EMBEDDINGS OF HAMILTONIAN PATHS IN FAULTY $k$-ARY 2-CUBES

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Abstract

It is well known that the $k$-ary $n$-cube has been one of the most efficient interconnection networks for distributed-memory parallel systems. A $k$-ary $n$-cube is bipartite if and only if $k$ is even. Let $(X, Y)$ be a bipartition of a $k$-ary 2-cube (even integer $k \geq 4$). In this paper, we prove that for any two healthy vertices $u \in X$, $v \in Y$, there exists a hamiltonian path from $u$ to $v$ in the faulty $k$-ary 2-cube with one faulty vertex in each part.

Keywords: complex networks, path embeddings, fault-tolerance, $k$-ary $n$-cubes.

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1. Introduction

The $k$-ary $n$-cube has many desired properties, such as easy of implementation, low-latency and high-bandwidth interprocessor communication. Therefore, a large number of distributed-memory parallel systems (also known as multicomputers) have been built with a $k$-ary $n$-cube forming the underlying topology, such as the iWarp [12], the J-machine [11] and the Cray T3D [9]. An interconnection network is often modeled as a graph, in which vertices and edges correspond to nodes and communication links, respectively. The $k$-ary $n$-cube, denoted by $Q_n^k$ ($k \geq 2$ and $n \geq 1$), is a graph consisting of $k^n$ vertices, each of which has the form $u = u_{n-1}u_{n-2} \ldots u_0$, where $0 \leq u_i \leq k - 1$ for $0 \leq i \leq n - 1$. Two vertices

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$u = u_{n-1}u_{n-2}\ldots u_0$ and $v = v_{n-1}v_{n-2}\ldots v_0$ are adjacent if and only if there exists an integer $j$, $0 \leq j \leq n - 1$, such that $u_j = v_j \pm 1 (\text{mod } k)$ and $u_i = v_i$, for every $i \in \{1,2,\ldots,n\} \setminus \{j\}$. For clarity of presentation, we omit writing “($\text{mod } k$)” in similar expressions for the remainder of the paper.

The graph embedding is a technique that maps a guest graph into a host graph. Many graph embeddings take paths and cycles as guest graphs because they are the common structures used to model linear arrays in parallel processing [2, 4, 15, 16, 17]. In recent years, the problem of path embeddings in an interconnection network has attracted a great deal of attention from the researchers. Since failures are inevitable, fault-tolerant is an important issue in distributed-memory parallel systems. Many works related to embeddings of various faulty interconnection networks have been studied previously, including hypercubes [3, 5, 7, 10, 14, 16, 19], $k$-ary $n$-cubes [1, 15, 17, 19] and stars [6, 13]. In particular, Yang et al. [19] proved that for arbitrary two healthy vertices of $Q^k_n$, with odd $k \geq 3$, there exists a fault-free hamiltonian path connecting these two vertices if the number of faults is at most $2n - 3$.

The parity of a vertex $u = u_{n-1}u_{n-2}\ldots u_0$ of $Q^k_n$ is defined to be $u_{n-1} + u_{n-2} + \cdots + u_0$ modulo 2. We speak of a vertex as being odd or even according to whether its parity is odd or even. Given any two distinct vertices $u$ and $v$. Let

$$e_{u,v} = \begin{cases} 1, & \text{if } u \text{ and } v \text{ have different parities;} \\ 0, & \text{if } u \text{ and } v \text{ have the same parity.} \end{cases}$$

For even $k \geq 4$, Stewart and Xiang [15] studied the problem of embedding long paths in the $k$-ary $n$-cube with faulty vertices and edges. They presented the following result.

**Theorem 1.1** [15]. Let $k \geq 4$ be even and let $f_v$ be the number of faulty vertices and $f_e$ be the number of faulty edges in $Q^k_2$ with $0 \leq f_v + f_e \leq 2$. Given any two healthy vertices $u$ and $v$ of $Q^k_2$, then there is a path from $u$ to $v$ of length at least $k^2 - 2f_v - 1$ if $e_{u,v} = 1$.

Let $X$ be the set of even vertices and $Y$ be the set of odd vertices of a $Q^k_2$ with even $k \geq 4$. Obviously, $(X,Y)$ is a bipartition of the $Q^k_2$. We denote the set of faulty vertices of the $Q^k_2$ by $F_v$. Let $f_v^{\text{max}} = \max\{|F_v \cap X|, |F_v \cap Y|\}$. In this paper, we prove that there is a path from $u$ to $v$ in the faulty $Q^k_2$ of length $k^2 - 2f_v^{\text{max}} - 1$ if $e_{u,v} = 1$. As $|F_v \cap X| + |F_v \cap Y| = f_v$, we have $f_v^{\text{max}} \leq f_v$. Obviously, $k^2 - 2f_v^{\text{max}} - 1 \geq k^2 - 2f_v - 1$. Therefore, our result improves the result noted above.

The rest of this paper is organized as follows. In the next section, some basic definitions are introduced. In Section 3, we construct a hamiltonian path connecting any two healthy vertices in different parts in the faulty $k$-ary 2-cube (even $k \geq 4$) with one faulty vertex in each part. Conclusions are covered in Section 4.
2. Basis Definition

Throughout this paper, we restrict our attention to \( n = 2 \) and even \( k \geq 4 \). For convenience, we write \( v_{a,b} \) as the vertex of \( Q_2^k \) with the form \( v_1v_0 = ab \), where \( 0 \leq a, b \leq k - 1 \). For \( 0 \leq i \leq j \leq k - 1 \), \( \text{Row}(i : j) \) of \( Q_2^k \) is the subgraph of \( Q_2^k \) induced by \( \{v_{a,b} : i \leq a \leq j, 0 \leq b \leq k - 1\} \), \( \text{Col}(i : j) \) of \( Q_2^k \) is the subgraph of \( Q_2^k \) induced by \( \{v_{a,b} : 0 \leq a \leq k - 1, i \leq b \leq j\} \).

Given \( 1 \leq k_1, k_2 \leq k - 1 \), the subgraph of \( Q_2^k \) induced by \( \{v_{a,b} : 0 \leq a \leq k_1 - 1, 0 \leq b \leq k_2 - 1\} \) is denoted by \( \text{Grid}(k_1, k_2) \). A vertex of \( \text{Grid}(k_1, k_2) \) is called a corner vertex if its degree in \( \text{Grid}(k_1, k_2) \) is 2. For \( 0 \leq i \leq j \leq k_1 - 1 \), \( \text{Row}(i : j) \) of \( \text{Grid}(k_1, k_2) \) is the subgraph of \( \text{Grid}(k_1, k_2) \) induced by \( \{v_{a,b} : i \leq a \leq j, 0 \leq b \leq k_2 - 1\} \). For \( 0 \leq i \leq j \leq k_2 - 1 \), \( \text{Col}(i : j) \) of \( \text{Grid}(k_1, k_2) \) is the subgraph of \( \text{Grid}(k_1, k_2) \) induced by \( \{v_{a,b} : 0 \leq a \leq k_1 - 1, i \leq b \leq j\} \).

