

VERTEX RAINBOW COLORINGS OF GRAPHS

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Abstract

In a properly vertex-colored graph G , a path P is a rainbow path if no two vertices of P have the same color, except possibly the two end-vertices of P . If every two vertices of G are connected by a rainbow path, then G is vertex rainbow-connected. A proper vertex coloring of a connected graph G that results in a vertex rainbow-connected graph is a vertex rainbow coloring of G . The minimum number of colors needed in a vertex rainbow coloring of G is the vertex rainbow connection number $\text{vrc}(G)$ of G . Thus if G is a connected graph of order $n \geq 2$, then $2 \leq \text{vrc}(G) \leq n$. We present characterizations of all connected graphs G of order n for which $\text{vrc}(G) \in \{2, n-1, n\}$ and study the relationship between $\text{vrc}(G)$ and the chromatic number $\chi(G)$ of G . For a connected graph G of order n and size m , the number $m - n + 1$ is the cycle rank of G . Vertex rainbow connection numbers are determined for all connected graphs of cycle rank 0 or 1 and these numbers are investigated for connected graphs of cycle rank 2.

Keywords: rainbow path, vertex rainbow coloring, vertex rainbow connection number.

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1. INTRODUCTION

A *proper vertex coloring* of a graph G is a function $c : V(G) \rightarrow \mathbb{N}$ having the property that whenever u and v are adjacent vertices of G , then $c(u) \neq c(v)$. A proper vertex coloring of G that uses k colors is a k -*coloring* of G and the minimum integer k for which there is a proper k -coloring of G is the *chromatic number* $\chi(G)$ of G .

If c is a proper vertex coloring of a graph G and u and v are two vertices of G with distance $d(u, v) = 2$, where $P = (u, w, v)$ is a $u - v$ path of length 2 in G , then $c(u) \neq c(w)$ and $c(w) \neq c(v)$. Either $c(u) = c(v)$ or $c(u) \neq c(v)$ is possible however. In other words, no two vertices on the path $P = (u, w, v)$ are colored the same except possibly the two end-vertices. This observation suggests a generalization of proper vertex colorings of graphs introduced in [4, p. 441].

For a proper vertex coloring c of a connected graph G , a $u - v$ path P is called a *rainbow path* if no two vertices of P have the same color, except possibly u and v . In this context, G is *vertex rainbow-connected* or simply *rainbow-connected* (with respect to c) if G contains a rainbow $u - v$ path for every two vertices u and v . In this case, c is a *vertex rainbow coloring* or simply a *rainbow coloring* of G . If k colors are used, then c is a *rainbow k -coloring* of G . The *vertex rainbow connection number* (or simply the *vrc-number*) $\text{vrc}(G)$ of G is the minimum positive integer k for which there exists a rainbow k -coloring of G . Since each rainbow coloring of a graph G is a proper coloring, $\text{vrc}(G) \geq \chi(G)$. The concepts studied here were suggested and inspired by the ideas introduced and studied in [1, 2, 3].

If the distance $d(u, v)$ between vertices u and v in a connected graph G equals the *diameter* $\text{diam}(G)$ of G (the largest distance between two vertices of G), then u and v are *antipodal vertices* of G . For each $u - v$ geodesic P of G , a rainbow coloring of G must assign at least $\text{diam}(G)$ distinct colors to P . Thus $\text{vrc}(G) \geq \text{diam}(G)$. Furthermore, if $\text{vrc}(G) = \text{diam}(G) = k$ and c is a rainbow k -coloring of G , then $c(u) = c(v)$ for every pair u, v of antipodal vertices of G . We summarize these observations below.

Observation 1.1. *For every nontrivial connected graph G ,*

$$\text{vrc}(G) \geq \max\{\chi(G), \text{diam}(G)\}.$$

Furthermore, if $\text{vrc}(G) = \text{diam}(G) = k$ and c is a rainbow k -coloring of G , then $c(u) = c(v)$ for every pair u, v of antipodal vertices of G .

To illustrate these concepts, consider the 3-cube Q_3 , whose vertices are labeled as shown in Figure 1(a). The rainbow 4-coloring of Q_3 in Figure 1(a) shows that $\text{vrc}(Q_3) \leq 4$. Since $\chi(Q_3) = 2$ and $\text{diam}(Q_3) = 3$, it follows by Observation 1.1 that $\text{vrc}(Q_3) \geq 3$. In fact, $\text{vrc}(Q_3) > 3$. To see this, suppose that there exists a rainbow 3-coloring c of Q_3 , where say $c(v_i) = i$ for $i = 1, 2$. Since $d(v_1, u_3) =$

$d(v_2, u_4) = 3$, it follows that $c(u_3) = 1$ and $c(u_4) = 2$ by Observation 1.1. On the other hand, observe that v_4 is adjacent to v_1 and u_4 while v_3 is adjacent to v_2 and u_3 and so $\{c(v_3), c(v_4)\} \cap \{1, 2\} = \emptyset$, that is, $c(v_3) = c(v_4) = 3$. However, this is impossible since c is a proper coloring. Hence, $\text{vrc}(Q_3) = 4$.

For the Petersen graph P , it follows since $\text{diam}(P) = 2$ and $\chi(P) = 3$ that $\text{vrc}(P) \geq 3$. Consider the proper 3-coloring c of P of Figure 1(b). Since $\text{diam}(P) = 2$, every two nonadjacent vertices u and v of P have a common neighbor w . Because $c(u) \neq c(w)$ and $c(v) \neq c(w)$, it follows that (u, w, v) is a rainbow $u - v$ path. Thus c is a rainbow coloring of P and so $\text{vrc}(P) \leq 3$, that is, $\text{vrc}(P) = 3$. This illustrates the following useful observation.

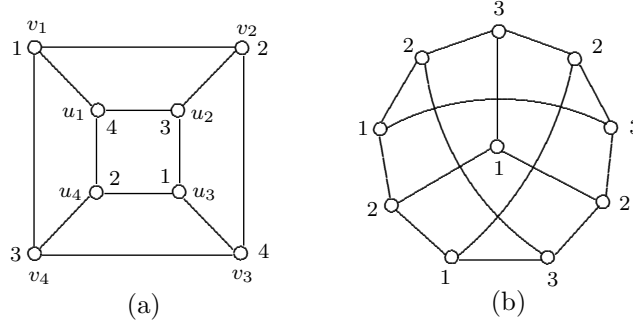


Figure 1. A rainbow 4-coloring of the 3-cube and a rainbow 3-coloring of the Petersen graph.

Observation 1.2. *If G is a connected graph with $\text{diam}(G) = 2$, then every proper coloring of G is a rainbow coloring and so $\text{vrc}(G) = \chi(G)$.*

2. GRAPHS WITH PRESCRIBED ORDER AND VERTEX RAINBOW CONNECTION NUMBER

For every connected graph G of order $n \geq 2$,

$$2 \leq \text{vrc}(G) \leq n.$$

We first characterize all connected graphs G of order n for which $\text{vrc}(G)$ attains one of these two extreme values.

Proposition 2.1. *Let G be a nontrivial connected graph of order n . Then*

- (a) $\text{vrc}(G) = n$ if and only if G is a complete graph.
- (b) $\text{vrc}(G) = 2$ if and only if G is a complete bipartite graph.

Proof. We first verify (a). Clearly $\text{vrc}(K_n) = n$ by Observation 1.1. For the converse, assume that $G \neq K_n$ and let u and v be antipodal vertices. Then $uv \notin E(G)$. Consider a proper coloring c of G assigning the color 1 to both u and v and assigning the $n-2$ colors $2, 3, \dots, n-1$ to the remaining $n-2$ vertices of G . Then every $u-v$ path is a rainbow path. Also, if $\{u', v'\} \neq \{u, v\}$, then every $u'-v'$ geodesic contains at most one of u and v and therefore is a rainbow path. Hence, c is a rainbow coloring of G , implying that $\text{vrc}(G) \leq n-1$.

