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EMBEDDINGS OF HAMILTONIAN PATHS IN FAULTY k-ARY 2-CUBES [1]

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Abstract

It is well known that the k-ary n-cube has been one of the most efficient interconnection networks for distributed-memory parallel systems. A k-ary n-cube is bipartite if and only if k is even. Let (X, Y) be a bipartition of a k-ary 2-cube (even integer $k \geq 4$). In this paper, we prove that for any two healthy vertices $u \in X,\ v \in Y$, there exists a hamiltonian path from u to v in the faulty k-ary 2-cube with one faulty vertex in each part.

Keywords: complex networks, path embeddings, fault-tolerance, k-ary n-cubes .

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1. Introduction

The k-ary n-cube has many desired properties, such as easy of implementation, low-latency and high-bandwidth interprocessor communication. Therefore, a large number of distributed-memory parallel systems (also known as multicomputers) have been built with a k-ary n-cube forming the underling topology, such as the iWarp [12], the J-machine [11] and the Cray T3D [9]. An interconnection network is often modeled as a graph, in which vertices and edges correspond to nodes and communication links, respectively. The k-ary n-cube, denoted by Q_n^k ($k \geq 2$ and $n \geq 1$), is a graph consisting of k^n vertices, each of which has the form $u = u_{n-1}u_{n-2} \ldots u_0$, where $0 \leq u_i \leq k-1$ for $0 \leq i \leq n-1$. Two vertices

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$u = u_{n-1}u_{n-2}\dots u_0$ and $v = v_{n-1}v_{n-2}\dots v_0$ are adjacent if and only if there exists an integer j, $0 \le j \le n-1$, such that $u_j = v_j \pm 1 (\mathrm{mod}\ k)$ and $u_i = v_i$, for every $i \in \{1, 2, \dots, n\} \setminus \{j\}$. For clarity of presentation, we omit writing "(mod k)" in similar expressions for the remainder of the paper.

The graph embedding is a technique that maps a guest graph into a host graph. Many graph embeddings take paths and cycles as guest graphs because they are the common structures used to model linear arrays in parallel processing [2, 4, 15, 16, 17]. In recent years, the problem of path embeddings in an interconnection network has attracted a great deal of attention from the researchers. Since failures are inevitable, fault-tolerant is an important issue in the distributed-memory parallel system. Many works related to embeddings of the longest paths in various faulty interconnection networks have been studied previously, including hypercubes [3, 5, 7, 10, 14, 16, 19], k-ary n-cubes [1, 15, 17, 19] and stars [6, 13]. In particular, Yang *et al.* [19] proved that for arbitrary two healthy vertices of Q_n^k with odd $k \ge 3$, there exists a fault-free hamiltonian path connecting these two vertices if the number of faults is at most $2n-3$.

The parity of a vertex $u = u_{n-1}u_{n-2}\dots u_0$ of Q_n^k is defined to be $u_{n-1} + u_{n-2} + \dots + u_0$ modulo 2. We speak of a vertex as being odd or even according to whether its parity is odd or even. Given any two distinct vertices u and v. Let

$$e_{u,v} = \begin{cases} 1, & \text{if } u \text{ and } v \text{ have different parities,} \\ 0, & \text{if } u \text{ and } v \text{ have the same parity.} \end{cases}$$

For even $k \ge 4$, Stewart and Xiang [15] studied the problem of embedding long paths in the k-ary n-cube with faulty vertices and edges. They presented the following result.

Theorem 1.1 [15]**.** *Let $k \ge 4$ be even and let f_v be the number of faulty vertices and f_e be the number of faulty edges in Q_2^k with $0 \le f_v + f_e \le 2$. Given any two healthy vertices u and v of Q_2^k, then there is a path from u to v of length at least $k^2 - 2f_v - 1$ if $e_{u,v} = 1$.*

Let X be the set of even vertices and Y be the set of odd vertices of a Q_2^k with even $k \ge 4$. Obviously, (X, Y) is a bipartition of the Q_2^k. We denote the set of faulty vertices of the Q_2^k by F_v. Let $f_v^{max} = \max\{|F_v \cap X|, |F_v \cap Y|\}$. In this paper, we prove that there is a path from u to v in the faulty Q_2^k of length $k^2 - 2f_v^{max} - 1$ if $e_{u,v} = 1$. As $|F_v \cap X| + |F_v \cap Y| = f_v$, we have $f_v^{max} \le f_v$. Obviously, $k^2 - 2f_v^{max} - 1 \ge k^2 - 2f_v - 1$. Therefore, our result improves the result noted above.

The rest of this paper is organized as follows. In the next section, some basic definitions are introduced. In Section 3, we construct a hamiltonian path connecting any two healthy verities in different parts in the faulty k-ary 2-cube (even $k \ge 4$) with one faulty vertex in each part. Conclusions are covered in Section 4.

2. Basis Definition

Throughout this paper, we restrict our attention to $n = 2$ and even $k \geq 4$. For convenience, we write $v_{a,b}$ as the vertex of Q_2^k with the form $v_1v_0 = ab$, where $0 \leq a, b \leq k-1$. For $0 \leq i \leq j \leq k-1$, $Row(i:j)$ of Q_2^k is the subgraph of Q_2^k induced by $\{v_{a,b} : i \leq a \leq j,\ 0 \leq b \leq k-1$, $Col(i:j)$ of Q_2^k is the subgraph of Q_2^k induced by $\{v_{a,b} : 0 \leq a \leq k-1, i \leq b \leq j\}$.

Given $1 \leq k_1, k_2 \leq k-1$, the subgraph of Q_2^k induced by $\{v_{a,b} : 0 \leq a \leq k_1 - 1, 0 \leq b \leq k_2 - 1\}$ is denoted by $Grid(k_1, k_2)$. A vertex of $Grid(k_1, k_2)$ is called a *corner vertex* if its degree in $Grid(k_1, k_2)$ is 2. For $0 \leq i \leq j \leq k_1 - 1$, $Row(i:j)$ of $Grid(k_1, k_2)$ is the subgraph of $Grid(k_1, k_2)$ induced by $\{v_{a,b} : i \leq a \leq j, 0 \leq b \leq k_2 - 1\}$. For $0 \leq i \leq j \leq k_2 - 1$, $Col(i:j)$ of $Grid(k_1, k_2)$ is the subgraph of $Grid(k_1, k_2)$ induced by $\{v_{a,b} : 0 \leq a \leq k_1 - 1, i \leq b \leq j\}$.

Instead of $Row(i:i)$ and $Col(j:j)$ of Q_2^k (resp. $Grid(k_1, k_2)$) we simply write $Row(i)$ and $Col(j)$ of Q_2^k (resp. $Grid(k_1, k_2)$). $Row(0:2)$ of Q_2^4 and $Grid(2,4)$ are shown in Figure 1 and Figure 2, respectively.

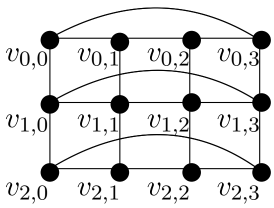

Figure 1. $Row(0:2)$ of Q_2^4

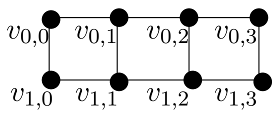

Figure 2. $Grid(2,4)$

Choose a vertex $u = v_{a,b}$ ($0 \leq a, b \leq k-1$) in $Row(a)$ of Q_2^k. The neighbour of u in $Row(a-1)$ (resp. $Row(a+1)$) is denoted by $n^{a-1}(u)$ (resp. $n^{a+1}(u)$), that is, $n^{a-1}(u) = v_{a-1,b}$ (resp. $n^{a+1}(u) = v_{a+1,b}$).

3. Path Embeddings in Faulty k-ary 2-cubes

We start with some useful lemmas.