Instead of \( \text{Row}(i : j) \) and \( \text{Col}(j : i) \) of \( Q_2^k \) (resp. \( \text{Grid}(k_1, k_2) \)) we simply write \( \text{Row}(i) \) and \( \text{Col}(j) \) of \( Q_2^k \) (resp. \( \text{Grid}(k_1, k_2) \)). \( \text{Row}(0 : 2) \) of \( Q_2^4 \) and \( \text{Grid}(2,4) \) are shown in Figure 1 and Figure 2, respectively.

Choose a vertex \( u = v_{a,b} \ (0 \leq a, b \leq k - 1 \) in \( \text{Row}(a) \) of \( Q_2^k \). The neighbour of \( u \) in \( \text{Row}(a - 1) \) (resp. \( \text{Row}(a + 1) \)) is denoted by \( n^{-1}(u) \) (resp. \( n^{a+1}(u) \)), that is, \( n^{-1}(u) = v_{a-1,b} \) (resp. \( n^{a+1}(u) = v_{a+1,b} \)).

3. Path Embeddings in Faulty \( k \)-ary 2-cubes

We start with some useful lemmas.

**Lemma 3.1** [8]. Given an integer \( n \geq 1 \), let \( u \) be a corner vertex of \( \text{Grid}(2,n) \). For any vertex \( v \neq u \) in \( \text{Grid}(2,n) \) such that \( e_{u,v} = 1 \), there exists a hamiltonian path of \( \text{Grid}(2,n) \) from \( u \) to \( v \).

**Lemma 3.2** [8]. Given even \( k_1,k_2 \geq 2 \), let \( u \) and \( v \) be vertices in \( \text{Row}(0) \) and \( \text{Row}(k_1-1) \) of \( \text{Grid}(k_1,k_2) \), respectively. If at least one of \( u \) and \( v \) is a corner vertex of \( \text{Grid}(k_1,k_2) \) and \( e_{u,v} = 1 \), then there is a hamiltonian path of \( \text{Grid}(k_1,k_2) \) from \( u \) to \( v \).
In [15], Stewart and Xiang constructed the long paths in \( \text{Row}(0 : p - 1) \) of \( Q_k^2 \) (even \( k \geq 4 \)), where \( 2 \leq p \leq k \). They present the following result.

**Lemma 3.3** [15]. Given an even \( k \geq 4 \), let \( u \) and \( v \) be any two distinct healthy vertices in \( \text{Row}(0 : p - 1) \) of \( Q_k^2 \), where \( 2 \leq p \leq k \). If \( e_{u,v} = 1 \), then there exists a hamiltonian path of \( \text{Row}(0 : p - 1) \) from \( u \) to \( v \) that contains at least one healthy edge of \( \text{Row}(0) \).

According to the proof of Lemma 1 in [15], we have the following lemma.

**Lemma 3.4** [15]. Given an even \( k \geq 4 \), let \( u \) and \( v \) be any two distinct healthy vertices and \( x \) be a faulty vertex in \( \text{Row}(0 : 1) \) of \( Q_k^2 \). If \( e_{x,u} = 1 \) and \( e_{u,v} = 0 \), then there exists a hamiltonian path of \( \text{Row}(0 : 1) \) from \( u \) to \( v \) that contains at least one healthy edge of \( \text{Row}(1) \).

**Lemma 3.5.** Given an even \( k \geq 4 \), let \( x \) be the only faulty vertex in \( \text{Row}(0 : 1) \) of \( Q_k^2 \) and let \( u,v \) be any two distinct healthy vertices in \( \text{Row}(0 : p - 1) \) of \( Q_k^2 \) such that \( e_{x,u} = 1 \) and \( e_{u,v} = 0 \), where \( p \) is even and \( 4 \leq p \leq k \). Then there exists a hamiltonian path of \( \text{Row}(0 : p - 1) \) from \( u \) to \( v \) if one of the following holds.

(i) \( u,v \in V(\text{Row}(0 : 1)) \).

(ii) \( u,v \in V(\text{Row}(p - 1)) \).

(iii) \( u \in V(\text{Row}(0 : 1)) \) and \( v \in V(\text{Row}(p - 1)) \).

**Proof.** Suppose that \( u,v \in V(\text{Row}(0 : 1)) \). As \( x \in V(\text{Row}(0 : 1)) \) and \( e_{x,u} = 1 \), \( e_{u,v} = 0 \), Lemma 3.4 implies that there is a hamiltonian path \( P_1 \) of \( \text{Row}(0 : 1) \) from \( u \) to \( v \) that contains an edge \((s,t)\) of \( \text{Row}(1) \). As \( n^2(s),n^2(t) = 1 \), by Lemma 3.3, there is a hamiltonian path \( P_2 \) of \( \text{Row}(2 : p - 1) \) from \( n^2(s) \) to \( n^2(t) \). Then, \( P_1 \cup P_2 = \{(s,t)\} + \{(s,n^2(s)),(t,n^2(t))\} \) is a hamiltonian path of \( \text{Row}(0 : p - 1) \) from \( u \) to \( v \).

Suppose that \( u,v \in V(\text{Row}(p - 1)) \). Let \( u = v_{p-1,j}, v = v_{p-1,j'} \), where \( 0 \leq j,j' \leq k - 1 \) and \( j \neq j' \). Without loss of generality, we assume that \( j < j' \). Let \( q \in \{j,j+1,j+2,\ldots, j'\} \) be odd and let \( G_1 = \text{Row}(2 : p - 1) \cap \text{Col}(0 : q) \) and \( G_2 = \text{Row}(2 : p - 1) \cap \text{Col}(q+1 : k - 1) \). Obviously, \( u \in V(G_1) \) and \( v \in V(G_2) \). As \( q \) is odd, we have \( e_{v_{2,q},v_{2,q+1}} = e_{v_{2,q+1},v_{2,q+2}} = 1 \). Thus one of \( e_{u,v_{2,q}} = 1 \) and \( e_{u,v_{2,q+1}} = 1 \) holds. Without loss of generality, we may assume that \( e_{u,v_{2,q}} = 1 \). As \( G_1 \) is isomorphic to \( \text{Grid}(p-2,q+1) \) and \( v_{2,q} \) is a corner vertex of \( G_1 \), Lemma 3.2 implies that there is a hamiltonian path \( P_1 \) of \( G_1 \) from \( v_{2,q} \) to \( u \). As \( e_{u,v} = 0 \), it is easy to see that \( e_{v_{2,q+1},v} = 1 \). As \( G_2 \) is isomorphic to \( \text{Grid}(p-2,k-q-1) \) and \( v_{2,q+1} \) is a corner vertex of \( G_2 \), Lemma 3.2 implies that there is a hamiltonian path \( P_2 \) of \( G_2 \) from \( v_{2,q+1} \) to \( v \). As \( e_{v_{2,q+1},v} = 1 \), we have \( e_{v_{1,q},v_{1,q+1}} = 0 \). Combining this with the fact that \( e_{x,u} = 1 \) and \( q \) is odd, we see that \( e_{v_{2,q+1},v_{1,q+1}} = 1 \) and \( e_{v_{1,q},v_{1,q+1}} = 0 \). By Lemma 3.4, there is a hamiltonian path \( P_3 \) of \( \text{Row}(0 : 1) \) from \( v_{1,q} \) to
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Therefore \( P_1 \cup P_2 \cup P_3 + \{(v_1,0, v_2,0), (v_1,q+1, v_2,q+1)\} \) is a hamiltonian path of \( \text{Row}(0 : p - 1) - x \) from \( u \) to \( v \).

