distinct integers a, b and c, we can consider the triple (a, b, c) to be a permutation of (d, e, f), where (d, e, f) is (a, b, c) in increasing order. We call (a, b, c) *even* (*odd*) if it is an even (odd) permutation of (d, e, f). Note that the even permutations of (a, b, c) are the cycles (a, b, c), (c, a, b) and (b, c, a).
For $k \geq 3$ we define the directed graphs H'_k and G'_k as follows:

$$\begin{aligned}
V(H'_k) &= V(G'_k) = \{(a, b) : 1 \leq a, b \leq k\}, \\
E(H'_k) &= \{(a, a)(a, b) : a < b \text{ and } b - a \text{ is odd }\} \cup \{(a, a)(a, b) : a > b \text{ and } a - b \\
&\quad \text{is even }\} \cup \{(b, a)(a, a) : b > a \text{ and } b - a \text{ is even }\} \cup \{(b, a)(a, a) : a > b \\
&\quad \text{and } a - b \text{ is odd }\}, \\
E(G'_k) &= E(H'_k) \cup E_k, \text{ where } [E_k =] \ \{(a, b)(b, c) : a, b, c \text{ distinct and } (a, b, c) \text{ is} \\
&\quad \text{even }\} \cup \{(a, b)(b, a) : a, b \leq k\}.
\end{aligned}$$

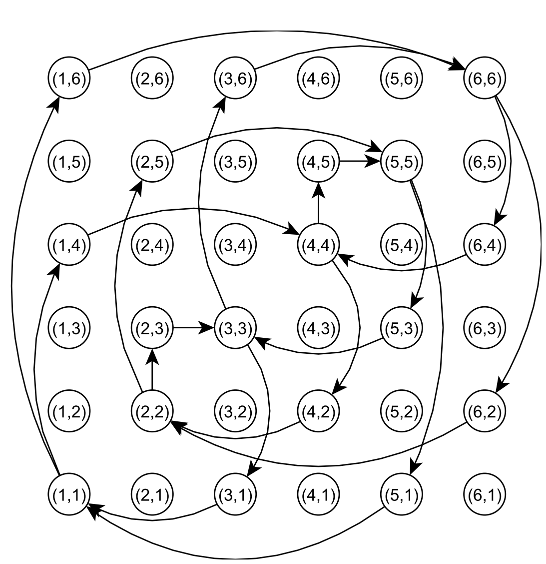


Figure 1. H_6.

If k is odd, we let $G_k = G'_k$ and $H_k = H'_k$. Otherwise, we let G_k and H_k be obtained from G'_k and H'_k, respectively, by removing the arcs $(1,1)(1,2)$, $(1,2)(2,2), (3,3)(3,4), (3,4)(4,4), \ldots, (k-1,k-1)(k-1,k), (k-1,k)(k,k)$. As examples, H_6 is depicted in Figure 1 and E_4 in Figure 2.

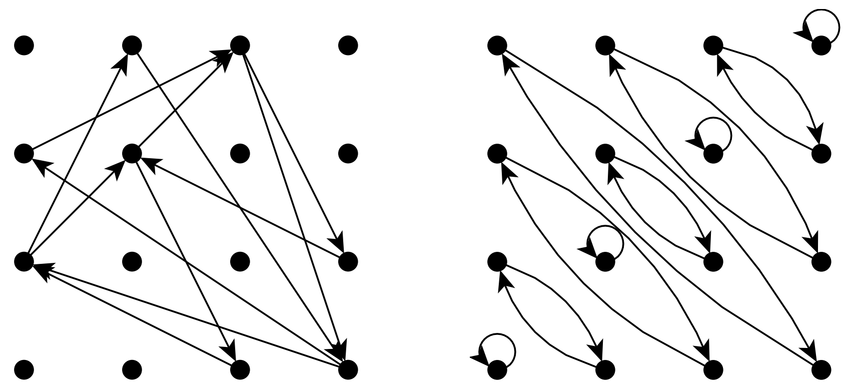
We will also write (a,b,c) for the arc $(a,b)(b,c)$. Vertices of the form (a,a) will be called *diagonal* vertices.