Lemma 3.1 [8]**.** *Given an integer $n \geq 1$, let u be a corner vertex of $Grid(2, n)$. For any vertex $v \neq u$ in $Grid(2, n)$ such that $e_{u,v} = 1$, there exists a hamiltonian path of $Grid(2, n)$ from u to v.*

Lemma 3.2 [8]**.** *Given even $k_1, k_2 \geq 2$, let u and v be vertices in $Row(0)$ and $Row(k_1 - 1)$ of $Grid(k_1, k_2)$, respectively. If at least one of u and v is a corner vertex of $Grid(k_1, k_2)$ and $e_{u,v} = 1$, then there is a hamiltonian path of $Grid(k_1, k_2)$ from u to v.*

In [15], Stewart and Xiang constructed the long paths in $Row(0 : p-1)$ of Q_2^k (even $k \geq 4$), where $2 \leq p \leq k$. They present the following result.

Lemma 3.3 [15]**.** *Given an even $k \geq 4$, let u and v be any two distinct healthy vertices in $Row(0 : p-1)$ of Q_2^k, where $2 \leq p \leq k$. If $e_{u,v} = 1$, then there exists a hamiltonian path of $Row(0 : p-1)$ from u to v that contains at least one healthy edge of $Row(0)$.*

According to the proof of Lemma 1 in [15], we have the following lemma.

Lemma 3.4 [15]**.** *Given an even $k \geq 4$, let u and v be any two distinct healthy vertices and x be a faulty vertex in $Row(0 : 1)$ of Q_2^k. If $e_{x,u} = 1$ and $e_{u,v} = 0$, then there exists a hamiltonian path of $Row(0 : 1) - x$ from u to v that contains at least one healthy edge of $Row(1)$.*

Lemma 3.5. *Given an even $k \geq 4$, let x be the only faulty vertex in $Row(0 : 1)$ of Q_2^k and let u, v be any two distinct healthy vertices in $Row(0 : p-1)$ of Q_2^k such that $e_{x,u} = 1$ and $e_{u,v} = 0$, where p is even and $4 \leq p \leq k$. Then there exists a hamiltonian path of $Row(0 : p-1) - x$ from u to v if one of the following holds.*
(i) *$u, v \in V(Row(0 : 1))$.*
(ii) *$u, v \in V(Row(p-1))$.*
(iii) *$u \in V(Row(0 : 1))$ and $v \in V(Row(p-1))$.*

Proof. Suppose that $u, v \in V(Row(0 : 1))$. As $x \in V(Row(0 : 1))$ and $e_{x,u} = 1$, $e_{u,v} = 0$, Lemma 3.4 implies that there is a hamiltonian path P_1 of $Row(0 : 1) - x$ from u to v that contains an edge (s,t) of $Row(1)$. As $e_{n^2(s),n^2(t)} = 1$, by Lemma 3.3, there is a hamiltonian path P_2 of $Row(2 : p-1)$ from $n^2(s)$ to $n^2(t)$. Then, $P_1 \cup P_2 - \{(s,t)\} + \{(s, n^2(s)), (t, n^2(t))\}$ is a hamiltonian path of $Row(0 : p-1) - x$ from u to v.

Suppose that $u, v \in V(Row(p-1))$. Let $u = v_{p-1,j}, v = v_{p-1,j'}$, where $0 \leq j, j' \leq k-1$ and $j \neq j'$. Without loss of generality, we assume that $j < j'$. Let $q \in \{j, j+1, j+2, \dots, j'\}$ be odd and let $G_1 = Row(2 : p-1) \cap Col(0 : q)$ and $G_2 = Row(2 : p-1) \cap Col(q+1 : k-1)$. Obviously, $u \in V(G_1)$ and $v \in V(G_2)$. As q is odd, we have $e_{v_{2,0},v_{2,q}} = e_{v_{2,q+1},v_{2,k-1}} = 1$. Thus one of $e_{u,v_{2,0}} = 1$ and $e_{u,v_{2,q}} = 1$ holds. Without loss of generality, we may assume that $e_{u,v_{2,0}} = 1$. As G_1 is isomorphic to $Grid(p-2, q+1)$ and $v_{2,0}$ is a corner vertex of G_1, Lemma 3.2 implies that there is a hamiltonian path P_1 of G_1 from $v_{2,0}$ to u. As $e_{u,v} = 0$, it is easy to see that $e_{v_{2,q+1},v} = 1$. As G_2 is isomorphic to $Grid(p-2, k-q-1)$ and $v_{2,q+1}$ is a corner vertex of G_2, Lemma 3.2 implies that there is a hamiltonian path P_2 of G_2 from $v_{2,q+1}$ to v. As $e_{v_{2,0},u} = 1$, we have $e_{v_{1,0},u} = 0$. Combining this with the fact that $e_{x,u} = 1$ and q is odd, we see that $e_{x,v_{1,0}} = 1$ and $e_{v_{1,0},v_{1,q+1}} = 0$. By Lemma 3.4, there is a hamiltonian path P_3 of $Row(0 : 1) - x$ from $v_{1,0}$ to

$v_{1,q+1}$. Therefore $P_1 \cup P_2 \cup P_3 + \{(v_{1,0}, v_{2,0}), (v_{1,q+1}, v_{2,q+1})\}$ is a hamiltonian path of $Row(0 : p-1) - x$ from u to v.

Suppose that $u \in V(Row(0 : 1))$ and $v \in V(Row(p-1))$. As $k \geq 4$, we may choose a vertex $s \in V(Row(1))$ such that $s \neq u$ and $e_{u,s} = 0$. Clearly $e_{x,s} = 1$. By Lemma 3.4, there is a hamiltonian path P_1 of $Row(0 : 1) - x$ from u to s. As $e_{u,s} = e_{u,v} = 0$ and $e_{s,n^2(s)} = 1$, we have $e_{n^2(s),v} = 1$. By Lemma 3.3, there is a hamiltonian path P_2 of $Row(2 : p-1)$ from $n^2(s)$ to v. Then, $P_1 \cup P_2 + \{(s, n^2(s))\}$ is a hamiltonian path of $Row(0 : p-1) - x$ from u to v. The proof is complete. ■

Given a graph G, let S and T be two subsets of $V(G)$. An (S,T)-*path* is a path which starts at a vertex of S, ends at a vertex of T, and whose internal vertices belong to neither S nor T.

Lemma 3.6. *Given an even $k \geq 4$, let $S = \{u, v\}$ be a set of two distinct vertices in $Row(0 : 1) - v_{0,0}$ of Q_2^k and let $T = \{v_{0,1}, v_{1,0}\}$. If $e_{u,v} = 1$, then there exists two vertex-disjoint (S,T)-paths in $Row(0 : 1) - v_{0,0}$ that contain all vertices of $Row(0 : 1) - v_{0,0}$.*

Proof. As $e_{u,v} = 1$, without loss of generality, assume that u is even and v is odd. We consider the following two cases. In each case, we will construct two vertex-disjoint (S,T)-paths P_1 and P_2 in $Row(0 : 1) - v_{0,0}$.

Case 1. $v = v_{1,0}$. In this case, u is in $G_1 = Row(0 : 1) \cap Col(1 : k-1)$ which is isomorphic to $Grid(2, k-1)$. As $v_{0,1}$ is odd and u is even, we have $e_{v_{0,1},u} = 1$. Combining this with the fact that $v_{0,1}$ is a corner vertex of G_1, Lemma 3.1 implies that there is a hamiltonian path P_1 of G_1 from u to $v_{0,1}$. Let $P_2 = v$. Clearly, P_1 and P_2 are vertex-disjoint (S,T)-paths in $Row(0 : 1)$ that contain all vertices of $Row(0 : 1) - v_{0,0}$.

Case 2. $v \neq v_{1,0}$. In this case, u and v are in $Row(0 : 1) \cap Col(1 : k-1)$. Let $u = v_{i,j}$ and $v = v_{i',j'}$, where $0 \leq i, i' \leq 1$ and $1 \leq j, j' \leq k-1$. Without loss of generality, we may assume that $j \leq j'$.