Suppose that \( u \in V(\text{Row}(0 : 1)) \) and \( v \in V(\text{Row}(p - 1)) \). As \( k \geq 4 \), we may choose a vertex \( s \in V(\text{Row}(1)) \) such that \( s \neq u \) and \( e_{u,s} = 0 \). Clearly \( e_{x,s} = 1 \). By Lemma 3.4, there is a hamiltonian path \( P_1 \) of \( \text{Row}(0 : 1) - x \) from \( u \) to \( s \). As \( e_{u,v} = e_{u,v} = 0 \) and \( e_s,n_2(s) = 1 \), we have \( e_{n_2(s),v} = 1 \). By Lemma 3.3, there is a hamiltonian path \( P_2 \) of \( \text{Row}(2 : p - 1) \) from \( n^2(s) \) to \( v \). Then, \( P_1 \cup P_2 + \{(s, n^2(s))\} \) is a hamiltonian path of \( \text{Row}(0 : p - 1) - x \) from \( u \) to \( v \). The proof is complete. 

Given a graph \( G \), let \( S \) and \( T \) be two subsets of \( V(G) \). An \((S,T)\)-path is a path which starts at a vertex of \( S \), ends at a vertex of \( T \), and whose internal vertices belong to neither \( S \) nor \( T \).

**Lemma 3.6.** Given an even \( k \geq 4 \), let \( S = \{u, v\} \) be a set of two distinct vertices in \( \text{Row}(0 : 1) - v_{0,0} \) of \( Q_k^2 \) and let \( T = \{v_{0,1}, v_{1,0}\} \). If \( e_{u,v} = 1 \), then there exists two vertex-disjoint \((S,T)\)-paths in \( \text{Row}(0 : 1) - v_{0,0} \) that contain all vertices of \( \text{Row}(0 : 1) - v_{0,0} \).

**Proof.** As \( e_{u,v} = 1 \), without loss of generality, assume that \( u \) is even and \( v \) is odd. We consider the following two cases. In each case, we will construct two vertex-disjoint \((S,T)\)-paths \( P_1 \) and \( P_2 \) in \( \text{Row}(0 : 1) - v_{0,0} \).

**Case 1.** \( v = v_{1,0} \). In this case, \( u \) is in \( G_1 = \text{Row}(0 : 1) \cap \text{Col}(1 : k - 1) \) which is isomorphic to \( \text{Grid}(2, k - 1) \). As \( v_{0,1} \) is odd and \( u \) is even, we have \( e_{v_{0,1},u} = 1 \). Combining this with the fact that \( v_{0,1} \) is a corner vertex of \( G_1 \), Lemma 3.1 implies that there is a hamiltonian path \( P_1 \) of \( G_1 \) from \( u \) to \( v_{0,1} \). Let \( P_2 = v \). Clearly, \( P_1 \) and \( P_2 \) are vertex-disjoint \((S,T)\)-paths in \( \text{Row}(0 : 1) - v_{0,0} \) that contain all vertices of \( \text{Row}(0 : 1) - v_{0,0} \).

**Case 2.** \( v \neq v_{1,0} \). In this case, \( u \) and \( v \) are in \( \text{Row}(0 : 1) \cap \text{Col}(1 : k - 1) \). Let \( u = v_{i,j} \) and \( v = v_{i,j'}, \) where \( 0 \leq i, i' \leq 1 \) and \( 1 \leq j, j' \leq k - 1 \). Without loss of generality, we may assume that \( j \leq j' \).

Suppose first that \( j \neq j' \). Let \( G_1 = \text{Row}(0 : 1) \cap \text{Col}(1 : j) \) and \( G_2 = \text{Row}(0 : 1) \cap \text{Col}(j + 1 : k - 1) \). Observe that \( G_1 \) is isomorphic to \( \text{Grid}(2, j) \) and \( G_2 \) is isomorphic to \( \text{Grid}(2, k - j - 1) \). As \( v_{0,1} \) is a vertex of \( G_1 \), \( v_{1,k-1} \) is a corner vertex of \( G_2 \) and \( e_{v_{0,1},u} = 1 \), \( e_{v_{1,k-1},v} = 1 \), Lemma 3.1 implies that \( G_1 \) has a hamiltonian path \( P_1 \) from \( u \) to \( v_{0,1} \) and \( G_2 \) has a hamiltonian path \( P_2 \) from \( v \) to \( v_{1,k-1} \). Let \( P_2 = P_2^1 + \{(v_{1,k-1}, v_{0,0})\} \). Then \( P_1 \) and \( P_2 \) are vertex-disjoint \((S,T)\)-paths in \( \text{Row}(0 : 1) - v_{0,0} \) that contain all vertices of \( \text{Row}(0 : 1) - v_{0,0} \).

Suppose next that \( j = j' \). If \( 2 \leq j = j' \leq k - 2 \), let \( G_1 = \text{Row}(0 : 1) \cap \text{Col}(1 : j' - 1) \) and \( G_2 = \text{Row}(0 : 1) \cap \text{Col}(j + 1 : k - 1) \). Recall that \( u = v_{i,j} \) is even and \( v = v_{i,j'} \) is odd. Then \( e_{v_{0,1},v_{i,j'-1}} = 1 \) and \( e_{v_{1,k-1},v_{i,j'+1}} = 1 \). Observe that \( G_1 \) and \( G_2 \) are isomorphic to \( \text{Grid}(2, j' - 1) \) and \( \text{Grid}(2, k - j - 1) \), respectively. By
Lemma 3.1, there is a hamiltonian path $P_1$ of $G_1$ from $v_{i,j'-1}$ to $v_{0,1}$ and there is a hamiltonian path $P_2$ of $G_2$ from $v_{i,j+1}$ to $v_{1,k-1}$. Let $P_1 = P_1^1 + \{(v,v_{i,j'-1})\}$ and $P_2 = P_2^1 + \{(u,v_{i,j+1}),(v_{1,k-1},v_0)\}$. If $j = j' = 1$, then $u = v_{1,1}$, $v = v_{0,1}$. Let $P_1 = v$ and $P_2 = P_2^1 + \{(u,v_{1,2}),(v_{1,k-1},v_0)\}$. If $j = j' = k - 1$, then $u = v_{1,k-1}$, $v = v_{0,k-1}$. Let $P_1 = P_1^1 + \{(v,v_{0,k-2})\}$ and $P_2 = uv_{1,0}$. Therefore, $P_1$ and $P_2$ are as required.