Next, we verify (b). If G is a complete bipartite graph, then $\text{vrc}(G) = 2$ by Observation 1.2. For the converse, suppose that G is a connected graph with $\text{vrc}(G) = 2$. It then follows by Observation 1.1 that $\chi(G) = \text{diam}(G) = 2$, implying that G is a complete bipartite graph. ■

The *clique number* $\omega(G)$ of a graph G is the order of a largest complete subgraph in G . It is well known that if G is a nontrivial graph of order n , then

$$(1) \quad \chi(G) = n-1 \text{ if and only if } \omega(G) = n-1.$$

While this is not true in general for the vertex rainbow connection number (for example, if $n \geq 4$, then $\text{vrc}(P_n) = n-1 \geq 3$ and $\omega(P_n) = 2$), this is, in fact, the case for 2-connected graphs.

Theorem 2.2. *If G is a 2-connected graph of order $n \geq 4$, then*

$$\text{vrc}(G) = n-1 \text{ if and only if } \omega(G) = n-1.$$

Proof. If $\omega(G) = n-1$, then $\text{diam}(G) = 2$. It then follows by Observation 1.2 and (1) that $\text{vrc}(G) = \chi(G) = n-1$.

For the converse, let G be a 2-connected graph of order n with $\omega(G) \leq n-2$. If $\text{diam}(G) = 2$, then $\text{vrc}(G) = \chi(G) \leq n-2$ by Observation 1.2 and (1). Thus we may assume that $\text{diam}(G) \geq 3$ and then $n \geq 6$. Let w_1 and w_2 be antipodal vertices. Since G is 2-connected, it follows by a well-known theorem of Whitney [6] that w_1 and w_2 lie on a common cycle C . Let

$$C = (v_1 = w_1, v_2, \dots, v_k = w_2, v_{k+1}, \dots, v_N, v_1)$$

where N is the length of C . Since $d(w_1, w_2) = \text{diam}(G) \geq 3$, it follows that $k \geq 4$ and neither v_1v_{k-1} nor v_2v_k is an edge of G . Let $w'_1 = v_{k-1}$, $w'_2 = v_2$ and $W = \{w_1, w_2, w'_1, w'_2\}$. Now we define a proper coloring $c : V(G) \rightarrow \mathbb{N}$ such that $c(w_i) = c(w'_i) = i$ for $i = 1, 2$ and $\{c(v) : v \in V(G) - W\} = \{3, 4, \dots, n-2\}$. We verify that c is a rainbow coloring by showing that for every pair u, v of distinct vertices there is a rainbow $u-v$ path.

If $\{u, v\} \subseteq W$, then at least one of the two $u-v$ paths along C is a rainbow $u-v$ path. Thus, suppose that $\{u, v\} \not\subseteq W$. Since G is 2-connected, there are two internally disjoint $u-v$ paths P_1 and P_2 . If one of P_1 and P_2 , say P_1 , contains

at most one of w_1 and w'_1 and at most one of w_2 and w'_2 , then P_1 is a rainbow $u - v$ path. If this is not the case, then we may assume that $\{w_i, w'_i\} \subseteq V(P_i)$ for $i = 1, 2$. Let $w_1^* \in \{w_1, w'_1\}$ and $w_2^* \in \{w_2, w'_2\}$ such that

$$\begin{aligned} d_{P_1}(u, w_1^*) &= \min\{d_{P_1}(u, w_1), d_{P_1}(u, w'_1)\}, \\ d_{P_2}(v, w_2^*) &= \min\{d_{P_2}(v, w_2), d_{P_2}(v, w'_2)\}. \end{aligned}$$

Recall that there exists a rainbow $w_1^* - w_2^*$ path along C , which we denote by P^* . Then the $u - w_1^*$ path along P_1 followed by P^* and then followed by the $w_2^* - v$ path along P_2 is a rainbow $u - v$ trail (in which no two vertices are assigned the same color, except possibly u and v). This rainbow $u - v$ trail contains a rainbow $u - v$ path. ■

Next, we characterize all connected graphs G of order n with $\text{vrc}(G) = n - 1$. By Theorem 2.2, it remains to consider connected graphs having cut-vertices. We first present some preliminary results. For a connected graph G with cut-vertices, an *end-block* of G contains exactly one cut-vertex of G . It is known that every connected graph G with cut-vertices has at least two end-blocks.

Lemma 2.3. *Let G be a connected graph of order $n \geq 3$ having cut-vertices. If G contains k end-blocks, then*

$$\text{vrc}(G) \leq n - k + 1.$$

In particular, if G contains k end-vertices, then $\text{vrc}(G) \leq n - k + 1$.

Proof. Let B_1, B_2, \dots, B_k be k end-blocks of G . For each integer i with $1 \leq i \leq k$, suppose that B_i contains the cut-vertex u_i and v_i is a vertex in B_i farthest from u_i . Thus $S = \{v_1, v_2, \dots, v_k\}$ is an independent set of vertices in G . Define a proper coloring c of G by assigning the color 1 to the vertices in S and assigning the $n - k$ colors $2, 3, \dots, n - k + 1$ to the remaining $n - k$ vertices in $V(G) - S$. For each pair x, y of the vertices of G , an $x - y$ geodesic is a rainbow $x - y$ path in G . Thus c is a rainbow coloring with $n - k + 1$ colors and so $\text{vrc}(G) \leq n - k + 1$. ■

A block of order 3 or more in a graph is called a *cyclic block*. By Lemma 2.3, if a graph G of order n contains two end-blocks, then $\text{vrc}(G) \leq n - 1$. In the case where two end-blocks of G are cyclic, more can be said.

Lemma 2.4. *Let G be a connected graph of order $n \geq 5$ having cut-vertices. If G contains two cyclic end-blocks, then*

$$\text{vrc}(G) \leq n - 2.$$

Proof. Let B_1 and B_2 be two cyclic end-blocks of G . For $i = 1, 2$, let u_i be the cut-vertex of B_i , let v_i be a vertex in B_i farthest from u_i and let $w_i \neq u_i$ be a vertex adjacent to v_i . Define a proper coloring c of G by assigning the color 1 to the vertices in $\{v_1, w_2\}$, the color 2 to the vertices in $\{v_2, w_1\}$ and assigning the $n-4$ colors $3, 4, \dots, n-2$ to the remaining $n-4$ vertices in $V(G) - \{v_1, v_2, w_1, w_2\}$. Let $V_i = V(B_i) - \{u_i\}$ for $i = 1, 2$. For each pair x, y of the vertices of G , let P be an $x - y$ geodesic in G . If $\{x, y\} \cap V_1 = \emptyset$ or $\{x, y\} \cap V_2 = \emptyset$, say $\{x, y\} \cap V_1 = \emptyset$, then P is a path in $G - V_1$ and so P is a rainbow $x - y$ path in G . Thus we may assume that $\{x, y\} \cap V_i \neq \emptyset$ for $i = 1, 2$, where say $x \in V_1$ and $y \in V_2$. There are two cases.

Case 1. $\{c(x), c(y)\} \cap \{1, 2\} = \emptyset$. It then follows by the definition of v_1 and v_2 that P is a path in $G - \{v_1, v_2\}$ and so P is a rainbow $x - y$ path in G .

Case 2. $\{c(x), c(y)\} \cap \{1, 2\} \neq \emptyset$. We consider two subcases.

Subcase 2.1. $|\{c(x), c(y)\}| = 1$, say $c(x) = c(y) = 1$. Thus $x = v_1$ and $y = w_2$. Again, it follows by the definition of v_2 that $v_2 \notin V(P)$ and so P is a rainbow $x - y$ path in G .