Suppose first that $j \neq j'$. Let $G_1 = Row(0 : 1) \cap Col(1 : j)$ and $G_2 = Row(0 : 1) \cap Col(j+1 : k-1)$. Observe that G_1 is isomorphic to $Grid(2, j)$ and G_2 is isomorphic to $Grid(2, k-j-1)$. As $v_{0,1}$ is a corner vertex of G_1, $v_{1,k-1}$ is a corner vertex of G_2 and $e_{v_{0,1},u} = 1$, $e_{v_{1,k-1},v} = 1$, Lemma 3.1 implies that G_1 has a hamiltonian path P_1 from u to $v_{0,1}$ and G_2 has a hamiltonian path P_2^1 from v to $v_{1,k-1}$. Let $P_2 = P_2^1 + \{(v_{1,k-1}, v_{1,0})\}$. Then P_1 and P_2 are vertex-disjoint (S,T)-paths in $Row(0 : 1) - v_{0,0}$ that contain all vertices of $Row(0 : 1) - v_{0,0}$.

Suppose next that $j = j'$. If $2 \leq j = j' \leq k-2$, let $G_1 = Row(0 : 1) \cap Col(1 : j'-1)$ and $G_2 = Row(0 : 1) \cap Col(j+1 : k-1)$. Recall that $u = v_{i,j}$ is even and $v = v_{i',j'}$ is odd. Then $e_{v_{0,1},v_{i',j'-1}} = 1$ and $e_{v_{1,k-1},v_{i,j+1}} = 1$. Observe that G_1 and G_2 are isomorphic to $Grid(2, j'-1)$ and $Grid(2, k-j-1)$, respectively. By

Lemma 3.1, there is a hamiltonian path P_1^1 of G_1 from $v_{i',j'-1}$ to $v_{0,1}$ and there is a hamiltonian path P_2^1 of G_2 from $v_{i,j+1}$ to $v_{1,k-1}$. Let $P_1 = P_1^1 + \{(v, v_{i',j'-1})\}$ and $P_2 = P_2^1 + \{(u, v_{i,j+1}), (v_{1,k-1}, v_{1,0})\}$. If $j = j' = 1$, then $u = v_{1,1}$, $v = v_{0,1}$. Let $P_1 = v$ and $P_2 = P_2^1 + \{(u, v_{1,2}), (v_{1,k-1}, v_{1,0})\}$. If $j = j' = k-1$, then $u = v_{1,k-1}$, $v = v_{0,k-1}$. Let $P_1 = P_1^1 + \{(v, v_{0,k-2})\}$ and $P_2 = uv_{1,0}$. Therefore, P_1 and P_2 are as required. ■

Lemma 3.7. *Let odd $v_{a,b}$ and odd $v_{a',b'}$ be two distinct vertices in $Row(0:1)$ of Q_2^4 and let $S = \{v_{1,1}, v_{1,3}\}$ and $T = \{v_{a',b'}, v_{0,2}\}$. Then there exist two vertex-disjoint (S,T)-paths in $Row(0:1) - v_{a,b}$ that contain all vertices of $Row(0:1) - v_{a,b}$.*

Proof. We distinguish four cases. In each case, we will construct two vertex-disjoint (S,T)-paths P_1 and P_2 in $Row(0:1) - v_{a,b}$.

Case 1. $v_{a,b}, v_{a',b'} \in V(Col(0:1))$. In this case $v_{a,b}, v_{a',b'} \in \{v_{0,1}, v_{1,0}\}$. Let $P_1 = v_{1,1}v_{1,2}v_{0,2}$. Then P_1 is a path from $v_{1,1}$ to $v_{0,2}$. If $v_{a,b} = v_{0,1}$ and $v_{a',b'} = v_{1,0}$, let $P_2 = v_{1,3}v_{0,3}v_{0,0}v_{1,0}$. If $v_{a,b} = v_{1,0}$ and $v_{a',b'} = v_{0,1}$, let $P_2 = v_{1,3}v_{0,3}v_{0,0}v_{0,1}$. Then P_2 is a path from $v_{1,3}$ to $v_{a',b'}$. Therefore, there exist two vertex-disjoint (S,T)-paths in $Row(0:1) - v_{a,b}$ that contain all vertices of $Row(0:1) - v_{a,b}$.

Case 2. $v_{a,b}, v_{a',b'} \in V(Col(2:3))$. In this case $v_{a,b}, v_{a',b'} \in \{v_{0,3}, v_{1,2}\}$. Let $P_1 = v_{1,1}v_{1,0}v_{0,0}v_{0,1}v_{0,2}$. Then P_1 is a path from $v_{1,1}$ to $v_{0,2}$. If $v_{a,b} = v_{0,3}$ and $v_{a',b'} = v_{1,2}$, let $P_2 = v_{1,3}v_{1,2}$. If $v_{a,b} = v_{1,2}$ and $v_{a',b'} = v_{0,3}$, let $P_2 = v_{1,3}v_{0,3}$. Then P_2 is a path from $v_{1,3}$ to $v_{a',b'}$. Therefore, P_1 and P_2 are as required.

Case 3. $v_{a',b'} \in V(Col(0:1))$ and $v_{a,b} \in V(Col(2:3))$. In this case $v_{a',b'} \in \{v_{0,1}, v_{1,0}\}$ and $v_{a,b} \in \{v_{0,3}, v_{1,2}\}$. If $v_{a',b'} = v_{0,1}$, let $P_1 = v_{1,1}v_{1,0}v_{0,0}v_{0,1}$. If $v_{a',b'} = v_{1,0}$, let $P_1 = v_{1,1}v_{0,1}v_{0,0}v_{1,0}$. Then P_1 is a path from $v_{1,1}$ to $v_{a',b'}$. Suppose first that $v_{a,b} = v_{0,3}$. Let $P_2 = v_{1,3}v_{1,2}v_{0,2}$. Suppose next that $v_{a,b} = v_{1,2}$. Let $P_2 = v_{1,3}v_{0,3}v_{0,2}$. Then P_2 is a path from $v_{1,3}$ to $v_{0,2}$. Therefore, P_1 and P_2 are as required.

Case 4. $v_{a,b} \in V(Col(0:1))$ and $v_{a',b'} \in V(Col(2:3))$. In this case $v_{a,b} \in \{v_{0,1}, v_{1,0}\}$ and $v_{a',b'} \in \{v_{0,3}, v_{1,2}\}$. If $v_{a,b} = v_{0,1}$, let $P_1^1 = v_{1,1}v_{1,0}v_{0,0}v_{0,3}$. If $v_{a,b} = v_{1,0}$, let $P_1^1 = v_{1,1}v_{0,1}v_{0,0}v_{0,3}$.

Suppose first that $v_{a',b'} = v_{0,3}$. Let $P_1 = P_1^1$ and let $P_2 = v_{1,3}v_{1,2}v_{0,2}$. Then P_1 is a path from $v_{1,1}$ to $v_{0,3} = v_{a',b'}$ and P_2 is path from $v_{1,3}$ to $v_{0,2}$. Suppose next that $v_{a',b'} = v_{1,2}$. Let $P_1 = P_1^1 + \{(v_{0,3}, v_{0,2})\}$ and let $P_2 = v_{1,3}v_{1,2}$. Then P_1 is a path from $v_{1,1}$ to $v_{0,2}$ and P_2 is path from $v_{1,3}$ to $v_{1,2} = v_{a',b'}$. Therefore, P_1 and P_2 are as required. ■

Lemma 3.8. *Given an even $k \geq 6$, let $S = \{v_{1,1}, v_{1,k-1}\}$ and let odd $v_{a,b}$ and odd $v_{a',b'}$ be two distinct vertices in $Row(0:1)$ of Q_2^k. Then there exists a set*

$T = \{v_{a',b'}, v_{0,c}\}$ (c is even) such that there are two vertex-disjoint (S,T)-paths in $Row(0:1) - v_{a,b}$ that contain all vertices of $Row(0:1) - v_{a,b}$.

Proof. Without loss of generality, we assume that $v_{a,b}$ is in $Col(0 : \frac{k}{2})$. We distinguish four cases. In each case, we will construct two vertex-disjoint (S,T)-paths P_1 and P_2 in $Row(0:1) - v_{a,b}$.