Lemma 3.7. Let odd $v_{a,b}$ and odd $v_{a',b'}$ be two distinct vertices in $\text{Row}(0:1)$ of $Q_k$ and let $S = \{v_{1,1}, v_{1,3}\}$ and $T = \{v_{a',b'}, v_{0,2}\}$. Then there exist two vertex-disjoint $(S,T)$-paths in $\text{Row}(0:1) - v_{a,b}$ that contain all vertices of $\text{Row}(0:1) - v_{a,b}$.

Proof. We distinguish four cases. In each case, we will construct two vertex-disjoint $(S,T)$-paths $P_1$ and $P_2$ in $\text{Row}(0:1) - v_{a,b}$.

Case 1. $v_{a,b}, v_{a',b'} \in V(\text{Col}(0:1))$. In this case $v_{a,b}, v_{a',b'} \in \{v_{0,1}, v_{1,0}\}$. Let $P_1 = v_{1,1}v_{1,2}v_{0,1}$. Then $P_1$ is a path from $v_{1,1}$ to $v_{0,2}$. If $v_{a,b} = v_{0,1}$ and $v_{a',b'} = v_{1,0}$, let $P_2 = v_{1,3}v_{0,3}v_{0,0}v_{1,0}$. If $v_{a,b} = v_{1,0}$ and $v_{a',b'} = v_{0,1}$, let $P_2 = v_{1,3}v_{0,3}v_{0,0}v_{1,0}$. Then $P_2$ is a path from $v_{1,3}$ to $v_{a',b'}$. Therefore, there exist two vertex-disjoint $(S,T)$-paths in $\text{Row}(0:1) - v_{a,b}$ that contain all vertices of $\text{Row}(0:1) - v_{a,b}$.

Case 2. $v_{a,b}, v_{a',b'} \in V(\text{Col}(2:3))$. In this case $v_{a,b}, v_{a',b'} \in \{v_{0,3}, v_{1,2}\}$. Let $P_1 = v_{1,1}v_{1,0}v_{0,0}v_{1,0}v_{0,2}$. Then $P_1$ is a path from $v_{1,1}$ to $v_{0,2}$. If $v_{a,b} = v_{0,3}$ and $v_{a',b'} = v_{1,2}$, let $P_2 = v_{1,3}v_{1,2}$. If $v_{a,b} = v_{1,2}$ and $v_{a',b'} = v_{0,3}$, let $P_2 = v_{1,3}v_{0,3}v_{0,0}v_{1,0}$. Then $P_2$ is a path from $v_{1,3}$ to $v_{a',b'}$. Therefore, $P_1$ and $P_2$ are as required.

Case 3. $v_{a',b'} \in V(\text{Col}(0:1))$ and $v_{a,b} \in V(\text{Col}(2:3))$. In this case $v_{a',b'} \in \{v_{0,1}, v_{1,0}\}$ and $v_{a,b} \in \{v_{0,3}, v_{1,2}\}$. If $v_{a',b'} = v_{0,1}$, let $P_1 = v_{1,1}v_{1,0}v_{0,0}v_{1,0}$. If $v_{a',b'} = v_{1,0}$, let $P_1 = v_{1,1}v_{1,0}v_{0,0}v_{1,0}$. Then $P_1$ is a path from $v_{1,1}$ to $v_{a',b'}$. Suppose first that $v_{a,b} = v_{0,3}$. Let $P_2 = v_{1,3}v_{1,2}v_{0,2}v_{0,0}$. Suppose next that $v_{a,b} = v_{1,2}$. Let $P_2 = v_{1,3}v_{0,3}v_{0,0}v_{1,0}$. Then $P_2$ is a path from $v_{1,3}$ to $v_{0,2}$. Therefore, $P_1$ and $P_2$ are as required.

Case 4. $v_{a,b} \in V(\text{Col}(0:1))$ and $v_{a',b'} \in V(\text{Col}(2:3))$. In this case $v_{a,b} \in \{v_{0,1}, v_{1,0}\}$ and $v_{a',b'} \in \{v_{0,3}, v_{1,2}\}$. If $v_{a,b} = v_{0,1}$, let $P_1 = v_{1,1}v_{1,0}v_{0,0}v_{0,3}$. If $v_{a,b} = v_{1,0}$, let $P_1 = v_{1,1}v_{1,1}v_{0,1}v_{0,0}v_{0,3}$. Suppose first that $v_{a',b'} = v_{0,3}$. Let $P_1 = P_1^1$ and let $P_2 = v_{1,3}v_{1,2}v_{0,2}$. Then $P_1$ is a path from $v_{1,1}$ to $v_{0,3} = v_{a',b'}$ and $P_2$ is path from $v_{1,3}$ to $v_{0,2}$. Suppose next that $v_{a',b'} = v_{1,2}$. Let $P_1 = P_1^1 + \{(v_{0,3}, v_{0,2})\}$ and let $P_2 = v_{1,3}v_{1,2}$. Then $P_1$ is a path from $v_{1,1}$ to $v_{0,3}$ and $P_2$ is path from $v_{1,3}$ to $v_{1,2} = v_{a',b'}$. Therefore, $P_1$ and $P_2$ are as required.

Lemma 3.8. Given an even $k \geq 6$, let $S = \{v_{1,1}, v_{1,k-1}\}$ and let odd $v_{a,b}$ and odd $v_{a',b'}$ be two distinct vertices in $\text{Row}(0:1)$ of $Q_k$. Then there exists a set
Let \( T = \{v_{a,b}, v_{c}\} \) (c is even) such that there are two vertex-disjoint \((S, T)\)-paths in \(\text{Row}(0 : 1) - v_{a,b}\) that contain all vertices of \(\text{Row}(0 : 1) - v_{a,b}\).

**Proof.** Without loss of generality, we assume that \( v_{a,b} \in \text{Col}(0 : \frac{k}{2}) \). We distinguish four cases. In each case, we will construct two vertex-disjoint \((S, T)\)-paths \(P_1\) and \(P_2\) in \(\text{Row}(0 : 1) - v_{a,b}\).