Subcase 2.2. $|\{c(x), c(y)\}| = 2$, say $c(x) = 1$ and $c(y) = 2$. Thus $x = v_1$ and $y = v_2$. Furthermore, $P : P_1, Q, P_2$, where $V(P_1) \subseteq V_1$, $V(Q) \subseteq V(G) - (V_1 \cup V_2)$ and $V(P_2) \subseteq V_2$. Since B_i ($i = 1, 2$) is 2-connected, there is a $v_1 - u_1$ path P'_1 in B_1 that does not contain w_1 and there is a $u_2 - v_2$ path P'_2 in B_2 that does not contain w_2 . Thus $P' : P'_1, Q, P'_2$ is a rainbow $x - y$ path in G . Therefore, c is a rainbow $(n-2)$ -coloring of G , which implies that $\text{vrc}(G) \leq n-2$. ■

Lemma 2.5. *If x is a cut-vertex of a nontrivial connected graph G and c is a rainbow coloring of G , then $c(x) \neq c(v)$ for every vertex v in $G - x$. In particular, if X is the set of cut-vertices, then*

$$(2) \quad \text{vrc}(G) \geq |X| + 1.$$

Proof. Let X be the set of cut-vertices. For a vertex $x \in X$, suppose that c is a rainbow coloring such that $c(x) = c(v)$ for some vertex $v \in V(G) - \{x\}$. Since x is a cut-vertex, there exists a vertex v' such that v and v' belong to different components in $G - x$. Then every $v - v'$ path contains x and so there is no rainbow $v - v'$ path, which is a contradiction. Since every graph contains at least one vertex (in fact two vertices) that is not a cut-vertex, it then follows that $\text{vrc}(G) \geq |X| + 1$. ■

For each integer $n \geq 3$, let \mathcal{F}_n be the class of connected graphs of order n such that $G \in \mathcal{F}_n$ if and only if either $\omega(G) = n-1$ or G is obtained by joining an end-vertex of a nontrivial path to some vertices of a nontrivial complete graph. Therefore, $P_n \in \mathcal{F}_n$ while $K_n \notin \mathcal{F}_n$. We show that these graphs in \mathcal{F}_n are the only connected graphs of order n having vertex rainbow connection number $n-1$.

Theorem 2.6. *If G is a connected graph of order $n \geq 3$, then $\text{vrc}(G) = n - 1$ if and only if $G \in \mathcal{F}_n$.*

Proof. Suppose that $G \in \mathcal{F}_n$. Then $\text{vrc}(G) \leq n - 1$ by Proposition 2.1(a) since G is not complete. It remains to show that $\text{vrc}(G) \geq n - 1$. If $\omega(G) = n - 1$, then $\text{vrc}(G) \geq \chi(G) = \omega(G) = n - 1$. Otherwise, we may assume that G is obtained from a path P_a and a complete graph K_b , where $a, b \geq 2$ and $n = a + b$, by joining an end-vertex of P_a to some vertices of K_b . Let c be a rainbow coloring of G . Then c must assign b distinct colors to the b vertices of K_b since c must be proper. Furthermore, c must assign $a - 1$ additional distinct colors to the $a - 1$ cut-vertices of G belonging to P_a by Lemma 2.5. This implies that c must use at least $(a - 1) + b = n - 1$ distinct colors and so $\text{vrc}(G) \geq n - 1$. Therefore, $\text{vrc}(G) = n - 1$ in each case.

For the converse, let G be a connected graph of order $n \geq 3$ with $\text{vrc}(G) = n - 1$. If G is 2-connected, then $\omega(G) = n - 1$ by Theorem 2.2 and so $G \in \mathcal{F}_n$.

Hence, suppose that G contains a cut-vertex. Then by Lemmas 2.3 and 2.4, there are exactly two end-blocks, at least one of which must be K_2 . Therefore, if G contains k blocks ($k \geq 2$), then we may label these blocks as B_1, B_2, \dots, B_k such that (i) B_1 and B_k are the two end-blocks and $B_1 = K_2$ and (ii) two blocks B_i and B_j share a cut-vertex if and only if $|i - j| = 1$. That is, the structure of G must be the one shown in Figure 2. Let $v_1 \in V(B_1)$ and $v_k \in V(B_k)$ such that v_1 is the end-vertex in G and v_k is a vertex in B_k farthest from v_1 . Also, let x_i be the cut-vertex belonging to B_i and B_{i+1} for $1 \leq i \leq k - 1$.

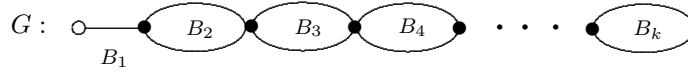


Figure 2. A step in the proof of Theorem 2.6.

We first claim that $B_i = K_2$ for $1 \leq i \leq k - 1$. Assume that $k \geq 3$ since $B_1 = K_2$. If there exists a vertex $u_i \in V(B_i) - \{x_{i-1}, x_i\}$ for some i ($2 \leq i \leq k - 1$), then $S = \{v_1, v_k, u_i\}$ is an independent set and so any coloring $c : V(G) \rightarrow \mathbb{N}$ such that $c(v) = 1$ if $v \in S$ and $\{c(v) : v \in V(G) - S\} = \{2, 3, \dots, n - 2\}$ is proper. Furthermore, it is straightforward to verify that c is a rainbow coloring of G . However, this is certainly a contradiction since $\text{vrc}(G) = n - 1$. Thus, $B_i = K_2$ for $1 \leq i \leq k - 1$, as claimed.

It remains to determine the structure of B_k . Note that $|V(B_k)| = n - k + 1$. If B_k is complete, then certainly $G \in \mathcal{F}_n$. Otherwise, since B_k is 2-connected, it follows that either $\omega(B_k) = n - k$ or $\text{vrc}(B_k) \leq n - k - 1$ by Theorem 2.2. If $\text{vrc}(B_k) = \ell \leq n - k - 1$, then let c' be a rainbow ℓ -coloring of B_k using the colors $1, 2, \dots, \ell$. Define $c : V(G) \rightarrow \mathbb{N}$ by $c(v) = c'(v)$ if $v \in V(B_k) - \{x_{k-1}\}$, $c(v_1) = c'(v_k)$, and $c(x_i) = \ell + i$ for $1 \leq i \leq k - 1$. Then c is not only a proper

coloring but also a rainbow coloring of G that uses $\ell + k - 1$ colors. However, $\ell + k - 1 \leq n - 2$ and so this cannot occur.

Hence, $\omega(B_k) = n - k$. If there are two vertices $w_1, w_2 \in V(B_k) - \{x_{k-1}\}$ such that $w_1 w_2 \notin E(G)$, then we may assume, without loss of generality, that $B_k - w_1$ is complete. Then one can verify that a coloring $c : V(G) \rightarrow \mathbb{N}$ such that $c(v_1) = c(w_1) = c(w_2) = 1$ and $\{c(v) : v \in V(G) - \{v_1, w_1, w_2\}\} = \{2, 3, \dots, n-2\}$ is a proper rainbow coloring of G , which is again impossible. Therefore, $B_k - x_{k-1}$ must be complete, implying that $G \in \mathcal{F}_n$. ■

By Observation 1.1, if G is a nontrivial connected graph of order n with $\chi(G) = a$ and $\text{vrc}(G) = b$, then $2 \leq a \leq b \leq n$. Define a triple (a, b, n) of integers to be *realizable* if $2 \leq a \leq b \leq n$ and there exists a connected graph G of order n such that $\chi(G) = a$ and $\text{vrc}(G) = b$. We determine all realizable triples.

Theorem 2.7. *A triple (a, b, n) of integers with $2 \leq a \leq b \leq n$ is realizable if and only if $(a, b, n) \notin \{(a, b, b) : a < b\}$.*

Proof. Let (a, b, n) be a realizable triple. Then there is a connected graph of order n such that $\chi(G) = a$ and $\text{vrc}(G) = b$. By Observation 1.1, $2 \leq \chi(G) \leq \text{vrc}(G) \leq n$ and so $2 \leq a \leq b \leq n$. If $b = n$, then $G = K_n$ by Proposition 2.1 and so $a = b = n$. Thus we may assume that $b < n$. Hence $(a, b, n) \notin \{(a, b, b) : a < b\}$.