Case 1. $v_{a,b} \in V(Col(0:1))$ and $v_{a',b'} \in V(Col(0:1))$. As both $v_{a,b}$ and $v_{a',b'}$ are odd, we have $v_{a,b}, v_{a',b'} \in \{v_{0,1}, v_{1,0}\}$. Let $v_{0,c} = v_{0,2}$. We will construct an (S,T)-path P_1 from $v_{1,1}$ to $v_{0,c}$ and an (S,T)-path P_2 from $v_{1,k-1}$ to $v_{a',b'}$. Let $G_1 = Row(0:1) \cap Col(3:k-1)$. Observe that G_1 is isomorphic to $Grid(2, k-3)$.

Let $P_1 = v_{1,1}v_{1,2}v_{0,2}$. As $v_{1,k-1}$ is a corner vertex of G_1 and $e_{v_{0,k-1},v_{1,k-1}} = 1$, Lemma 3.1 implies that there is a hamiltonian path P_2^1 of G_1 from $v_{1,k-1}$ to $v_{0,k-1}$. Then $P_2 = P_2^1 + \{(v_{0,k-1}, v_{0,0}), (v_{0,0}, v_{a',b'})\}$ is as required.

Case 2. $v_{a,b} \in V(Col(0:1))$ and $v_{a',b'} \in V(Col(2:k-1))$. In this case, let $v_{0,c} = v_{0,0}$. As the odd $v_{a,b}$ is in $G_1 = Row(0:1) \cap Col(0:1)$, we have $v_{a,b} \in \{v_{0,1}, v_{1,0}\}$. If $v_{a,b} = v_{0,1}$, let $P_1 = v_{1,1}v_{1,0}v_{0,0}$. If $v_{a,b} = v_{1,0}$, let $P_1 = v_{1,1}v_{0,1}v_{0,0}$. Then P_1 is a hamiltonian path of $G_1 - v_{a,b}$ from $v_{1,1}$ to $v_{0,c}$. Observe that $G_2 = Row(0:1) \cap Col(2:k-1)$ is isomorphic to $Grid(2, k-2)$. Combining this with the fact that $v_{1,k-1}$ is a corner vertex of G_2 and $e_{v_{a',b'},v_{1,k-1}} = 1$, there is a hamiltonian path P_2 of G_2 from $v_{1,k-1}$ to $v_{a',b'}$. It can be seen that P_1 and P_2 are vertex-disjoint (S,T)-paths in $Row(0:1) - v_{a,b}$ that contain all vertices of $Row(0:1) - v_{a,b}$.

Case 3. $v_{a,b} \in V(Col(2 : \frac{k}{2}))$ and $v_{a',b'} \in V(Col(0:1))$. As $G_1 = Row(0:1) \cap Col(0:1)$ is isomorphic to $Grid(2,2)$, $v_{1,1}$ is a corner vertex of G_1 and $e_{v_{a',b'},v_{1,1}} = 1$, Lemma 3.1 implies that there is a hamiltonian path P_1 of G_1 from $v_{1,1}$ to $v_{a',b'}$. Let $v_{0,c} = v_{0,2}$. It is enough to construct a hamiltonian path P_2 of $G_2 - v_{a,b} = Row(0:1) \cap Col(2:k-1) - v_{a,b}$ from $v_{1,k-1}$ to $v_{0,c} = v_{0,2}$.

Suppose first that $v_{a,b}$ is in $Col(2)$. Then $v_{a,b} = v_{1,2}$. As $Row(0:1) \cap Col(3:k-1)$ is isomorphic to $Grid(2, k-3)$, $v_{0,3}$ is a corner vertex of $Row(0:1) \cap Col(3:k-1)$ and $e_{v_{0,3},v_{1,k-1}} = 1$, Lemma 3.1 implies that there is a hamiltonian path P_2^1 of $Row(0:1) \cap Col(3:k-1)$ from $v_{1,k-1}$ to $v_{0,3}$. Then $P_2 = P_2^1 + \{(v_{0,3}, v_{0,2})\}$ is as required.

Suppose next that $v_{a,b}$ is not in $Col(2)$. Then $Row(0:1) \cap Col(2:b-1)$ is isomorphic to $Grid(2, b-2)$ and $Row(0:1) \cap Col(b+1:k-1)$ is isomorphic to $Grid(2, k-b-1)$. If $a = 0$ then $\bar{a} = 1$, and if $a = 1$ then $\bar{a} = 0$. As $v_{a,b}$ is odd, it can be seen that both $v_{\bar{a},b-1}$ and $v_{\bar{a},b+1}$ are odd. Thus $e_{v_{0,2},v_{\bar{a},b-1}} = 1$ and $e_{v_{1,k-1},v_{\bar{a},b+1}} = 1$. As $v_{0,2}$ is a corner vertex of $Row(0:1) \cap Col(2:b-1)$ and $v_{1,k-1}$ is a corner vertex of $Row(0:1) \cap Col(b+1:k-1)$, Lemma 3.1 implies that there is a hamiltonian path P_2^1 of $Row(0:1) \cap Col(2:b-1)$ from $v_{\bar{a},b-1}$ to $v_{0,2}$ and a hamiltonian path P_2^2 in $Row(0:1) \cap Col(b+1:k-1)$ from $v_{1,k-1}$ to

$v_{\bar{a},b+1}$. Combining P_2^1 with P_2^2 as well as the edges $(v_{\bar{a},b-1}, v_{\bar{a},b})$ and $(v_{\bar{a},b}, v_{\bar{a},b+1})$, we may obtain the required path P_2.

Case 4. $v_{a,b} \in V(Col(2 : \frac{k}{2}))$ and $v_{a',b'} \in V(Col(2 : k-1))$.

Case 4.1. $v_{a,b}$ is in $Row(0)$, that is, $v_{a,b} = v_{0,b}$. Suppose first that $b' > b$. As b is odd, we have that $v_{1,b-1}$ is odd and $v_{0,b+1}$ is even. Let $v_{0,c} = v_{0,b+1}$. Observe that $G_1 = Row(0:1) \cap Col(0:b-1)$ is isomorphic to $Grid(2,b)$ and $G_2 = Row(0:1) \cap Col(b+2:k-1)$ is isomorphic to $Grid(2,k-b-2)$. As $v_{1,b-1}$ is a corner vertex of G_1 and $e_{v_{1,1},v_{1,b-1}} = 1$, there is a hamiltonian path P_1^1 of G_1 from $v_{1,1}$ to $v_{1,b-1}$. If $v_{a',b'} = v_{1,b+1}$, let $P_1 = P_1^1 + \{(v_{1,b-1}, v_{1,b}), (v_{1,b}, v_{1,b+1})\}$. As $v_{1,k-1}$ is a corner vertex of G_2 and $e_{v_{1,k-1},v_{0,b+2}} = 1$, there is a hamiltonian path P_2^1 of G_2 from $v_{1,k-1}$ to $v_{0,b+2}$. Let $P_2 = P_2^1 + \{(v_{0,b+2}, v_{0,b+1})\}$. Then P_1 is an (S,T)-path from $v_{1,1}$ to $v_{a',b'}$ and P_2 is an (S,T)-path from $v_{1,k-1}$ to $v_{0,b+1} = v_{0,c}$. If $v_{a',b'} \neq v_{1,b+1}$, let $P_1 = P_1^1 + \{(v_{1,b-1}, v_{1,b}), (v_{1,b}, v_{1,b+1}), (v_{1,b+1}, v_{0,b+1})\}$. Then P_1 is an (S,T)-path from $v_{1,1}$ to $v_{0,b+1} = v_{0,c}$. Note that now $v_{a',b'}$ is in G_2. As $e_{v_{1,k-1},v_{a',b'}} = 1$, there is a hamiltonian (S,T)-path P_2 of G_2 from $v_{1,k-1}$ to $v_{a',b'}$. Furthermore, it can be seen that P_1 and P_2 are vertex-disjoint (S,T)-paths and contain all vertices of $Row(0:1) - v_{a,b}$.