**Case 1.** \( v_{a,b} \in V(\text{Col}(0 : 1)) \) and \( v_{a',b'} \in V(\text{Col}(0 : 1)) \). As both \( v_{a,b} \) and \( v_{a',b'} \) are odd, we have \( v_{a,b}, v_{a',b'} \in \{v_{0,1}, v_{1,0}\} \). Let \( v_{0,c} = v_{0,2} \). We will construct an \((S, T)\)-path \(P_1\) from \(v_{1,1}\) to \(v_{0,c}\) and an \((S, T)\)-path \(P_2\) from \(v_{1,k-1}\) to \(v_{a',b'}\). Let \( G_1 = \text{Row}(0 : 1) \cap \text{Col}(3 : k - 1) \). Observe that \( G_1 \) is isomorphic to \(\text{Grid}(2, k - 3)\).

Let \( P_1 = v_{1,1}v_{1,0}v_{0,0} \). As \( v_{1,k-1} \) is a corner vertex of \( G_1 \) and \( e_{v_{0,k-1},v_{1,k-1}} = 1 \), Lemma 3.1 implies that there is a hamiltonian path \( P_1^1 \) of \( G_1 \) from \( v_{1,k-1} \) to \( v_{0,k-1} \). Then \( P_2 = P_1^1 + \{(v_{0,k-1}, v_{0,0}), (v_{0,0}, v_{a',b'})\} \) is as required.

**Case 2.** \( v_{a,b} \in V(\text{Col}(0 : 1)) \) and \( v_{a',b'} \in V(\text{Col}(2 : k - 1)) \). In this case, let \( v_{0,c} = v_{0,0} \). As the odd \( v_{a,b} \) is in \( G_1 = \text{Row}(0 : 1) \cap \text{Col}(0 : 1) \), we have \( v_{a,b} \in \{v_{0,1}, v_{1,0}\} \). If \( v_{a,b} = v_{0,1} \), let \( P_1 = v_{1,1}v_{1,0}v_{0,0} \). If \( v_{a,b} = v_{1,0} \), let \( P_1 = v_{1,1}v_{0,1}v_{0,0} \). Then \( P_1 \) is a hamiltonian path of \( G_1 - v_{a,b} \) from \( v_{1,1} \) to \( v_{0,c} \). Observe that \( G_2 = \text{Row}(0 : 1) \cap \text{Col}(2 : k - 1) \) is isomorphic to \(\text{Grid}(2, k - 2)\). Combining this with the fact that \( v_{1,k-1} \) is a corner vertex of \( G_2 \) and \( e_{v_{0,k-1},v_{1,k-1}} = 1 \), there is a hamiltonian path \( P_2 \) of \( G_2 \) from \( v_{1,k-1} \) to \( v_{a',b'} \). It can be seen that \( P_1 \) and \( P_2 \) are vertex-disjoint \((S, T)\)-paths in \(\text{Row}(0 : 1) - v_{a,b}\) that contain all vertices of \(\text{Row}(0 : 1) - v_{a,b}\).

**Case 3.** \( v_{a,b} \in V(\text{Col}(2 : \frac{k}{2})) \) and \( v_{a',b'} \in V(\text{Col}(0 : 1)) \). As \( G_1 = \text{Row}(0 : 1) \cap \text{Col}(0 : 1) \) is isomorphic to \(\text{Grid}(2, 2)\), \( v_{1,1} \) is a corner vertex of \( G_1 \) and \( e_{v_{a',b'},v_{1,1}} = 1 \), Lemma 3.1 implies that there is a hamiltonian path \( P_1 \) of \( G_1 \) from \( v_{1,1} \) to \( v_{a',b'} \). Let \( v_{0,c} = v_{0,2} \). It is enough to construct a hamiltonian path \( P_2 \) of \( G_2 = \text{Row}(0 : 1) \cap \text{Col}(2 : k - 1) \) from \( v_{1,k-1} \) to \( v_{0,c} \).

Suppose first that \( v_{a,b} \) is in \(\text{Col}(2)\). Then \( v_{a,b} = v_{1,2} \). As \( \text{Row}(0 : 1) \cap \text{Col}(3 : k - 1) \) is isomorphic to \(\text{Grid}(2, k - 3)\), \( v_{0,3} \) is a corner vertex of \(\text{Row}(0 : 1) \cap \text{Col}(3 : k - 1) \) and \( e_{v_{0,3},v_{1,k-1}} = 1 \), Lemma 3.1 implies that there is a hamiltonian path \( P_2^1 \) of \(\text{Row}(0 : 1) \cap \text{Col}(3 : k - 1) \) from \( v_{1,k-1} \) to \( v_{0,3} \). Then \( P_2 = P_2^1 + \{(v_{0,3}, v_{0,2})\} \) is as required.

Suppose next that \( v_{a,b} \) is not in \(\text{Col}(2)\). Then \(\text{Row}(0 : 1) \cap \text{Col}(2 : b - 1) \) is isomorphic to \(\text{Grid}(2, b - 2)\) and \(\text{Row}(0 : 1) \cap \text{Col}(b + 1 : k - 1) \) is isomorphic to \(\text{Grid}(2, k - b - 1)\). If \( a = 0 \) then \( a = 1 \), and if \( a = 1 \) then \( a = 0 \). As \( v_{a,b} \) is odd, it can be seen that both \( v_{0,b-1} \) and \( v_{a,b+1} \) are odd. Thus \( e_{v_{0,2},v_{0,b-1}} = 1 \) and \( e_{v_{1,k-1},v_{a,b+1}} = 1 \). As \( v_{0,2} \) is a corner vertex of \(\text{Row}(0 : 1) \cap \text{Col}(2 : b - 1) \) and \( v_{1,k-1} \) is a corner vertex of \(\text{Row}(0 : 1) \cap \text{Col}(b + 1 : k - 1) \), Lemma 3.1 implies that there is a hamiltonian path \( P_2^1 \) of \(\text{Row}(0 : 1) \cap \text{Col}(2 : b - 1) \) from \( v_{a,b-1} \) to \( v_{0,2} \) and a hamiltonian path \( P_2^2 \) in \(\text{Row}(0 : 1) \cap \text{Col}(b + 1 : k - 1) \) from \( v_{1,k-1} \) to \( v_{0,2} \).
Combining $P_1^2$ with $P_2^2$ as well as the edges $(v_{a,b-1}, v_{a,b})$ and $(v_{a,b}, v_{a,b+1})$, we may obtain the required path $P_2$.