For the converse, assume that $(a, b, n) \neq (a, b, b)$ where $a < b$. Thus either $a = b = n$ or $2 \leq a \leq b < n$. If $a = b = n$, then $G = K_n$ has the desired properties. Thus we may assume that $2 \leq a \leq b < n$. Let $G_1 = K_{1,1,\dots,1,n-b}$ be complete a -partite graph of order $n - (b - a + 1)$ and let G_2 be the path $P_{b-a+1} = (v_1, v_2, \dots, v_{b-a+1})$ of order $b - a + 1$. The graph G is obtained from G_1 and G_2 by joining v_1 of G_2 to a vertex u of G_1 . Then the order of G is n . Let V_1, V_2, \dots, V_a be partite sets of G_1 , where $u \in V_a$ say. We show that $\chi(G) = a$ and $\text{vrc}(G) = b$. Since $\chi(G_1) = a$ and $G_1 \subset G$, it follows that $\chi(G) \geq a$. Define a coloring c_1 of G_1 by $c_1(v) = i$ if $v \in V_i$ ($1 \leq i \leq a$) and a coloring c_2 of G_2 by $c_2(v_i) = 1$ if i is odd and $c_2(v_i) = 2$ if i is even. Then the coloring of G defined by $c(v) = c_j(v)$ if $v \in V(G_j)$ for $j = 1, 2$ is a proper coloring of G using a distinct colors. Thus, $\chi(G) = a$. Next, we show that $\text{vrc}(G) = b$. Define a coloring $c'_2 : V(G_2) \rightarrow \{a+1, a+2, \dots, b\}$ by

$$c'_2(v_i) = \begin{cases} a+i & \text{if } 1 \leq i \leq b-a, \\ 1 & \text{if } i = b-a+1. \end{cases}$$

Now we define a coloring $c' : V(G) \rightarrow \{1, 2, \dots, b\}$ by

$$c'(v) = \begin{cases} c_1(v) & \text{if } v \in V(G_1), \\ c'_2(v) & \text{if } v \in V(G_2). \end{cases}$$

Observe that c' is a rainbow coloring of G using b distinct colors, which implies that $\text{vrc}(G) \leq b$. On the other hand, any rainbow coloring of G must assign a distinct colors to the vertices of G_1 and additional $b - a$ distinct colors to the $b - a$ cut-vertices of G_2 by Lemma 2.5. This implies that $\text{vrc}(G) \geq a + b - a = b$. Therefore, $\text{vrc}(G) = b$. ■

By Observation 1.1, if G is a connected graph of order n and diameter d with $\text{vrc}(G) = k$, then $1 \leq d \leq k \leq n$. We now determine all triples (d, k, n) of positive integers with $d \leq k \leq n$ that can be realized as the diameter, vertex rainbow number and order, respectively, of a connected graph.

Theorem 2.8. *A triple (d, k, n) of positive integers with $d \leq k \leq n$ is realizable as the diameter, vertex rainbow number and order, respectively, of a connected graph if and only if $(d, k, n) \notin \{(d, k, k) : d \geq 2\}$.*

Proof. If G is a connected graph of order n and diameter d with $\text{vrc}(G) = k$, then $1 \leq d \leq k \leq n$ by Observation 1.1. Furthermore, if $d = 1$, then $G = K_n$ and $\text{vrc}(G) = k = n$ by Proposition 2.1(a), while if $d \geq 2$, then $k \leq n - 1$ by Proposition 2.1(a) again. Thus $(d, k, n) \notin \{(d, k, k) : d \geq 2\}$.

For the converse, let (d, k, n) be a triple of positive integers with $d \leq k \leq n$ such that $(d, k, n) \notin \{(d, k, k) : d \geq 2\}$. We show that there is a connected graph G of order n and diameter d such that $\text{vrc}(G) = k$. If $d = 1$, then $k = n$. Let $G = K_n$ and so $\text{vrc}(G) = k = n$ by Proposition 2.1(a). Thus we may assume that $d \geq 2$ and then $2 \leq k \leq n - 1$. If $d = n - 1$, then $k = n - 1$. Let $G = P_n \in \mathcal{F}_n$ and so $\text{vrc}(G) = n - 1$ by Theorem 2.2. For $2 \leq d \leq n - 2$, we begin with three graphs G_1 , G_2 and G_3 . First, let $G_1 = K_{k-d+2}$ be a complete graph of order $k - d + 2 \geq 2$ where $V(G_1) = \{u_1, u_2, \dots, u_{k-d+2}\}$. If $k = n - 1$, let $G_2 = K_1$ with $V(K_1) = \{v_0\}$; while if $k \leq n - 2$, then $n - k - 1 \geq 1$ and let $G_2 = K_{1, n-k-1}$ be a star where $V(G_2) = \{v_0, v_1, \dots, v_{n-k-1}\}$ and v_0 is the central vertex of G_2 . Next, let $G_3 = P_d = (w_0, w_1, \dots, w_{d-1})$ be a path of order $d \geq 2$. The graph G is obtained from G_1, G_2 and G_3 by identifying the three vertices u_1, v_0, w_0 and labeling the resulting vertex in G by u_1 . Then the order of G is $(k - d + 2) + (n - k - 1) + (d - 1) = n$ and diameter of G is d . It remains to show that $\text{vrc}(G) = k$. If $k = n - 1$, then $G \in \mathcal{F}_n$ and so $\text{vrc}(G) = n - 1$ by Theorem 2.2. Thus we assume that $k \leq n - 2$. We first show that $\text{vrc}(G) \geq k$. If $d = 2$, then a rainbow coloring of G must assign k distinct colors to the vertices of K_k and so $\text{vrc}(G) \geq k$. If $d \geq 3$, then a rainbow coloring of G must assign $k - d + 2$ distinct colors to the vertices of K_{k-d+2} and an additional $d - 2$ distinct colors to the cut-vertices w_2, \dots, w_{d-2} of G by Lemma 2.5. Thus $\text{vrc}(G) \geq (k - d + 2) + (d - 2) = k$. To show that $\text{vrc}(G) \leq k$, define a proper coloring $c : V(G) \rightarrow \{1, 2, \dots, k\}$ as follows. If $d = 2$, then define

$$c(x) = \begin{cases} i & \text{if } x = u_i \text{ for } 1 \leq i \leq k, \\ k & \text{if } x \in \{v_1, v_2, \dots, v_{n-k-1}, w_{d-1}\}; \end{cases}$$

while if $d \geq 3$, then define

$$c(x) = \begin{cases} i & \text{if } x = u_i \text{ for } 1 \leq i \leq k - d + 2, \\ k - d + 2 + j & \text{if } x = w_j \text{ for } 1 \leq j \leq d - 2, \\ k - d + 2 & \text{if } x \in \{v_1, v_2, \dots, v_{n-k-1}, w_{d-1}\}. \end{cases}$$

In either case, c is a rainbow k -coloring of G and so $\text{vrc}(G) \leq k$. Therefore, $\text{vrc}(G) = k$. \blacksquare

3. GRAPHS WITH CYCLE RANK AT MOST 2

For a connected graph G of order n and size m , the number of edges that must be deleted from G to obtain a spanning tree of G is $m - n + 1$. The number $m - n + 1$ is called the *cycle rank* of G . Thus the cycle rank of a tree is 0, the cycle rank of a unicyclic graph (a connected graph containing exactly one cycle) is 1 and the cycle rank of a connected graph of order n and size $m = n + 1$ is 2. In this section we study the vertex rainbow connection numbers of connected graphs having cycle rank at most 2. As an immediate consequence of (2) and Lemma 2.3, we obtain a formula for the vertex rainbow connection number of a tree.

Corollary 3.1. *If T is a tree of order n with k end-vertices, then $\text{vrc}(T) = n - k + 1$.*

By Corollary 3.1, it remains to consider connected graphs having either cycle rank 1 (unicyclic graphs) or cycle rank 2. First, we present a useful lemma.