We now prove a few properties satisfied by the graphs G_k and H_k that are necessary for the proof of Theorem 2.

Lemma 3. *For all integers $k \geq 3$:*

1. *For all $a,b,c \leq k$, exactly one element of $\{(a,b,c),(c,b,a)\}$ is an arc of G'_k.*
2. *Every vertex of H_k has equal in- and out-degree.*
3. *G_k and $H_k[E(H_k)]$ (H_k minus its isolated vertices) are connected.*
4. *If k is odd, $|E(G_k)| = (k^3+k^2)/2$. Otherwise, $|E(G_k)| = (k^3+k^2-2k)/2$.*
5. *If k is odd, $|E(H_k)| = k(k-1)$. Otherwise, $|E(H_k)| = k(k-2)$.*
6. *For all distinct a, b and c, $(a,b,c) \in E_k$ if and only if $(c,a,b) \in E_k$ if and only if $(b,c,a) \in E_k$.*

Proof. 1. If a, b and c are distinct, (a,b,c) is even iff (c,b,a) is odd, and the result follows. All triples of the form (a,b,a) (where possibly $a = b$) are arcs, so the result is true if $a = c$. Otherwise we have $a = b$ and $b \neq c$, or $b = c$ and $a \neq b$. In either case we must show that for all $a \neq b$, exactly one of (a,a,b) and (b,a,a) is an arc. This is clear from the definition of $E(H'_k)$.



(a) Arcs of the form $(a,b)(b,c)$ with a,b,c distinct and (a,b,c) even.

(b) Arcs of the form $(a,b)(b,a)$.

Figure 2. E_4. Labels are omitted, but the vertex arrangement is similar to that of Figure 1.

2. Let $v = (a,b)$, with $a \neq b$. The only possible arcs incident with v are the in-edge (a,a,b) and the out-edge (a,b,b). Since $(a,a,b) \in E(H_k)$ if and only if $(a,b,b) \in E(H_k)$, either v is isolated in H_k or v has in- and out-degree one.

Next, suppose v is a diagonal vertex (a,a). If k is odd, v has in- and out-degree $(k-1)/2$ in H_k. If k and a are even, v has in-degree $k/2$ in H'_k and out-degree $k/2-1$ in H'_k, hence v has in- and out-degree $k/2-1$ in H_k. If k is even and a is odd, v has out-degree $k/2$ in H'_k and in-degree $k/2-1$ in H'_k, hence v has in- and out-degree $k/2-1$ in H_k.

3. For every $a < k$, (a,a) is connected to (k,k) in the underlying undirected graph of H_k via (a,k) or (k,a), unless k is even and $a = k-1$, in which case we have the path $(k,k)(k,k-2)(k-2,k-2)(k-2,k-1)(k-1,k-1)$. Since every arc of H_k is incident with a diagonal vertex, $H_k[E(H_k)]$ is connected.

For the connectedness of G_k, note that for all $a \neq b$, (a,b) or (b,a) is adjacent to a diagonal vertex, unless k is even and (a,b) has the form $(2m-1,2m)$. In this case we can still find a path from (a,b) to a diagonal vertex: If $m < k/2$, we have the path $(2m-1,2m)(2m,2m+1)(2m+1,2m+1)$. If $m = k/2$, we have $(k-2,k-2)(k-2,k-1)(k-1,k)$.

Now, for all $a \neq b$, one of (a,b) and (b,a) is connected to a diagonal vertex. Since $(a,b,a) \in E(G_k)$ for all a and b, and all diagonal vertices are connected to each other, G_k is connected.

4. From the discussion on de Bruijn graphs, there is a one-to-one correspondence between color codes for vertices of degree two and sets $\{(a,b,c),(c,b,a)\}$. From (1) it follows that G'_k has as many arcs as there are color codes for degree-2 vertices. By Observation 1 this equals $(k^3+k^2)/2$. If k is odd, $|E(G_k)| =$

$|E(G'_k)|$, otherwise exactly k arcs are removed from G'_k to obtain G_k, hence $|E(G_k)| = (k^3 + k^2 - 2k)/2$.