**Case 4.** $v_{a,b} \in V(Col(2 : \frac{k}{2}))$ and $v_{a',b'} \in V(Col(2 : k - 1))$.

**Case 4.1.** $v_{a,b} \in \text{Row}(0)$, that is, $v_{a,b} = v_{0,b}$. Suppose first that $b' > b$.

As $b$ is odd, we have that $v_{1,b-1}$ is odd and $v_{0,b+1}$ is even. Let $v_{0,c} = v_{0,b+1}$. Observe that $G_1 = \text{Row}(0 : 1) \cap \text{Col}(0 : b - 1)$ is isomorphic to $\text{Grid}(2, b)$ and $G_2 = \text{Row}(0 : 1) \cap \text{Col}(b + 2 : k - 1)$ is isomorphic to $\text{Grid}(2, k - b - 2)$. As $v_{1,b-1}$ is a corner vertex of $G_1$ and $e_{v_{1,k-1},v_{1,k-1}} = 1$, there is a hamiltonian path $P_1^2$ of $G_1$ from $v_{1,1}$ to $v_{1,b-1}$. If $v_{a',b'} = v_{1,b+1}$, let $P_1 = P_1^2 + \{(v_{1,b-1}, v_{1,b}) \cdot (v_{1,b}, v_{1,b+1})\}$. As $v_{1,k-1}$ is a corner vertex of $G_2$ and $e_{v_{1,k-1},v_{0,k+2}} = 1$, there is a hamiltonian path $P_2^2$ of $G_2$ from $v_{1,k-1}$ to $v_{0,b+2}$. Let $P_2 = P_2^2 + \{(v_{0,b+2}, v_{0,b+1})\}$. Then $P_1$ is an $(S,T)$-path from $v_{1,1}$ to $v_{a',b'}$ and $P_2$ is an $(S,T)$-path from $v_{1,k-1}$ to $v_{0,b+1} = v_{0,c}$. If $v_{a',b'} \neq v_{1,b+1}$, let $P_1 = P_1^1 \cdot v_{1,b-1} \cdot v_{1,b} \cdot (v_{1,b}, v_{1,b+1}) \cdot (v_{1,b+1}, v_{0,b+1})$. Then $P_1$ is an $(S,T)$-path from $v_{1,1}$ to $v_{0,b+1} = v_{0,c}$. Note that now $v_{a',b'}$ is in $G_2$. As $e_{v_{1,k-1},v_{a',b'}} = 1$, there is a hamiltonian $(S,T)$-path $P_2$ of $G_2$ from $v_{1,k-1}$ to $v_{a',b'}$. Furthermore, it can be seen that $P_1$ and $P_2$ are vertex-disjoint $(S,T)$-paths and contain all vertices of $\text{Row}(0 : 1) - v_{a,b}$.

Suppose next that $b' < b$. As $b$ is odd, we have $v_{1,b+1}$ is odd and $v_{0,b+1}$ is even. Let $v_{0,c} = v_{0,b+1}$. By a similar proof above, we may obtain two required $(S,T)$-paths.

**Case 4.2.** $v_{a,b}$ is in $\text{Row}(1)$, that is, $v_{a,b} = v_{1,b}$. We only consider the case that $b' > b$ since the proof for $b' < b$ is similar. Let $G_1 = \text{Row}(0 : 1) \cap \text{Col}(0 : b - 1)$ and $G_2 = \text{Row}(0 : 1) \cap \text{Col}(b + 1 : k - 1)$. Observe that $G_1$ is isomorphic to $\text{Grid}(2,b)$ and $G_2$ is isomorphic to $\text{Grid}(2, k - b - 1)$. As $v_{1,b} = v_{a,b}$ is odd, we have $v_{0,b-1}$ is odd and $v_{0,b}$ is even. Let $v_{0,c} = v_{0,b}$. As $e_{v_{0,b-1},v_{0,b}} = 1$ and $v_{0,b-1}$ is a corner vertex of $G_1$, Lemma 3.1 implies that there is a hamiltonian path $P_1^1$ of $G_1$ from $v_{1,1}$ to $v_{0,b-1}$. Let $P_1 = P_1^1 + \{(v_{0,b-1}, v_{0,b})\}$. Then $P_1$ is an $(S,T)$-path from $v_{1,1}$ to $v_{0,b} = v_{0,c}$. As $e_{v_{1,b-1},v_{1,k-1}} = 1$ and $v_{1,k-1}$ is a corner vertex of $G_2$, Lemma 3.1 implies that there is a hamiltonian path $P_2$ of $G_2$ from $v_{1,k-1}$ to $v_{a',b'}$. It can be seen that $P_1$ and $P_2$ are vertex-disjoint $(S,T)$-paths in $\text{Row}(0 : 1) - v_{a,b}$ that contain all vertices of $\text{Row}(0 : 1) - v_{a,b}$.

**Lemma 3.9.** Let $S = \{v_{1,1}, v_{1,5}\}$ and let odd $v_{1,b}$ and odd $v_{a',b'}$ be two distinct vertices in $\text{Row}(0 : 1)$ of $Q_2^6$. Then there exists a set $T = \{v_{a',b'}, v_{0,c}\} \quad (c = 2$ or $4)$ such that there are two vertex-disjoint $(S,T)$-paths in $\text{Row}(0 : 1) - v_{1,b}$ that contain all vertices of $\text{Row}(0 : 1) - v_{1,b}$.

**Proof.** As $v_{1,b}$ is odd, we have $v_{1,b} \in \{v_{1,0}, v_{1,2}, v_{1,4}\}$. If $v_{1,b} = v_{1,2}$ (resp. $v_{1,4}$), let $v_{0,c} = v_{0,2}$ (resp. $v_{0,4}$). Using similar proofs of Case 3 and Case 4.2 in Lemma 3.8, we may obtain two vertex-disjoint $(S,T)$-paths in $\text{Row}(0 : 1) - v_{1,b}$ that contain all vertices of $\text{Row}(0 : 1) - v_{1,b}$.
Suppose that \( v_{1,0} = v_{1,0} \). Let \( v_{0,0} = v_{0,2} \). If \( v_{a,b} \in V(Col(1)) \), then \( v_{a,b} = v_{0,1} \). Similar to Case 1 of Lemma 3.8, we may obtain two vertex-disjoint \((S,T)\)-paths in \( Row(0 : 1) - v_{1,0} \) that contain all vertices of \( Row(0 : 1) - v_{1,0} \). If \( v_{a,b} \in V(Col(5)) \), then \( v_{a,b} = v_{0,5} \), let \( P_1 = v_{1,1}v_{1,0}v_{0,0}v_{0,5} \) and \( P_2 = v_{1,5}v_{1,4}v_{0,4}v_{0,3}v_{1,3}v_{1,2}v_{0,2} \). Obviously, \( P_1 \) and \( P_2 \) are as required. If \( v_{a,b} \in V(Col(2 : 4)) \), then \( v_{a,b} \in \{ v_{1,2}, v_{0,3}, v_{1,4} \} \). Let \( P_1 = v_{0,5}v_{0,0}v_{0,1}v_{0,2} \) and \( G = Row(0 : 1) \cap Col(b' + 1 : 5) \). Observe that \( G \) is isomorphic to \( Grid(2, 5 - b') \). As \( v_{1,5} \) is a corner vertex of \( G \) and \( e_{v_{1,5}v_{0,5}} = 1 \), Lemma 3.1 implies that there is a hamiltonian path \( P_2 \) of \( G \) from \( v_{1,5} \) to \( v_{0,5} \). Then \( P_1 = P_1^1 \cup P_2^2 \) is an \((S,T)\)-path from \( v_{1,5} \) to \( v_{0,2} = v_{0,0} \). If \( v_{a,b} = v_{1,2} \), then \( P_1 = v_{1,1}v_{1,2} \). If \( v_{a,b} = v_{0,3} \), then \( P_1 = v_{1,1}v_{1,2}v_{1,3}v_{0,3} \). If \( v_{a,b} = v_{1,4} \), then \( P_1 = v_{1,1}v_{1,2}v_{1,3}v_{0,3}v_{0,4}v_{1,4} \). Hence \( P_2 \) is an \((S,T)\)-path from \( v_{1,1} \) to \( v_{1,b}^a \). Therefore, \( P_1 \) and \( P_2 \) are as required.