Lemma 3.2. *Let G be a connected graph that is not a tree. If the cycle rank of G is at most 2, then any rainbow coloring of G assigns the same color to at most two vertices belonging to a common cycle in G .*

Proof. Assume, to the contrary, that there is a rainbow coloring c of a connected graph G of cycle rank 1 or 2 that assigns the same color to three distinct vertices v_1, v_2 , and v_3 belonging to a cycle C in G . Since c is a proper coloring, $C - \{v_1, v_2, v_3\}$ consists of three vertex-disjoint subpaths of C . Suppose that $C - \{v_1, v_2, v_3\} = P_1 \cup P_2 \cup P_3$ where v_i is adjacent to a vertex in $V(P_j)$ if and only if $i \neq j$ for $1 \leq i, j \leq 3$. Since there exists a rainbow $u - v_1$ path in G for every $u \in V(P_1)$, there exist vertices $u_1 \in V(P_1)$ and $u_2 \in V(P_2) \cup V(P_3) \cup \{v_1\}$ such that G contains a $u_1 - u_2$ path whose edges do not belong to C . (Hence, the cycle rank must be 2.) Without loss of generality, suppose that $u_2 \in V(P_2) \cup \{v_1\}$. Then for every vertex $u \in V(P_3)$, there is no rainbow $u - v_3$ path in G . \blacksquare

We now determine the vertex rainbow connection numbers of unicyclic graphs.

Theorem 3.3. *If G is a connected unicyclic graph of order $n \geq 4$ containing k cut-vertices and ℓ end-vertices, then $\text{vrc}(G) = \lceil (n + k - \ell)/2 \rceil + \delta$, where $\delta \in \{0, 1\}$ and $\delta = 0$ if and only if (i) G is a cycle of order n or (ii) the subgraph induced by the set of vertices that are neither cut-vertices nor end-vertices is isomorphic to a path of odd order.*

Proof. First, assume that $G = C_n = (v_1, v_2, \dots, v_n, v_1)$. Then $k = \ell = 0$. By Lemma 3.2, $\text{vrc}(C_n) \geq \lceil n/2 \rceil$. On the other hand, the coloring $c : V(C_n) \rightarrow \mathbb{N}$ defined by

$$c(v_i) = \begin{cases} i & \text{if } 1 \leq i \leq \lceil n/2 \rceil, \\ i - \lceil n/2 \rceil & \text{if } \lceil n/2 \rceil + 1 \leq i \leq n \end{cases}$$

is an $\lceil n/2 \rceil$ -rainbow coloring. Therefore, $\text{vrc}(C_n) = \lceil n/2 \rceil$ and so the statement holds when $G = C_n$.

Next, assume that $G \neq C_n$. Then $k, \ell \geq 1$. Let $C = (v_1, v_2, \dots, v_n, v_1)$ be the cycle in G . Also let X and Y be the sets of cut-vertices and end-vertices, respectively, and assume, without loss of generality, that $v_1 \in X$. If $n > k + \ell$, then let $U = \{u_1, u_2, \dots, u_{n-k-\ell}\} \subseteq V(C)$ be the set of vertices that are not cut-vertices. Furthermore, assume that $u_i = v_{\phi(i)}$ for $1 \leq i \leq n - k - \ell$, where $2 \leq \phi(1) < \phi(2) < \dots < \phi(n - k - \ell) \leq N$.

Consider the coloring $c : V(G) \rightarrow \mathbb{N}$ such that $\{c(v) : v \in X\} = \{1, 2, \dots, k\}$,

$$c(u_i) = \begin{cases} k + i & \text{if } 1 \leq i \leq \lceil (n - k - \ell)/2 \rceil, \\ k + i - \lceil (n - k - \ell)/2 \rceil & \text{if } \lceil (n - k - \ell)/2 \rceil + 1 \leq i \leq n - k - \ell \end{cases}$$

and for each $v \in Y$,

$$c(v) = \begin{cases} \lceil (n + k - \ell)/2 \rceil & \text{if } \langle U \rangle \text{ is a path of odd order,} \\ \lceil (n + k - \ell)/2 \rceil + 1 & \text{otherwise.} \end{cases}$$

One can verify that c is a rainbow coloring of G using $\lceil (n + k - \ell)/2 \rceil + \delta$ colors. Thus the result is immediate if $U = \emptyset$ since $\text{vrc}(G) \geq k + 1$ by Lemma 2.5 and $n = k + \ell$. If $U \neq \emptyset$, on the other hand, then by Lemmas 2.5 and 3.2,

$$\text{vrc}(G) \geq |\{c(v) : v \in U \cup X\}| \geq \lceil (n - k + \ell)/2 \rceil + k = \lceil (n + k - \ell)/2 \rceil.$$

Therefore, if $\text{vrc}(G) = \lceil (n + k - \ell)/2 \rceil$ and c is a rainbow coloring of G using $\lceil (n + k - \ell)/2 \rceil$ colors, then there exists an end-vertex $y_1 \in Y$ such that $c(y_1) = c(u)$ for some $u \in U \subseteq V(C)$. Without loss of generality, assume that $\min\{d(v, y_1) : v \in V(C)\} = d(v_1, y_1)$. If there are two vertices $u, u' \in U$ such that $c(y_1) = c(u) = c(u')$, then let $u = v_{i_1}$ and $u' = v_{i_2}$ where $2 \leq i_1 < i_1 + 2 \leq i_2 \leq N$ and observe that there is no rainbow $y_1 - v_{i_1+1}$ path, which is a contradiction. Therefore, there exists exactly one vertex $u_0 \in U$ that is assigned the color $c(y_1)$,

which then implies that $|U|$ must be odd. This also implies that $c(y) = c(u_0)$ for every end-vertex y .

Finally, if the induced subgraph $\langle U \rangle$ is not connected, then we may assume that there exists a cut-vertex $x \in V(C) - \{v_1\}$ and a vertex $u \in U$ such that $u = v_{i_1}$, $x = v_{i_2}$ and $u_0 = v_{i_3}$, where $2 \leq i_1 < i_2 < i_3 \leq N$. Furthermore, let y_2 be an end-vertex such that $\min\{d(v, y_2) : v \in V(C)\} = d(v_{i_2}, y_2)$. Since $|U| \geq 3$, there exists $u' \in U - \{u_0, u\}$ such that $c(u) = c(u')$.

If $u' = v_i$ for some i with $2 \leq i \leq i_2 - 1$, then there is no rainbow $y_1 - y_2$ path; otherwise, either a rainbow $y_1 - u'$ path or a rainbow $y_2 - u'$ path does not exist. Thus, c cannot be a rainbow coloring of G . ■

Theorems 2.6 and 3.3 illustrate an interesting feature of vertex rainbow colorings of a graph. It is well known that $\chi(H) \leq \chi(G)$ if H is a subgraph of G . In general, this is not the case for the vertex rainbow connection numbers of graphs. For example, $P_n \subseteq C_n$ for every $n \geq 3$ but $\text{vrc}(P_n) = n - 1$ by Theorem 2.6; while $\text{vrc}(C_n) = \lceil n/2 \rceil$ by Theorem 3.3.

Therefore, $\text{vrc}(P_n) > \text{vrc}(C_n)$ for $n \geq 4$.

We now consider connected graphs of cycle rank 2. If G is a connected graph whose cycle rank is 2, then either (i) G contains exactly two cycles with at most one vertex in common or (ii) G contains a unique pair v_α, v_β of distinct vertices and three internally disjoint $v_\alpha - v_\beta$ paths. If G satisfies (i), then G is said to be of *type I*; while if G satisfies (ii), then G is of *type II*. We next determine the vertex rainbow connection numbers of graphs of type I.