5. Every arc of H_k is incident with a diagonal vertex. No two diagonal vertices are adjacent, hence $|E(H_k)| = \sum_{a=1}^{k}[\mathrm{id}(a,a) + \mathrm{od}(a,a)]$. From the proof of (2), $\mathrm{id}(a,a) = \mathrm{od}(a,a) = (k-1)/2$ if k is odd, so $|E(H_k)| = k(k-1)$. If k is even, $\mathrm{id}(a,a) = \mathrm{od}(a,a) = (k-2)/2$, so $|E(H_k)| = k(k-2)$.

6. As noted before, the even permutations of (a, b, c) are the cyclic permutations. ■

***Proof of Theorem* 2.** We first show that, for all $k \geq 3$ and n such that $|E(H_k)| \leq n \leq |E(G_k)|$, G_k contains a circuit of length n. Such a circuit corresponds directly to a recognizable k-coloring of C_n.

Since $H_k \subseteq G_k$ and $H_k[E(H_k)]$ is Eulerian by Lemma 3, we have a circuit of length n for $n = |E(H_k)|$.

From (6) of the lemma it follows that the arcs of G_k not in H_k, i.e., E_k, can be partitioned into a set C of 3-cycles $\{(a,b,c),(b,c,a),(c,a,b)\}$, 2-cycles $\{(a,b,a),(b,a,b)\}$ and loops $\{(a,a,a)\}$. Since $H_k[E(H_k)]$ and G_k are connected, we can form a sequence $H_k[E(H_k)] = H^0 \subset H^1 \subset \cdots \subset H^m \subset G_k$ of connected digraphs, where H^{i+1} is obtained from H^i by adding the arcs (and possibly some vertices) of a 2- or 3-cycle in C, and $m = |C| - k$.

H^0 is Eulerian, and each H^{i+1} is connected and obtained from H^i by adding a cycle. Therefore each H^i is Eulerian. Since $|E(H^{i+1})| - |E(H^i)| \leq 3$, for all i, we can add loops to some H^i, as necessary, to obtain an Eulerian digraph $G \subseteq G_k$ of size n, for any n such that $|E(H_k)| \leq n \leq |E(G_k)|$.

Now, if k is even (so $k \geq 4$) we have that $|E(H_k)| = k(k-2) \leq ((k-1)^3 + (k-1)^2 + 2)/2$ (the lower bound on n). If k is odd and $k \geq 5$, we have $|E(H_k)| = k(k-1) \leq ((k-1)^3 + (k-1)^2 - 2(k-1) + 2)/2$. Since $|E(G_k)|$ equals the upper bound on n, we are done for $k \geq 4$. If $k = 3$, then $|E(H_k)| = 6$, so to complete the proof we need a recognizable 3-coloring of C_5, which is easy to find. ■

Corresponding to the arcs that are removed from G'_k to obtain G_k when k is even we define *special* codes for vertices of degree 2 to be the codes $(1;1,2)$, $(2;1,2)$, $(3;3,4)$, $(4;3,4), \ldots$. From the proof above we have the following:

Theorem 4. *If k is even and $n = (k^3 + k^2 - 2k)/2$, there is a recognizable k-coloring of C_n such that none of the special codes occur, while every other possible code does occur.*

3. Paths

As mentioned in Section 2, the following monotonicity property of $\mathrm{rn}(P_n)$ enables us to determine the recognition number of paths by considering extreme cases only. These extreme cases are proved using Theorem 2.

Theorem 5. $\mathrm{rn}(P_{n-1}) \leq \mathrm{rn}(P_n)$, *for all integers* $n \geq 2$.