Lemma 3.10. Given an integer \( k \in \{ 4, 6 \} \), let even \( u \) be a vertex in \( Row(0 : 1) - v_{0,0} \) of \( Q_k^b \). Let \( S = \{ u, v_{0,k-1} \} \) and \( T = \{ v_{1,2}, v_{0,1} \} \). Then there are two vertex-disjoint \((S,T)\)-paths in \( Row(0 : 1) - v_{0,0} \) that contain all vertices of \( Row(0 : 1) - v_{0,0} \).

Proof. As \( u \neq v_{0,0} \) is even, we have \( u \in V(Col(1 : k-1)) \). If \( u \in V(Col(1)) \), then \( u = v_{1,1} \). Let \( P_1 = Row(1) - \{ (v_{1,1}, v_{1,2}) \} \) and \( P_2 = Row(0) - v_{0,0} \). Obviously, \( P_1 \) and \( P_2 \) are two vertex-disjoint \((S,T)\)-paths in \( Row(0 : 1) - v_{0,0} \) that contain all vertices of \( Row(0 : 1) - v_{0,0} \). If \( u \in V(Col(k - 1)) \), then \( u = v_{1,k-1} \). Let \( P_1 = v_{1,k-1}v_{1,0}v_{1,1} \). If \( k = 4 \), let \( P_2 = v_{0,3}v_{0,2} \). If \( k = 6 \), let \( P_2 = v_{0,5}v_{0,4}v_{1,4}v_{1,3}v_{0,3}v_{0,2}v_{0,1} \). Then \( P_1 \) and \( P_2 \) are as required. If \( u \in V(Col(2 : k - 2)) \), let \( G = Row(0 : 1) \cap Col(2 : k - 2) \). Observe that \( G \) is isomorphic to \( Grid(2, k - 3) \). As odd \( v_{1,2} \) is a corner vertex of \( G \) and \( u \) is even, Lemma 3.1 implies that there is a hamiltonian path \( P_1 \) of \( G \) from \( u \) to \( v_{1,2} \). Let \( P_2 = v_{0,k-1}v_{1,k-1}v_{1,0}v_{1,1} \). Clearly, \( P_1 \) and \( P_2 \) are as required.

Note that in a \( Q_k^b \), \( Col(1 : 3) \) and \( Col(3 : 5) \) are isomorphic. By a similar proof above, we have following corollary.

Corollary 3.11. Let even \( u \) be a vertex in \( Row(0 : 1) - v_{0,0} \) of \( Q_k^b \) and let \( S = \{ u, v_{0,5} \}, T = \{ v_{1,4}, v_{0,1} \} \). Then there are two vertex-disjoint \((S,T)\)-paths in \( Row(0 : 1) - v_{0,0} \) that contain all vertices of \( Row(0 : 1) - v_{0,0} \).

We define the following paths in \( Row(i : i+1) \) of a \( Q_k^b \). Let \( i \leq a \leq i + 1, 0 \leq b, m \leq k - 1 \) and \( m \neq b \). If \( a = i \) then \( a = i + 1 \), and if \( a = i + 1 \) then \( a = i \).

\[
C_{m}^{+}(v_{a,b}, v_{\bar{a},\bar{b}}) = v_{a,b}v_{a,b+1}v_{a,b+2} \cdots v_{a,m-1}v_{a,m}v_{\bar{a},m-1}v_{\bar{a},m-2} \cdots v_{\bar{a},b+1}v_{\bar{a},b}.
\]

\[
C_{m}^{-}(v_{a,b}, v_{\bar{a},\bar{b}}) = v_{a,b}v_{a,b-1}v_{a,b-2} \cdots v_{a,m+1}v_{a,m}v_{\bar{a},m+1}v_{\bar{a},m+2} \cdots v_{\bar{a},b-1}v_{\bar{a},b}.
\]

In addition, if \( m = b \), we define \( C_{b}^{+}(v_{a,b}, v_{\bar{a},\bar{b}}) = C_{b}^{-}(v_{a,b}, v_{\bar{a},\bar{b}}) = (v_{a,b}, v_{\bar{a},\bar{b}}) \).
Theorem 3.12. Given an even $k \geq 4$, let $F_v = \{u^*, v^*\}$ be a set of faulty vertices of $Q_2^k$ such that $e_{u^*, v^*} = 1$ and let $u$ and $v$ be any two healthy vertices of $Q_2^k$ such that $e_{u, v} = 1$. Then there exists a hamiltonian path of $Q_2^k - F_v$ from $u$ to $v$.

Proof. Without loss of generality, we may assume that $u^* = v_0$ and $u^* = v_0$ is even, we see that $v^*$ is odd. Let $v^* = u_{a,b}$ where $0 \leq a, b \leq k - 1$.

As $Row(1 : k - 1)$ is isomorphic to $Col(1 : k - 1)$, it is enough to consider $v^*$ is in $Row(1 : k - 1)$. Furthermore, we may assume that $v^*$ is in $Row(\frac{k}{2} : k - 1)$ because $Row(1 : \frac{k}{2})$ and $Row(\frac{k}{2} : k - 1)$ are isomorphic.

If $a$ is odd, let $p = a - 2$. If $a$ is even, let $p = a - 1$. Clearly, $p$ is odd and $v^* = u_{a,b} \in V(Row(p + 1 : p + 2))$. Let $u = u_{i,j}$ and $v = v_{j',j''}$. We consider the following five cases.

Case 1. $u, v \in V(Row(0 : 1))$. Let $S = \{u, v\}$ and $T = \{v_{0,1}, v_{1,0}\}$. As $e_{u, v} = 1$, Lemma 3.6 implies that there exists two vertex-disjoint $(S,T)$-paths $P_1, P_2$ in $Row(0 : 1) - v_0$ that contain all vertices of $Row(0 : 1) - v_0$. Recall that odd $v^*$ is in $Row(p + 1 : p + 2)$. As even $v_{p+1,0}$ and even $v_{k-1,1}$ are two distinct vertices in $Row(p + 1 : k - 1)$, Lemma 3.4 and Lemma 3.5(iii) imply that there exists a hamiltonian path $P_3$ of $Row(p + 1 : k - 1) - v^*$ from $v_{p+1,0}$ to $v_{k-1,1}$.