Theorem 3.4. *Let G be a connected graph having cycle rank 2 and of type I. For the two cycles C_1 and C_2 of G , let $U_i \subseteq V(C_i)$ be the set of vertices that are not cut-vertices for $i = 1, 2$. If G is of order n containing k cut-vertices and ℓ end-vertices, then $\text{vrc}(G) = \lceil (n + k - \ell)/2 \rceil + \delta$, where $\delta \in \{0, 1\}$ and $\delta = 0$ if and only if one of the following two situations (A) and (B) occurs:*

- (A) *One of U_1 and U_2 is empty and the other induces a path of odd order.*
- (B) *Both U_1 and U_2 are nonempty and at least one of $\langle U_1 \rangle$ and $\langle U_2 \rangle$ is connected, say $\langle U_1 \rangle$ is connected. Moreover, either $|U_1| \not\equiv |U_2| \pmod{2}$ or $\langle U_2 \rangle$ is connected.*

Proof. Let X and Y be the sets of cut-vertices and end-vertices, respectively. Also, let $U = U_1 \cup U_2$ and $V(C) = V(C_1) \cup V(C_2)$. Necessarily, U is a proper subset of $V(C)$. Suppose that c is a rainbow coloring of G . Then it is straightforward to verify that no three distinct vertices belonging to $V(C)$ can be assigned the same color by c . Hence $\text{vrc}(G) \geq |X| + \lceil |U|/2 \rceil = \lceil (n + k - \ell)/2 \rceil$.

First assume, without loss of generality, that $U_1 = \emptyset$. Then $\ell \geq 1$. Let $e \in E(C_1)$ and consider the graph $H = G - e$, which is a connected unicyclic graph. Since $U_1 = \emptyset$, it follows that c is a rainbow coloring of G if and only if c

is a rainbow coloring of H and so $\text{vrc}(G) = \text{vrc}(H)$. Therefore, the result follows by Theorem 3.3.

Next assume that neither U_1 nor U_2 is empty. Let $C_i = (v_{i,1}, v_{i,2}, \dots, v_{i,N_i}, v_{i,1})$, where N_i is the length of C_i for $i = 1, 2$. Assume also that $d(v_{1,1}, v_{2,1}) = \min\{d(u, v) : (u, v) \in V(C_1) \times V(C_2)\}$. Necessarily, both $v_{1,1}$ and $v_{2,1}$ belong to X (where it is possible that $v_{1,1} = v_{2,1}$).

Let $U_i = \{u_{i,1}, u_{i,2}, \dots, u_{i,|U_i|}\}$ such that $u_{i,j} = v_{i,\phi_i(j)}$ for $1 \leq j \leq |U_i|$, where $2 \leq \phi_i(1) < \phi_i(2) < \dots < \phi_i(|U_i|) \leq N_i$, for $i = 1, 2$.

If $|U_i|$ is odd, then for $1 \leq j \leq |U_i|$ and $j \neq \lceil |U_i|/2 \rceil$, let

$$f(u_{i,j}) = \begin{cases} j & \text{if } 1 \leq j \leq \lfloor |U_i|/2 \rfloor, \\ j - \lceil |U_i|/2 \rceil & \text{if } \lceil |U_i|/2 \rceil + 1 \leq j \leq |U_i|. \end{cases}$$

If $|U_i|$ is even, then for $1 \leq j \leq |U_i|$ and $j \notin \{|U_i|/2, |U_i|/2 + 1\}$, let

$$g(u_{i,j}) = \begin{cases} j & \text{if } 1 \leq j \leq |U_i|/2 - 1, \\ j - |U_i|/2 - 1 & \text{if } |U_i|/2 + 2 \leq j \leq |U_i|. \end{cases}$$

Observe that $1 \leq f(u_{i,j}) \leq \lfloor |U_i|/2 \rfloor$ and $1 \leq g(u_{i,j}) \leq |U_i|/2 - 1$. We now consider the following two cases.

Case 1. $\langle U_1 \rangle$ is connected. Then $\langle U_1 \rangle = (u_{1,1}, u_{1,2}, \dots, u_{1,|U_1|})$ is a path.

Subcase 1.1. $|U_1| \not\equiv |U_2| \pmod{2}$. If $|U_1|$ is odd, then let $c_0 : V(G) \rightarrow \mathbb{N}$ be a coloring such that $\{c_0(v) : v \in X\} = \{1, 2, \dots, k\}$ and

$$\begin{aligned} c_0(u_{1,i}) &= k + f(u_{1,i}) && \text{if } 1 \leq i \leq |U_1| \text{ and } i \neq \lceil |U_1|/2 \rceil, \\ c_0(u_{2,i}) &= \begin{cases} k + \lfloor |U_1|/2 \rfloor + i & \text{if } 1 \leq i \leq |U_2|/2, \\ k + \lfloor |U_1|/2 \rfloor - |U_2|/2 + i & \text{if } |U_2|/2 + 1 \leq i \leq |U_2|, \end{cases} \\ c_0(v) &= \lceil (n + k - \ell)/2 \rceil && \text{if } v \in \{u_{1,\lceil |U_1|/2 \rceil}\} \cup Y. \end{aligned}$$

If $|U_1|$ is even, then let $c_0 : V(G) \rightarrow \mathbb{N}$ be a coloring such that $\{c_0(v) : v \in X\} = \{1, 2, \dots, k\}$ and

$$\begin{aligned} c_0(u_{1,i}) &= k + g(u_{1,i}) && \text{if } 1 \leq i \leq |U_1| \text{ and } i \notin \{|U_1|/2, |U_1|/2 + 1\}, \\ c_0(u_{2,i}) &= k + |U_1|/2 - 1 + f(u_{2,i}) && \text{if } 1 \leq i \leq |U_2| \text{ and } i \neq \lceil |U_2|/2 \rceil, \\ c_0(v) &= \begin{cases} \lceil (n + k - \ell)/2 \rceil - 1 & \text{if } v \in \{u_{1,|U_1|/2}, u_{2,\lceil |U_2|/2 \rceil}\}, \\ \lceil (n + k - \ell)/2 \rceil & \text{if } v \in \{u_{1,|U_1|/2+1}\} \cup Y. \end{cases} \end{aligned}$$

One can verify that c_0 is a rainbow coloring of G using $\lceil (n + k - \ell)/2 \rceil$ colors. Therefore, $\text{vrc}(G) = \lceil (n + k - \ell)/2 \rceil$.

Subcase 1.2. $\langle U_2 \rangle$ is connected. We may assume that $|U_1| \equiv |U_2| \pmod{2}$. If both $|U_1|$ and $|U_2|$ are odd, then let $c_0 : V(G) \rightarrow \mathbb{N}$ be a coloring such that $\{c_0(v) : v \in X\} = \{1, 2, \dots, k\}$ and

$$\begin{aligned} c_0(u_{1,i}) &= k + f(u_{1,i}) && \text{if } 1 \leq i \leq |U_1| \text{ and } i \neq \lceil |U_1|/2 \rceil, \\ c_0(u_{2,i}) &= k + \lfloor |U_1|/2 \rfloor + f(u_{2,i}) && \text{if } 1 \leq i \leq |U_2| \text{ and } i \neq \lceil |U_2|/2 \rceil, \\ c_0(v) &= \lceil (n + k - \ell)/2 \rceil && \text{if } v \in \{u_{1, \lceil |U_1|/2 \rceil}, u_{2, \lceil |U_2|/2 \rceil}\} \cup Y. \end{aligned}$$

If both $|U_1|$ and $|U_2|$ are even, then let $c_0 : V(G) \rightarrow \mathbb{N}$ be a coloring such that $\{c_0(v) : v \in X\} = \{1, 2, \dots, k\}$ and

$$\begin{aligned} c_0(u_{1,i}) &= k + g(u_{1,i}) && \text{if } 1 \leq i \leq |U_1| \text{ and } i \notin \{|U_1|/2, |U_1|/2 + 1\}, \\ c_0(u_{2,i}) &= k + |U_1|/2 - 1 + g(u_{2,i}) && \text{if } 1 \leq i \leq |U_2| \text{ and } i \notin \{|U_2|/2, |U_2|/2 + 1\}, \\ c_0(v) &= \begin{cases} (n + k - \ell)/2 - 1 & \text{if } v \in \{u_{1, |U_1|/2}, u_{2, |U_2|/2}\}, \\ (n + k - \ell)/2 & \text{if } v \in \{u_{1, |U_1|/2 + 1}, u_{2, |U_2|/2 + 1}\} \cup Y. \end{cases} \end{aligned}$$

Then in each case c_0 is a rainbow coloring of G using $\lceil (n + k - \ell)/2 \rceil$ colors.