Proof. For $n \leq 6$, it is easily verified that $\mathrm{rn}(P_n) = 2$, so assume $n \geq 7$. Let c be a recognizable coloring of $P = P_n = u_1, u_2, \ldots, u_{n-3}, v_3, v_2, v_1$. If c is a recognizable coloring of $P - u_1$, we are done. Otherwise, $c(u_2) = c(v_1)$ and $c(u_3) = c(v_2)$. Similarly, if c is not a recognizable coloring of $P - v_1$, then $c(v_2) = c(u_1)$ and $c(v_3) = c(u_2)$. Remove u_1 and u_2 from P, and add a vertex w and the edge v_1w to form a path $P' = P_{n-1}$. Set $c(w) = c(u_1)$. Only the color codes of u_3, v_1 and w are affected. Note that $c(u_4) \neq c(v_1)$, since u_3 and v_2 have different codes in P. Therefore w and u_3 have different codes in P'. Since the code of v_1 in P' is the same as the code of u_2 in P, and u_2 is removed, c is a recognizable k-coloring of P'. ■

Theorem 6. *Let* $k \geq 3$ *be an integer. Then* $\mathrm{rn}(P_n) = k$ *for all integers* n *such that*

$$\begin{array}{ccccl} \frac{(k-1)^3+(k-1)^2-2(k-1)+10}{2} & \leq & n & \leq & \frac{k^3+k^2+4}{2} \qquad \text{if } k \text{ is odd,} \\ \frac{(k-1)^3+(k-1)^2+6}{2} & \leq & n & \leq & \frac{k^3+k^2-2k+8}{2} \qquad \text{if } k \text{ is even.} \end{array}$$

Proof. That $\mathrm{rn}(P_n) \geq k$ if n is as given is proved in [8]. For the upper bounds, we need only prove the maximal cases, by Theorem 5. First suppose k is odd, and let $n = (k^3 + k^2 + 4)/2$. By Theorem 2 there is a recognizable k-coloring c of C_{n-2}. Let uv be any edge of C_{n-2} such that $c(u) \neq c(v)$. Remove uv and add vertices u' and v' together with the edges uv' and vu'. Setting $c(u') = c(u)$ and $c(v') = c(v)$ yields a recognizable k-coloring of P_n.

Next, suppose k is even and let $n = (k^3+k^2-2k+8)/2$. By Theorem 4, there is a recognizable k-coloring c of C_{n-4} such that every code except the special codes occurs. In particular, neither of the codes $(1; 1, 2)$ and $(2; 1, 2)$ occurs. Let uv be an edge of C_{n-4} such that $c(u) = 1$ and $c(v) = 2$. (Such a u and v exist since, for example, the code $(1; 2, 3)$ occurs.) Remove uv and add vertices u_1, u_2, v_1 and v_2, and edges uv_1, v_1v_2, vu_1 and u_1u_2. Set $c(u_1) = c(u_2) = 1$ and $c(v_1) = c(v_2) = 2$. Then c is a recognizable k-coloring of P_n. ■

In the first paragraph of the preceding proof we can take uv such that $c(u) = 1$ and $c(v) = 2$, since every possible color code occurs. In the second paragraph,

c is a recognizable k-coloring of C_n such that no special code occurs. The construction then adds only the codes $(1;1,2)$, $(2;1,2)$ and two degree one codes, while preserving all other codes. We therefore have:

Theorem 7. *Let $n \geq 4$ and $P = P_n$ with end-vertices u_1 and v_1. Let the neighbors of u_1 and v_1 be u_2 and v_2, respectively.*

If k is odd and $n = (k^3 + k^2 + 4)/2$, there is a recognizable k-coloring c of P such that $c(u_1) = c(v_2) = 1$ and $c(v_1) = c(u_2) = 2$.

If k is even and $n = (k^3 + k^2 - 2k + 8)/2$, there is a recognizable k-coloring c of P such that $c(u_1) = c(u_2) = 1$ and $c(v_1) = c(v_2) = 2$. Moreover, every color code for vertices of degree two occurs, except for the special codes other than $(1;1,2)$ and $(2;1,2)$.

4. Trees

For the proof of the next result, it will be useful to define the following operation: Let uv be an edge of a graph G, with $deg(u) = 3$. By $G/(u,v)$ we denote the graph obtained from $G - \{u,v\}$ by adding an edge between the other two neighbours of u. (In our constructions they will never be adjacent in G.)