Suppose next that $b' < b$. As b is odd, we have $v_{1,b+1}$ is odd and $v_{0,b-1}$ is even. Let $v_{0,c} = v_{0,b-1}$. By a similar proof above, we may obtain two required (S,T)-paths.

Case 4.2. $v_{a,b}$ is in $Row(1)$, that is, $v_{a,b} = v_{1,b}$. We only consider the case that $b' > b$ since the proof for $b' < b$ is similar. Let $G_1 = Row(0:1) \cap Col(0:b-1)$ and $G_2 = Row(0:1) \cap Col(b+1:k-1)$. Observe that G_1 is isomorphic to $Grid(2,b)$ and G_2 is isomorphic to $Grid(2,k-b-1)$. As $v_{1,b} = v_{a,b}$ is odd, we have $v_{0,b-1}$ is odd and $v_{0,b}$ is even. Let $v_{0,c} = v_{0,b}$. As $e_{v_{0,b-1},v_{1,1}} = 1$ and $v_{0,b-1}$ is a corner vertex of G_1, Lemma 3.1 implies that there is a hamiltonian path P_1^1 of G_1 from $v_{1,1}$ to $v_{0,b-1}$. Let $P_1 = P_1^1 + \{(v_{0,b-1}, v_{0,b})\}$. Then P_1 is an (S,T)-path from $v_{1,1}$ to $v_{0,b} = v_{0,c}$. As $e_{v_{a',b'},v_{1,k-1}} = 1$ and $v_{1,k-1}$ is a corner vertex of G_2, Lemma 3.1 implies that there is a hamiltonian path P_2 of G_2 from $v_{1,k-1}$ to $v_{a',b'}$. It can be seen that P_1 and P_2 are vertex-disjoint (S,T)-paths in $Row(0:1) - v_{a,b}$ that contain all vertices of $Row(0:1) - v_{a,b}$. ∎

Lemma 3.9. *Let $S = \{v_{1,1}, v_{1,5}\}$ and let odd $v_{1,b}$ and odd $v_{a',b'}$ be two distinct vertices in $Row(0:1)$ of Q_2^6. Then there exists a set $T = \{v_{a',b'}, v_{0,c}\}$ ($c = 2$ or 4) such that there are two vertex-disjoint (S,T)-paths in $Row(0:1) - v_{1,b}$ that contain all vertices of $Row(0:1) - v_{1,b}$.*

Proof. As $v_{1,b}$ is odd, we have $v_{1,b} \in \{v_{1,0}, v_{1,2}, v_{1,4}\}$. If $v_{1,b} = v_{1,2}$ (resp. $v_{1,4}$), let $v_{0,c} = v_{0,2}$ (resp. $v_{0,4}$). Using similar proofs of Case 3 and Case 4.2 in Lemma 3.8, we may obtain two vertex-disjoint (S,T)-paths in $Row(0:1) - v_{1,b}$ that contain all vertices of $Row(0:1) - v_{1,b}$.

Suppose that $v_{1,b} = v_{1,0}$. Let $v_{0,c} = v_{0,2}$. If $v_{a',b'} \in V(Col(1))$, then $v_{a',b'} = v_{0,1}$. Similar to Case 1 of Lemma 3.8, we may obtain two vertex-disjoint (S,T)-paths in $Row(0:1)-v_{1,b}$ that contain all vertices of $Row(0:1)-v_{1,b}$. If $v_{a',b'} \in V(Col(5))$, then $v_{a',b'} = v_{0,5}$, let $P_1 = v_{1,1}v_{1,0}v_{0,0}v_{0,5}$ and $P_2 = v_{1,5}v_{1,4}v_{0,4}v_{0,3}v_{1,3}v_{1,2}v_{0,2}$. Obviously, P_1 and P_2 are as required. If $v_{a',b'} \in V(Col(2:4))$, then $v_{a',b'} \in \{v_{1,2}, v_{0,3}, v_{1,4}\}$. Let $P_1^1 = v_{0,5}v_{0,0}v_{0,1}v_{0,2}$ and $G = Row(0:1) \cap Col(b'+1:5)$. Observe that G is isomorphic to $Grid(2, 5-b')$. As $v_{1,5}$ is a corner vertex of G and $e_{v_{1,5},v_{0,5}} = 1$, Lemma 3,1 implies that there is a hamiltonian path P_1^2 of G from $v_{1,5}$ to $v_{0,5}$. Then $P_1 = P_1^1 \cup P_1^2$ is an (S,T)-path from $v_{1,5}$ to $v_{0,2} = v_{0,c}$. If $v_{a',b'} = v_{1,2}$, then $P_2 = v_{1,1}v_{1,2}$. If $v_{a',b'} = v_{0,3}$, then $P_2 = v_{1,1}v_{1,2}v_{1,3}v_{0,3}$. If $v_{a',b'} = v_{1,4}$, then $P_2 = v_{1,1}v_{1,2}v_{1,3}v_{0,3}v_{0,4}v_{1,4}$. Hence P_2 is an (S,T)-path from $v_{1,1}$ to $v_{a',b'}$. Therefore, P_1 and P_2 are as required. ■

Lemma 3.10. *Given an integer $k \in \{4,6\}$, let even u be a vertex in $Row(0:1) - v_{0,0}$ of Q_2^k. Let $S = \{u, v_{0,k-1}\}$ and $T = \{v_{1,2}, v_{0,1}\}$. Then there are two vertex-disjoint (S,T)-paths in $Row(0:1)-v_{0,0}$ that contain all vertices of $Row(0:1) - v_{0,0}$.*

Proof. As $u \neq v_{0,0}$ is even, we have $u \in V(Col(1:k-1))$. If $u \in V(Col(1))$, then $u = v_{1,1}$. Let $P_1 = Row(1) - \{(v_{1,1}, v_{1,2})\}$ and $P_2 = Row(0) - v_{0,0}$. Obviously, P_1 and P_2 are two vertex-disjoint (S,T)-paths in $Row(0:1) - v_{0,0}$ that contain all vertices of $Row(0:1) - v_{0,0}$. If $u \in V(Col(k-1))$, then $u = v_{1,k-1}$. Let $P_1 = v_{1,k-1}v_{1,0}v_{1,1}v_{1,2}$. If $k=4$, let $P_2 = v_{0,3}v_{0,2}v_{0,1}$. If $k=6$, let $P_2 = v_{0,5}v_{0,4}v_{1,4}v_{1,3}v_{0,3}v_{0,2}v_{0,1}$. Then P_1 and P_2 are as required. If $u \in V(Col(2:k-2))$, let $G = Row(0:1) \cap Col(2:k-2)$. Observe that G is isomorphic to $Grid(2, k-3)$. As odd $v_{1,2}$ is a corner vertex of G and u is even, Lemma 3,1 implies that there is a hamiltonian path P_1 of G from u to $v_{1,2}$. Let $P_2 = v_{0,k-1}v_{1,k-1}v_{1,0}v_{1,1}v_{0,1}$. Clearly, P_1 and P_2 are as required. ■

Note that in a Q_2^6, $Col(1:3)$ and $Col(3:5)$ are isomorphic. By a similar proof above, we have following corollary.

Corollary 3.11. *Let even u be a vertex in $Row(0:1) - v_{0,0}$ of Q_2^6 and let $S = \{u, v_{0,5}\}$, $T = \{v_{1,4}, v_{0,1}\}$. Then there are two vertex-disjoint (S,T)-paths in $Row(0:1) - v_{0,0}$ that contain all vertices of $Row(0:1) - v_{0,0}$.*

We define the following paths in $Row(i:i+1)$ of a Q_2^k. Let $i \le a \le i+1$, $0 \le b$, $m \le k-1$ and $m \neq b$. If $a = i$ then $\bar{a} = i+1$, and if $a = i+1$ then $\bar{a} = i$.
$C_m^+(v_{a,b}, v_{\bar{a},b}) = v_{a,b}v_{a,b+1}v_{a,b+2}\cdots v_{a,m-1}v_{a,m}v_{\bar{a},m}v_{\bar{a},m-1}v_{\bar{a},m-2}\cdots v_{\bar{a},b+1}v_{\bar{a},b}$.
$C_m^-(v_{a,b}, v_{\bar{a},b}) = v_{a,b}v_{a,b-1}v_{a,b-2}\cdots v_{a,m+1}v_{a,m}v_{\bar{a},m}v_{\bar{a},m+1}v_{\bar{a},m+2}\cdots v_{\bar{a},b-1}v_{\bar{a},b}$.
In addition, if $m = b$, we define $C_b^+(v_{a,b}, v_{\bar{a},b}) = C_b^-(v_{a,b}, v_{\bar{a},b}) = (v_{a,b}, v_{\bar{a},b})$.