Subcase 1.3. $\langle U_2 \rangle$ is disconnected and $|U_1| \equiv |U_2| \pmod{2}$. Let c_0 be the coloring considered in Subcase 1.2 according to the parity of $|U_1|$ and $|U_2|$. The coloring $c : V(G) \rightarrow \mathbb{N}$ defined by $c(v) = c_0(v)$ for $v \in V(G) - Y$ and $c(v) = (n + k - \ell)/2 + 1$ for $v \in Y$ is a rainbow coloring of G . Thus $\text{vrc}(G) \leq (n + k - \ell)/2 + 1$.

Now assume, to the contrary, that there exists a rainbow coloring c of G using $(n + k - \ell)/2$ colors. We may then assume that $\{c(v) : v \in V(G) - X\} = \{1, 2, \dots, (n - k - \ell)/2\}$ and let $V_1, V_2, \dots, V_{(n - k - \ell)/2}$ be the color classes partitioning the set $V(G) - X$. Then $|V_i \cap V(C)| = 2$ for $1 \leq i \leq (n - k - \ell)/2$. Also, since $\langle U_2 \rangle$ is disconnected, we may assume that there are vertices $u_1 \in U_1$, $u_2, u'_2 \in U_2$ and an end-vertex $y \in Y$, as shown in Figure 3.

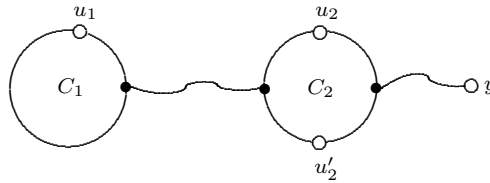


Figure 3. Illustrating the proof in Subcase 1.3.

If $c(y) = 1$, say, then observe that no two distinct vertices belonging to a common cycle can be colored 1. Therefore, we may assume that $c(u_1) = c(u_2) = 1$ and $c(u'_2) = 2$. Let $u \in U - \{u_1, u_2, u'_2\}$. Then observe that $c(u) \neq 2$ or c is not a rainbow coloring of G , which contradicts our assumption. Thus, there is no such c and so $\text{vrc}(G) = (n + k - \ell)/2 + 1$.

Case 2. Both $\langle U_1 \rangle$ and $\langle U_2 \rangle$ are disconnected. Then $|U_1|, |U_2| \geq 2$. To see that $\text{vrc}(G) \leq \lceil (n + k - \ell)/2 \rceil + 1$, let $c : V(G) \rightarrow \mathbb{N}$ be a coloring of G such that $c(v) = c_0(v)$ for $v \in V(G) - Y$, where c_0 is the coloring considered in Subcases 1.1 and 1.2 according to the parity of $|U_1|$ and $|U_2|$, and $c(v) = \lceil (n + k - \ell)/2 \rceil + 1$ for $v \in Y$. Then c is a rainbow coloring of G using $\lceil (n + k - \ell)/2 \rceil + 1$ colors.

If $|U_1| \equiv |U_2| \pmod{2}$, then the argument used in Subcase 1.3 shows that there is no rainbow coloring of G using $(n + k - \ell)/2$ colors. Therefore, suppose that $|U_1|$ is odd and $|U_2|$ is even and assume, to the contrary, that there exists a rainbow coloring c of G using $\lceil (n + k - \ell)/2 \rceil$ colors. Assume further that $\{c(v) : v \in V(G) - X\} = \{1, 2, \dots, \lceil (n - k - \ell)/2 \rceil\}$ and let $V_1, V_2, \dots, V_{\lceil (n - k - \ell)/2 \rceil}$ be the color classes partitioning the set $V(G) - X$. Then we may assume that $|V_1 \cap V(C)| = 1$ and $|V_i \cap V(C)| = 2$ for $2 \leq i \leq \lceil (n - k - \ell)/2 \rceil$. Since neither $\langle U_1 \rangle$ nor $\langle U_2 \rangle$ is connected, there are vertices $u_1, u'_1 \in U_1$, $u_2, u'_2 \in U_2$ and $y_1, y_2 \in Y$, as shown in Figure 4.

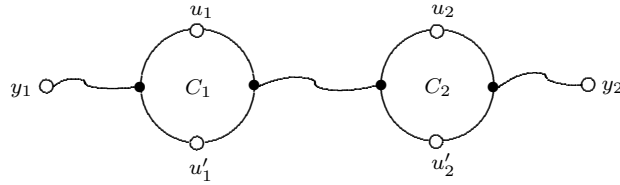


Figure 4. Illustrating the proof in Case 2.

First suppose that $c(u_1) = c(y_1) = c(y_2) = 1$ and $c(u'_1) = 2$. Then it is straightforward to verify that $u \in U$ and $c(u) = 2$ if and only if $u = u'_1$ in order for c to be a rainbow coloring. However, this contradicts our assumption that $|V_2 \cap V(C)| = 2$. Therefore, suppose that $c(y_1) \neq 1$, say $c(y_1) = 2$. Then we may assume that $c(u_1) = c(u_2) = 2$. Furthermore, if $u \in U$, then by the argument used in Subcase 1.3 we see that $c(u) = c(u'_1)$ if and only if $u = u'_1$ and so $c(u'_1) = 1$. Since there must be a rainbow $y_1 - y_2$ path, we may assume that $c(y_2) = 3$ and $c(u'_2) = 4$. However then, either at least one of $|V_3 \cap V(C)|$ and $|V_4 \cap V(C)|$ is less than 2 or c is not a rainbow coloring. Hence, $\text{vrc}(G) > \lceil (n + k - \ell)/2 \rceil$, which is the desired result. ■

We now turn to the vertex rainbow connection numbers of connected graphs having cycle rank 2 that are of type II. We first consider those graphs without end-vertices (and so without cut-vertices), that is, the subdivisions of $K_{2,3}$.

Theorem 3.5. *Let G be a connected graph of order n having cycle rank 2 and of type II containing no end-vertices. Let v_α and v_β be the vertices having degree 3 with three internally disjoint $v_\alpha - v_\beta$ paths P_1 , P_2 and P_3 such that $|V(P_1)| \geq$*

$|V(P_2)| \geq |V(P_3)|$. Then

$$\text{vrc}(G) = \begin{cases} \lceil n/2 \rceil & \text{if } |V(P_3)| = 2 \text{ and } |V(P_1)| > |V(P_2)|, \\ \lceil n/2 \rceil + 1 & \text{if } |V(P_3)| = 2 \text{ and } |V(P_1)| = |V(P_2)|, \\ \lfloor n/2 \rfloor & \text{otherwise.} \end{cases}$$

Proof. We consider two cases, according to the value of $|V(P_3)|$.

Case 1. $|V(P_3)| = 2$. Let G be constructed from a cycle $C = (v_1, v_2, \dots, v_n, v_1)$ by adding a chord $v_\alpha v_\beta$. We may assume that $\alpha = 1$ and $3 \leq \beta = |V(P_2)| \leq \lfloor n/2 \rfloor + 1$. Then the coloring $c : V(G) \rightarrow \mathbb{N}$ defined by

$$c(v_i) = \begin{cases} i & \text{if (i) } 1 \leq i \leq \lfloor n/2 \rfloor \text{ or} \\ & \text{(ii) } |V(P_1)| = |V(P_2)| \text{ and } i = n/2 + 1, \\ 1 & \text{if } i = \lfloor n/2 \rfloor + 1 \text{ and } |V(P_1)| > |V(P_2)|, \\ i - \lfloor n/2 \rfloor & \text{if } \lfloor n/2 \rfloor + 2 \leq i \leq n \end{cases}$$

is a rainbow coloring of G .