Theorem 8. *For each integer $n \geq 3$, the minimum recognition number among all trees of order n is the unique integer k such that*

$$\frac{(k-1)^3 + 5(k-1)^2 - 2}{2} \;\leq\; n \;\leq\; \frac{k^3 + 5k^2 - 4}{2}.$$

Proof. That a tree of order at least $((k-1)^3 + 5(k-1)^2 - 2)/2$ has recognition number at least k is shown in [8]. For completeness we give the proof: It is known that if T is a tree of order n with n_i vertices of degree i, then $n_1 = 2 + n_3 + 2n_4 + 3n_5 + \cdots$. If T has recognition number k we have by Observation 1 that $n_1 \leq k^2$ and $n_2 \leq \frac{k^3+k^2}{2}$. Combining these gives $n \leq n_1 + n_2 + n_1 - 2 \leq 2k^2 + \frac{k^3+k^2}{2} - 2 = \frac{k^3+5k^2-4}{2}$.

For the upper bound, given k and n, we construct a tree T of order n and recognition number k.

Case 1. k is odd. First, let $n = (k^3 + 5k^2 - 4)/2$. Let $P = P_{(k^3+k^2+4)/2}$ with end-vertices u and v. By Theorem 7 there is a recognizable k-coloring c of P such that u and the neighbor of v have color 1 and v and the neighbor of u have color 2.

From P we first construct T' as follows: For each l, where $1 \leq l \leq k$, let v_1 and v_2 be any two adjacent vertices of color l. Remove v_1v_2, add a path $w_l^1, w_l^2, \ldots, w_l^k$ and add edges $v_1w_l^1$ and $v_2w_l^k$. For each w_l^i, $1 \leq l, i \leq k$, add a vertex x_l^i and the edge $x_l^iw_l^i$. Set $c(w_l^i) = l$ and $c(x_l^i) = i$. Let $T = (T'/(w_1^2, x_1^2))/(w_2^1, x_2^1)$.

The vertices in T of degree two are precisely the vertices of degree two in P and their color codes are unchanged. There are $k^2 - 2$ new vertices of degree one, and it is easy to see that each of the k^2 color codes for vertices of degree one occurs exactly once. There are $k^2 - 2$ new vertices of degree three and it is easily checked that they have distinct color codes. Lastly, T has order $(k^3 + k^2 + 4)/2 + 2(k^2 - 2) = (k^3 + 5k^2 - 4)/2$.

If $n \leq (k^3 + k^2 + 4)/2$, we can take $T = P_n$ by Theorem 6, so suppose that $(k^3+k^2+4)/2 < n < (k^3+5k^2-4)/2$. From the tree T constructed in the previous paragraph we can obtain a tree T' of order n as follows: For any l and i, c is a recognizable k-coloring of $T/(w_l^i, x_l^i)$. Repeatedly removing pairs (w_l^i, x_l^i) using this operation, we obtain the required T' for all n of the form $(k^3+k^2+4)/2+2m$. Any such T' has vertices x, y and z of color 1, such that x has degree two and is adjacent to y and z. If we remove x and add yz, we have a tree of order one less than the order of T' for which c is a recognizable k-coloring, which covers the remaining cases.

Case 2. k is even. For $n = (k^3 + 5k^2 - 4)/2$ we take c to be a recognizable k-coloring of $P_{(k^3+k^2-2k+8)/2}$ according to Theorem 7. Let T^* be obtained from $P_{(k^3+k^2-2k+8)/2}$ in the same way that T' is obtained in Case 1 and set $T = (T^*/(w_1^1, x_1^1))/(w_2^2, x_2^2)$.

T has vertices with every possible color code for vertices of degree one and two, except for the special codes other than $(1; 1, 2)$ and $(2; 1, 2)$. For every odd $l \geq 3$, we add vertices y_l and y_{l+1}, edges $x_l^l y_l$ and $x_{l+1}^{l+1} y_{l+1}$, and set $c(x_l^l) = c(y_l) = l+1$ and $c(x_{l+1}^{l+1}) = c(y_{l+1}) = l$. The tree obtained in this way has order n and is recognizably k-colored by c.

For $n < (k^3 + 5k^2 - 4)/2$ the proof is analogous to the proof for *Case* 1. ■

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