Theorem 3.12. *Given an even $k \geq 4$, let $F_v = \{u^*, v^*\}$ be a set of faulty vertices of Q_2^k such that $e_{u^*,v^*} = 1$ and let u and v be any two healthy vertices of Q_2^k such that $e_{u,v} = 1$. Then there exists a hamiltonian path of $Q_2^k - F_v$ from u to v.*

Proof. Without loss of generality, we may assume that $u^* = v_{0,0}$. As $e_{u^*,v^*} = 1$ and $u^* = v_{0,0}$ is even, we see that v^* is odd. Let $v^* = v_{a,b}$ where $0 \leq a, b \leq k-1$. As $Row(1:k-1)$ is isomorphic to $Col(1:k-1)$, it is enough to consider v^* is in $Row(1:k-1)$. Furthermore, we may assume that v^* is in $Row(\frac{k}{2}:k-1)$ because $Row(1:\frac{k}{2})$ and $Row(\frac{k}{2}:k-1)$ are isomorphic.

If a is odd, let $p = a-2$. If a is even, let $p = a-1$. Clearly, p is odd and $v^* = v_{a,b} \in V(Row(p+1:p+2))$. Let $u = v_{i,j}$ and $v = v_{i',j'}$. We consider the following five cases.

Case 1. $u, v \in V(Row(0:1))$. Let $S = \{u, v\}$ and $T = \{v_{0,1}, v_{1,0}\}$. As $e_{u,v} = 1$, Lemma 3.6 implies that there exists two vertex-disjoint (S,T)-paths P_1, P_2 in $Row(0:1) - v_{0,0}$ that contain all vertices of $Row(0:1) - v_{0,0}$. Recall that odd v^* is in $Row(p+1:p+2)$. As even $v_{p+1,0}$ and even $v_{k-1,1}$ are two distinct vertices in $Row(p+1:k-1)$, Lemma 3.4 and Lemma 3.5(iii) imply that there exists a hamiltonian path P_3 of $Row(p+1:k-1) - v^*$ from $v_{p+1,0}$ to $v_{k-1,1}$.

If $p = 1$, then $P_1 \cup P_2 \cup P_3 + \{(v_{1,0}, v_{2,0}), (v_{0,1}, v_{k-1,1})\}$ is a hamiltonian path of $Q_2^k - F_v$ from u to v. Suppose that odd $p \geq 3$. As $e_{v_{2,0},v_{p,0}} = 1$, Lemma 3.3 implies that there exists a hamiltonian path P_4 of $Row(2:p)$ from $v_{2,0}$ to $v_{p,0}$. Then $\bigcup_{d=1}^4 P_d + \{(v_{1,0}, v_{2,0}), (v_{0,1}, v_{k-1,1}), (v_{p,0}, v_{p+1,0})\}$ is a hamiltonian path of $Q_2^k - F_v$ from u to v.

Case 2. $u \in V(Row(0:1))$ and $v \in V(Row(2:p))$. As $v \in V(Row(2:p))$, it is easy to see that odd $p \geq 3$. Noting that $v^* = v_{a,b} \in V(Row(p+1, p+2))$, we see that $Row(p+2)$ exists. Then $k-1 \geq p+2 \geq 5$, and so $k \geq 6$. We distinguish two cases.

Case 2.1. u is even and v is odd. Let $G_1 = Row(0:1) \cap Col(1:j)$. Observe that G_1 is isomorphic to $Grid(2, j)$. As $e_{v_{0,1},u} = 1$ and $v_{0,1}$ is a corner vertex of G_1, Lemma 3.1 implies that there is a hamiltonian path P_1 of G_1 from u to $v_{0,1}$. Let $P_2 = C_{j+1}^-(v_{0,k-1}, v_{1,k-1}) + \{(v_{1,0}, v_{1,k-1})\}$. Then P_1 and P_2 are two vertex-disjoint paths in $Row(0:1) - v_{0,0}$ that contain all vertices of $Row(0:1) - v_{0,0}$. Noting that v is odd, we have $e_{v_{2,0},v} = 1$. By Lemma 3.3, there is a hamiltonian path P_3 of $Row(2:p)$ from $v_{2,0}$ to v. As k is even and v^* is odd, we have $e_{v_{k-1,1},v_{k-1,k-1}} = 0$ and $e_{v_{k-1,1},v^*} = 1$. Combining this with the fact that $v^* \in V(Row(p+1:p+2))$, Lemma 3.4 and Lemma 3.5(ii) imply that there is a hamiltonian path P_4 of $Row(p+1:k-1) - v^*$ from $v_{k-1,1}$ to $v_{k-1,k-1}$. Then $\bigcup_{d=1}^4 P_d + \{(v_{0,1}, v_{k-1,1}), (v_{1,0}, v_{2,0}), (v_{0,k-1}, v_{k-1,k-1})\}$ is a hamiltonian path of $Q_2^k - F_v$ from u to v.

Case 2.2. u is odd and v is even. Noting that v is even and p is odd, we have $e_{v_{p,0},v} = 1$. By Lemma 3.3, there exists a hamiltonian path P_1 of $Row(2:p)$

from $v_{p,0}$ to v. As $k \geq 6$, we may choose a vertex $w \in V(Row(0))$ such that $w \neq u$ and $e_{w,u} = 0$. Combining this with the fact that $e_{v_{0,0},u} = 1$, Lemma 3.4 implies that there exists a hamiltonian path P_2 of $Row(0:1) - v_{0,0}$ from u to w. By $e_{w,n^{k-1}(w)} = 1$, we have $n^{k-1}(w)$ is even. Note that $v_{p+1,0}$ is even and $v^* \in V(Row(p+1:p+2))$ is odd. By Lemma 3.4 and Lemma 3.5(iii), there is a hamiltonian path P_3 of $Row(p+1:k-1) - v^*$ from $n^{k-1}(w)$ to $v_{p+1,0}$. Then $P_1 \cup P_2 \cup P_3 + \{(w, n^{k-1}(w)), (v_{p,0}, v_{p+1,0})\}$ is a hamiltonian path of $Q_2^k - F_v$ from u to v.

Case 3. $u \in V(Row(0:1))$ and $v \in V(Row(p+1:p+2))$.

Case 3.1. u is odd and v is even. Suppose first that $k = 4$. Then $Row(p+1:p+2) = Row(2:3)$. Let v' be the neighbour of v in $Row(0:1)$. It is easy to see that we may choose an odd u' in $Row(0:1) - u$ such that $u' \neq v'$. Denote the neighbour of u' in $Row(2:3)$ by u''. As u^* is even and both u and u' are odd, Lemma 3.4 implies that there is a hamiltonian path P_1 of $Row(0:1) - u^*$ from u to u'. Similarly, there is a hamiltonian path P_2 of $Row(2:3) - v^*$ from u'' to v. Then $P_1 \cup P_2 + \{(u', u'')\}$ is a hamiltonian path of $Q_2^4 - F_v$ from u to v.