On the other hand, consider an arbitrary rainbow coloring of G resulting in s color classes V_1, V_2, \dots, V_s . Then $|V_i| \leq 2$ for $1 \leq i \leq s$ by Lemma 3.2, implying that $n \leq 2s$ and so $s \geq \lceil n/2 \rceil$. Thus, $\text{vrc}(G) \geq \lceil n/2 \rceil$. In particular, if $|V(P_1)| = |V(P_2)|$, then the order of G is even and so $\text{vrc}(G) \geq n/2$. Assume, to the contrary, that there exists a rainbow $(n/2)$ -coloring of G . Then every color class consists of two vertices. Suppose that $V_1 = \{v_\alpha, u_1\}$, where $u_1 \in V(P_1) - \{v_\alpha, v_\beta\}$. Then there must be a color class that is a subset of $V(P_2) - \{v_\alpha\}$, say $V_2 = \{u_2, u'_2\} \subseteq V(P_2) - \{v_\alpha\}$. However then, it is impossible to have both a rainbow $u_1 - u_2$ path and a rainbow $u_1 - u'_2$ path, which is a contradiction. Therefore, $\text{vrc}(G) \geq n/2 + 1$ if $|V(P_1)| = |V(P_2)|$, which verifies the result in this case.

Case 2. $|V(P_3)| \geq 3$. We first show that $\text{vrc}(G) \geq \lfloor n/2 \rfloor$. Consider an arbitrary rainbow coloring resulting in s color classes V_1, V_2, \dots, V_s . Then $|V_i| \leq 3$ for $1 \leq i \leq s$ by Lemma 3.2. Furthermore, if $|V_1| = 3$ and $V_1 = \{v_1, v_2, v_3\}$, say, then we may assume that $v_i \in V(P_i) - \{v_\alpha, v_\beta\}$ for $1 \leq i \leq 3$. We claim that if $|V_1| = 3$, then $|V_i| \leq 2$ for $2 \leq i \leq s$. If this is not the case, then suppose that $V_2 = \{u_1, u_2, u_3\}$, where $u_i \in V(P_i) - \{v_\alpha, v_\beta\}$ for $1 \leq i \leq 3$. We may further assume that there are two distinct integers $i_1, i_2 \in \{1, 2, 3\}$ such that the unique $u_{i_1} - u_{i_2}$ path not containing v_α contains neither v_{i_1} nor v_{i_2} . However, this implies that there is no rainbow $u_{i_1} - v_{i_2}$ path, which cannot occur. Therefore, $|V_1| \leq 3$ and $|V_i| \leq 2$ for $2 \leq i \leq s$. This then implies that $n \leq 3 + 2(s - 1)$, that is, $s \geq \lfloor n/2 \rfloor$. Hence, $\text{vrc}(G) \geq \lfloor n/2 \rfloor$.

We next show that there exists a rainbow $\lfloor n/2 \rfloor$ -coloring of G . Let $|V(P_1)| + |V(P_2)| - 2 = N$ ($\leq n - 1$), which is the *circumference* of G (the length of a

largest cycle in G) by assumption. Then G is constructed from a cycle $C = (v_1, v_2, \dots, v_N, v_1)$ and a path $P = (u_1, u_2, \dots, u_{n-N})$, where $V(C) \cap V(P) = \emptyset$, by adding the edges $v_1 u_1$ and $v_{\lfloor N/2 \rfloor} u_{n-N}$. Define the coloring $c : V(G) \rightarrow \mathbb{N}$ by

$$c(v_i) = \begin{cases} i & \text{if } 1 \leq i \leq \lfloor N/2 \rfloor, \\ 1 & \text{if } i = \lfloor N/2 \rfloor + 1, \\ i - 1 & \text{if } (N \text{ is odd or } n \geq N + 2) \text{ and} \\ & \lfloor N/2 \rfloor + 2 \leq i \leq \lfloor n/2 \rfloor + 1, \\ i - \lceil N/2 \rceil & \text{if } \lfloor n/2 \rfloor + 2 \leq i \leq N, \end{cases}$$

$$c(u_i) = \begin{cases} i + \lfloor N/2 \rfloor & \text{if } (N \text{ is odd or } n \geq N + 2) \text{ and} \\ & 1 \leq i \leq \lfloor n/2 \rfloor - \lfloor N/2 \rfloor, \\ n - N + 2 - i & \text{if } (N \text{ is even or } n \geq N + 2) \text{ and} \\ & \lfloor n/2 \rfloor - \lfloor N/2 \rfloor + 1 \leq i \leq n - N. \end{cases}$$

Observe that c is an $\lfloor n/2 \rfloor$ -coloring. Since the restriction of c to each of the three cycles in G is a rainbow coloring of that cycle, it follows that c is a rainbow coloring of G . Therefore, $\text{vrc}(G) \leq \lfloor n/2 \rfloor$, concluding that $\text{vrc}(G) = \lfloor n/2 \rfloor$. ■

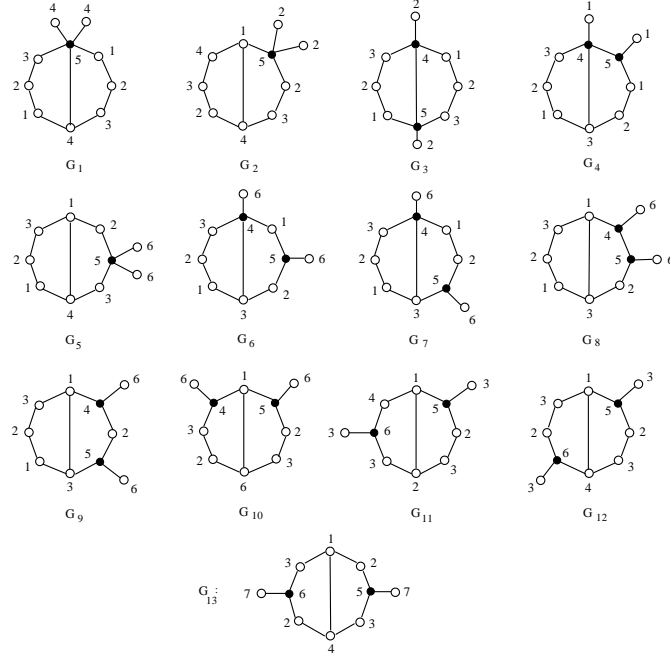


Figure 5. Type II graphs G_i ($1 \leq i \leq 13$) of order 10 containing H as a subgraph.

Let us next consider graphs of type II with end-vertices and cut-vertices. For example, let H be the graph of order 8 obtained from an 8-cycle by joining two

antipodal vertices. Hence, $\text{vrc}(H) = 5$ by Theorem 3.5. Now let G be a graph obtained from H by adding two pendant edges. Therefore, $n = 10$, $k = 1$ or $k = 2$, and $\ell = 2$. There are 13 such graphs (up to symmetry), all shown in Figure 5. It turns out that the vertex rainbow connection number of each of these graphs can be expressed as $\lceil (n + k - \ell)/2 \rceil + \delta = 5 + \delta$, where $\delta \in \{0, 1, 2\}$. In particular, $\text{vrc}(G) = 5$ if and only if $G = G_i$ ($1 \leq i \leq 4$) while $\text{vrc}(G) = 7$ if and only if $G = G_{13}$. For each graph G_i ($1 \leq i \leq 11$), a rainbow coloring using $\text{vrc}(G_i)$ colors is shown in Figure 5.

We conclude with the following conjecture.

Conjecture 3.6. *If G is a connected type II graph of order n containing k cut-vertices and ℓ end-vertices, where $k, \ell \geq 1$, then*

$$(3) \quad \text{vrc}(G) = \lceil (n + k - \ell)/2 \rceil + \delta, \text{ where } \delta \in \{0, 1, 2\}.$$

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