Suppose next that $k \geq 6$. As $\frac{k}{2} - 2 \geq 3 - 2 = 1$, we may choose an odd x in $Row(p)$ such that $x \neq u$ and $n^{p+1}(x) \neq v$. Then $e_{x,u} = 0$. Note that $e_{u^*,u} = 1$ and $u^* \in V(Row(0:1))$. By Lemma 3.4 and Lemma 3.5(iii), there exists a hamiltonian path P_1 in $Row(0:p) - u^*$ from u to x. As x is odd, we have $n^{p+1}(x)$ is even. Recalling that $v^* \in V(Row(p+1:p+2))$ is odd and v is even, Lemma 3.4 and Lemma 3.5(i) imply that there is a hamiltonian path P_2 of $Row(p+1:k-1) - v^*$ from $n^{p+1}(x)$ to v. Then $P_1 \cup P_2 + \{(x, n^{p+1}(x))\}$ is a hamiltonian path of $Q_2^k - F_v$ from u to v.

Case 3.2. u is even and v is odd.

Case 3.2.1. $k = 4$. In this case, $Row(p+1:p+2) = Row(2:3)$. Let $S = \{u, v_{0,3}\}$ and $T = \{v_{1,2}, v_{0,1}\}$. By Lemma 3.10, there exist a $uv_{1,2}$-path P_1 and a $v_{0,3}v_{0,1}$-path P_2 in $Row(0:1) - v_{0,0}$. Moreover, P_1 and P_2 are two vertex-disjoint (S,T)-paths that contain all vertices of $Row(0:1) - v_{0,0}$.

Let $S = \{v_{3,1}, v_{3,3}\}$ and $T = \{v, v_{2,2}\}$. Recall that both v and v^* are odd. By Lemma 3.7, there are two vertex-disjoint (S,T)-paths P_3 and P_4 in $Row(2:3) - v^*$ that contain all vertices of $Row(2:3) - v^*$. Then $\bigcup_{d=1}^{4} P_d + \{(v_{0,1}, v_{3,1}), (v_{0,3}, v_{3,3}), (v_{1,2}, v_{2,2})\}$ is a hamiltonian path of $Q_2^4 - F_v$ from u to v.

Case 3.2.2. $k \geq 6$. If $p = 1$, then $v^* = v_{a,b} \in V(Row(2:3))$ and so $2 \leq a \leq 3$. Recall that $v^* = v_{a,b}$ is in $Row(\frac{k}{2}:k-1)$ and $k \geq 6$. Therefore $a \geq \frac{k}{2} \geq 3$. So $a = 3$ and $k = 2 \times 3 = 6$. Let $S = \{v_{3,1}, v_{3,5}\}$ and $T = \{v, v_{2,c}\}(c = 2$ or $4)$. By Lemma 3.9, there are two vertex-disjoint (S,T)-paths P_1, P_2 in $Row(2:3) - v^*$ that contain all vertices of $Row(2:3) - v^*$. As $v_{1,c} \in \{v_{1,2}, v_{1,4}\}$ and even u is in $Row(0:1) - v_{0,0}$, Lemma 3.10 and Corollary 3.11 imply that there exist a path P_3 from u to $v_{1,c}$ and a path P_4 from $v_{0,5}$ to $v_{0,1}$. Moreover, P_1 and

P_2 are two vertex-disjoint paths in $Row(0:1) - v_{0,0}$ that contain all vertices of $Row(0:1) - v_{0,0}$.

Let $P_5 = C_0^-(v_{4,1}, v_{5,1})$ and $P_6 = C_2^-(v_{4,5}, v_{5,5})$. Clearly, P_5 and P_6 are vertex-disjoint paths in $Row(4:5)$ that contain all vertices of $Row(4:5)$. Then $\bigcup_{d=1}^{6} P_d + \{(v_{1,c}, v_{2,c}), (v_{0,5}, v_{5,5}), (v_{0,1}, v_{5,1}), (v_{3,5}, v_{4,5}), (v_{3,1}, v_{4,1})\}$ is a hamiltonian path of $Q_2^6 - F_v$ from u to v.

Suppose that $p \geq 3$. We will choose an odd $u' \in V(Row(1))$ and construct a uu'-path P_1 and a $v_{0,k-1}v_{0,1}$-path P_2 in $Row(0:1) - v_{0,0}$. Suppose first that $u \in V(Row(0))$. As $u = v_{0,j}$ is even, we have $u' = v_{1,j}$ is odd. Let $P_1 = uu'$ and $P_2 = C_{j-1}^+(v_{1,1}, v_{0,1}) \cup C_{j+1}^-(v_{1,k-1}, v_{0,k-1}) + \{(v_{1,1}, v_{1,0}), (v_{1,0}, v_{1,k-1})\}$. Then P_1 is a path from u to u' and P_2 is a path from $v_{0,k-1}$ to $v_{0,1}$. Obviously, P_1 and P_2 are vertex-disjoint paths that contain all vertices of $Row(0:1) - v_{0,0}$. Suppose next that $u \in V(Row(1))$. As $u = v_{1,j}$ is even, we have $u' = v_{1,j-1} \in V(Row(1))$ is odd, where $1 \leq j \leq k-1$. Let $P_1 = Row(1) - \{(v_{1,j-1}, v_{1,j})\}$ and $P_2 = v_{0,k-1}v_{0,k-2}v_{0,k-3}\ldots v_{0,1}$. Then P_1 is a path from u to u' and P_2 is a path from $v_{0,k-1}$ to $v_{0,1}$. Clearly, P_1 and P_2 are vertex-disjoint paths that contain all vertices of $Row(0:1) - v_{0,0}$.

Noting that p is odd and k is even, we have both $v_{p+2,1}$ and $v_{p+2,k-1}$ are even. Let $S = \{v_{p+2,1}, v_{p+2,k-1}\}$. As odd $v^*, v \in V(Row(p+1:p+2))$, Lemma 3.8 implies that there exists a set $T = \{x, v\}$ ($x \in V(Row(p+1))$ is even), such that there are two vertex-disjoint (S,T)-paths P_3, P_4 in $Row(p+1:p+2) - v^*$ that contain all vertices of $Row(p+1:p+2) - v^*$.

Note that $x \in V(Row(p+1))$ and $u' \in V(Row(1))$. As x is even and u' is odd, it is easy to see that $e_{n^p(x), n^2(u')} = 1$. By Lemma 3.3, there exists a hamiltonian path P_5 of $Row(2:p)$ from $n^2(u')$ to $n^p(x)$.

We will construct a hamiltonian path of $Q_2^k - F_v$ from u to v in the following. Noting that $p+2$ is odd, we consider the following two cases. If $p+2 = k-1$, then $\bigcup_{d=1}^{5} P_d + \{(u', n^2(u')), (v_{0,1}, v_{p+2,1}), (v_{0,k-1}, v_{p+2,k-1}), (n^p(x), x)\}$ is a hamiltonian path of $Q_2^k - F_v$ from u to v. If $p+2 \leq k-3$, let $G_1 = Row(p+3:k-1) \cap Col(0:1)$ and $G_2 = Row(p+3:k-1) \cap Col(2:k-1)$. Observe that G_1 is isomorphic to $Grid(k-p-3, 2)$ and G_2 is isomorphic to $Grid(k-p-3, k-2)$. As p is odd and k is even, we have $e_{v_{p+3,1}, v_{k-1,1}} = e_{v_{p+3,k-1}, v_{k-1,k-1}} = 1$. As $v_{p+3,1}$ and $v_{p+3,k-1}$ are corner vertices of G_1 and G_2, respectively, Lemma 3.2 implies that there are a hamiltonian path P_6 of G_1 from $v_{p+3,1}$ to $v_{k-1,1}$ and a hamiltonian path P_7 of G_2 from $v_{p+3,k-1}$ to $v_{k-1,k-1}$. Then $\bigcup_{d=1}^{7} P_d + \{(u', n^2(u')), (v_{0,1}, v_{k-1,1}), (v_{0,k-1}, v_{k-1,k-1}), (n^p(x), x), (v_{p+2,1}, v_{p+3,1}), (v_{p+2,k-1}, v_{p+3,k-1})\}$ is a hamiltonian path of $Q_2^k - F_v$ from u to v.

Case 4. $u, v \in V(Row(2:p))$. As $u, v \in V(Row(2:p))$, it is easy to see that odd $p \geq 3$. Noting that $v^* = v_{a,b} \in V(Row(p+1, p+2))$, we see that $Row(p+2)$ exists. Then $k-1 \geq p+2 \geq 5$, and so $k \geq 6$. As $e_{u,v} = 1$, by Lemma 3.3, there exists a hamiltonian path P_1 of $Row(2:p)$ from u to v that contains an edge

(s,t) of $Row(2)$. As $e_{n^1(s),n^1(t)} = 1$, without loss of generality, we may assume that $n^1(s)$ is odd and $n^1(t)$ is even. Let $n^1(s) = v_{1,m}$ and $n^1(t) = v_{1,m+1}$.

If $m = 0$, then $n^1(s) = v_{1,0}$ and $n^1(t) = v_{1,1}$. Let $P_2 = v_{1,0}v_{1,k-1}v_{0,k-1}$ and $P_3 = C^+_{k-2}(v_{1,1}, v_{0,1})$. If $m \neq 0$, let $P_2 = v_{1,m}v_{0,m}v_{0,m+1}\dots v_{0,k-1}$, $P_3^1 = v_{1,m+1}v_{m+2}v_{m+4}\dots\ v_{1,k-1}v_{1,0}v_{1,1}$ and $P_3 = P_3^1 \cup C^+_{m-1}(v_{1,1}, v_{0,1})$. Then P_2 is a path from $n^1(s)$ to $v_{0,k-1}$ and P_3 is a path from $n^1(t)$ to $v_{0,1}$. Obviously, P_2 and P_3 are vertex-disjoint paths in $Row(0:1) - v_{0,0}$ that contain all vertices of $Row(0:1) - v_{0,0}$.

As $v_{k-1,1}, v_{k-1,k-1} \in V(Row(k-1))$ are even and $v^* \in V(Row(p+1 : p+2))$ is odd, Lemma 3.4 and Lemma 3.5(ii) imply that there is a hamiltonian path P_4 of $Row(p+1 : k-1) - v^*$ from $v_{k-1,1}$ to $v_{k-1,k-1}$. Then $\bigcup_{d=1}^4 P_d - \{(s,t)\} + \{(s,n^1(s)),(t,n^1(t)),(v_{0,1},\ v_{k-1,1}),(v_{0,k-1},v_{k-1,k-1})\}$ is a hamiltonian path of $Q_2^k - F_v$ from u to v.

Case 5. $u \in V(Row(2:p))$ and $v \in V(Row(p+3:k-1))$. As $u \in V(Row(2:p))$, it is easy to see that odd $p \geq 3$. Noting that $v \in V(Row(p+3:k-1))$, we have $k-1 \geq p+3$ and so $k \geq p+4 \geq 7$. As k is even, we have $k \geq 8$. Recall that $v = v_{i',j'}$. If i' is odd, let $q = i'-1$. If i' is even, let $q = i'$. Clearly, $q \geq p+3$ is even and $v \in V(Row(q:q+1))$. Now we consider the following two cases.

Case 5.1. $v \in V(Row(q))$. As $e_{u,v} = 1$, without loss of generality, we assume that u is even and v is odd. Choose an odd $w \in V(Row(p))$. Then $e_{u,w} = 1$. By Lemma 3.3, there is a hamiltonian path P_1 of $Row(2:p)$ from u to w that contains an edge (s,t) of $Row(2)$. Similar to Case 4, there exist an $n^1(s)v_{0,k-1}$-path P_2 and an $n^1(t)v_{0,1}$-path P_3 in $Row(0:1) - v_{0,0}$. Moreover, P_2 and P_3 are vertex-disjoint paths that contain all vertices of $Row(0:1) - v_{0,0}$.

As $v_{k-1,1}, v_{k-1,k-1} \in V(Row(k-1))$ are even and $v \in V(Row(q))$ is odd, Lemma 3.4 and Lemma 3.5(ii) imply that there is a hamiltonian path P_4 of $Row(q:k-1) - v$ from $v_{k-1,1}$ to $v_{k-1,k-1}$. As both w and v are odd, we have both $n^{p+1}(w)$ and $n^{q-1}(v)$ are even. Note that the odd v^* is in $Row(p+1:p+2)$. By Lemma 3.4 and Lemma 3.5(iii), there is a hamiltonian path P_5 of $Row(p+1 : q-1) - v^*$ from $n^{p+1}(w)$ to $n^{q-1}(v)$.
Then $\bigcup_{d=1}^5 P_d - \{(s,t)\} + \{(s,n^1(s)),(t,n^1(t)),(v_{0,1},v_{k-1,1}),\ (v_{0,k-1},v_{k-1,k-1}),\ (w,n^{p+1}(w)),(v,n^{q-1}(v))\}$ is a hamiltonian path of $Q_2^k - F_v$ from u to v.

Case 5.2. $v \in V(Row(q+1))$. As $e_{u,v} = 1$, without loss of generality, we assume that u is odd and v is even. Choose an even $w \neq v$ in $Row(q+1)$. As $e_{w,v} = 0$ and $e_{v,v^*} = 1$, Lemma 3.5(ii) implies that there is a hamiltonian path P_1 of $Row(p+1 : q+1) - v^*$ from w to v. Choose an odd $x \in V(Row(1))$ and an odd $y \in V(Row(0))$. Then $n^2(x)$ is even. Noting that u is odd, we have $e_{u,n^2(x)} = 1$. By Lemma 3.3, there is a hamiltonian path P_2 of $Row(2:p)$ from u to $n^2(x)$. Note that u^* is even and both x and y are odd. By Lemma 3.4, there is a hamiltonian path P_3 of $Row(0:1) - u^*$ from x to y.

We will construct a hamiltonian path of $Q_2^k - F_v$ from u to v in the following. Noting that $q+1$ is odd, we consider the following two cases. Suppose first that $q+1 = k-1$. As w is even, we have $n^0(w)$ is odd. Let $y = n^0(w)$. Then $P_1 \cup P_2 \cup P_3 + \{(w, n^0(w)), (x, n^2(x))\}$ is a hamiltonian path of $Q_2^k - F_v$ from u to v. Suppose next that $q+1 \leq k-3$. As w is even and y is odd, we have $n^{q+2}(w)$ is odd and $n^{k-1}(y)$ is even. By Lemma 3.3, there exists a hamiltonian path P_4 of $Row(q+2 : k-1)$ from $n^{q+2}(w)$ to $n^{k-1}(y)$. Then $\bigcup_{d=1}^{4} P_d + \{(y, n^{k-1}(y)), (x, n^2(x)), (w, n^{q+2}(w))\}$ is a hamiltonian path of $Q_2^k - F_v$ from u to v. The proof of this theorem is complete. ■

Given an even $k \geq 4$, let F_v be the set of faulty vertices of a Q_2^k. Recall that $f_v^{max} = \max\{|F_v \cap X|, |F_v \cap Y|\}$, where X be the set of even vertices and Y be the set of odd vertices of the Q_2^k. The following result is a direct consequence of Theorem 1.1 and 3.12.

Corollary 3.13. *Let $k \geq 4$ be even and let f_v be the number of faulty vertices and f_e be the number of faulty edges in Q_2^k with $0 \leq f_v + f_e \leq 2$. Given any two healthy vertices u and v of Q_2^k, then there is a path from u to v of length $k^2 - 2f_v^{max} - 1$ if $e_{u,v} = 1$.*

4. Conclusions

In this paper, we investigate the problem of embedding hamiltonian paths into faulty k-ary 2-cubes, where $k \geq 4$ is even. For any two healthy vertices u, v with $e_{u,v} = 1$, we proved that the faulty k-ary n-cube admits a path of length $k^2 - 2f_v^{max} - 1$ if $f_v + f_e \leq 2$. The above result show that the fault-tolerant capability of the k-ary 2-cube is nice in terms of the path embeddings. The work will help engineers to develop corresponding applications on the distributed-memory parallel system that employs the k-ary 2-cube as the interconnection network.

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