If $p = 1$, then $P_1 \cup P_2 \cup P_3 + \{(v_{0,1}, v_{2,0}), (v_{0,1}, v_{k-1,1})\}$ is a hamiltonian path of $Q_2^k - F_v$ from $u$ to $v$. Suppose that odd $p \geq 3$. As $e_{v_{2,0}, v_{p,0}} = 1$, Lemma 3.3 implies that there exists a hamiltonian path $P_3$ of $Row(2 : p)$ from $v_{2,0}$ to $v_{p,0}$. Then $\bigcup_{d=1}^4 P_d + \{(v_{0,1}, v_{2,0}), (v_{0,1}, v_{k-1,1}), (v_{p,0}, v_{p+1,0})\}$ is a hamiltonian path of $Q_2^k - F_v$ from $u$ to $v$.

Case 2. $u \in V(Row(0 : 1))$ and $v \in V(Row(2 : p))$. As $v \in V(Row(2 : p))$, it is easy to see that odd $p \geq 3$. Noting that $v^* = u_{a,b} \in V(Row(p + 1, p + 2))$, we see that $Row(p + 2)$ exists. Then $k - 1 \geq p + 2 \geq 5$, and so $k \geq 6$. We distinguish two cases.

Case 2.1. $u$ is even and $v$ is odd. Let $G_1 = Row(0 : 1) \cap Col(1 : j)$. Observe that $G_1$ is isomorphic to $Grid(2, j)$. As $e_{v_{0,1}, u} = 1$ and $v_{0,1}$ is a corner vertex of $G_1$, Lemma 3.1 implies that there is a hamiltonian path $P_1$ of $G_1$ from $u$ to $v_{0,1}$. Let $P_2 = C_{j+1}(v_{0,k-1}, v_{1,k-1}) + \{(v_{0,1}, v_{1,k-1})\}$. Then $P_1$ and $P_2$ are two vertex-disjoint paths in $Row(0 : 1) - v_0$ that contain all vertices of $Row(0 : 1) - v_0$. Noting that $v$ is odd, we have $e_{v_{2,0}, v} = 1$. By Lemma 3.3, there is a hamiltonian path $P_3$ of $Row(2 : p)$ from $v_{2,0}$ to $v$. As $k$ is even and $v^*$ is odd, we have $e_{v_{2k-1,1}, v_{k-1,k-1}} = 0$ and $e_{v_{k-1,1}, v_{k-1,k-1}} = 1$. Combining this with the fact that $v^* \in V(Row(p + 1 : p + 2))$, Lemma 3.4 and Lemma 3.5(ii) imply that there is a hamiltonian path $P_4$ of $Row(p + 1 : k - 1) - v^*$ from $v_{k-1,1}$ to $v_{k-1,k-1}$. Then $\bigcup_{d=1}^4 P_d + \{(v_{0,1}, v_{k-1,1}), (v_{1,0}, v_{2,0}), (v_{0,k-1}, v_{k-1,k-1})\}$ is a hamiltonian path of $Q_2^k - F_v$ from $u$ to $v$.

Case 2.2. $u$ is odd and $v$ is even. Noting that $v$ is even and $p$ is odd, we have $e_{v_{p,0}, v} = 1$. By Lemma 3.3, there exists a hamiltonian path $P_1$ of $Row(2 : p)$
from $v_{p,0}$ to $v$. As $k \geq 6$, we may choose a vertex $w \in V(\text{Row}(0))$ such that $w \neq u$ and $e_{w,u} = 0$. Combining this with the fact that $e_{v_{0,0},u} = 1$, Lemma 3.4 implies that there exists a Hamiltonian path $P_2$ of $\text{Row}(0 : 1) - v_{0,0}$ from $u$ to $w$. By $e_{w,nk-1}(w) = 1$, we have $n^{k-1}(w)$ is even. Note that $v_{p+1,0}$ is even and $v^* \in V(\text{Row}(p + 1 : p + 2))$ is odd. By Lemma 3.4 and Lemma 3.5(iii), there is a Hamiltonian path $P_3$ of $\text{Row}(p + 1 : k - 1) - v^*$ from $n^{k-1}(w)$ to $v_{p+1,0}$. Then $P_1 \cup P_2 \cup P_3 + \{(w, n^{k-1}(w)), (v_{p,0}, v_{p+1,0})\}$ is a Hamiltonian path of $Q^k_a - F_v$ from $u$ to $v$.

**Case 3.** $u \in V(\text{Row}(0 : 1))$ and $v \in V(\text{Row}(p + 1 : p + 2))$.

**Case 3.1.** $u$ is odd and $v$ is even. Assume first that $k = 4$. Then $\text{Row}(p + 1 : p + 2) = \text{Row}(2 : 3)$. Let $v'$ be the neighbour of $v$ in $\text{Row}(0 : 1)$. It is easy to see that we may choose an odd $u'$ in $\text{Row}(0 : 1) - u$ such that $u' \neq v'$. Denote the neighbour of $u'$ in $\text{Row}(2 : 3)$ by $u''$. As $u^*$ is even and both $u$ and $u'$ are odd, Lemma 3.4 implies that there is a Hamiltonian path $P_1$ of $\text{Row}(0 : 1) - u^*$ from $u$ to $u'$. Similarly, there is a Hamiltonian path $P_2$ of $\text{Row}(2 : 3) - v^*$ from $u''$ to $v$. Then $P_1 \cup P_2 + \{(u, u')\}$ is a Hamiltonian path of $Q^2_a - F_v$ from $u$ to $v$.

Suppose next that $k \geq 6$. As $\frac{k}{2} - 2 \geq 3 - 2 = 1$, we may choose an odd $x$ in $\text{Row}(p)$ such that $x \neq u$ and $n^{p+1}(x) \neq v$. Then $e_{x,u} = 0$. Note that $e_{u^*,u} = 1$ and $u^* \in V(\text{Row}(0 : 1))$. By Lemma 3.4 and Lemma 3.5(iii), there exists a Hamiltonian path $P_1$ in $\text{Row}(0 : p) - u^*$ from $u$ to $x$. As $x$ is odd, we have $n^{p+1}(x)$ is even. Recalling that $v^* \in V(\text{Row}(p + 1 : p + 2))$ is odd and $v$ is even, Lemma 3.4 and Lemma 3.5(i) imply that there is a Hamiltonian path $P_2$ of $\text{Row}(p + 1 : k - 1) - v^*$ from $n^{p+1}(x)$ to $v$. Then $P_1 \cup P_2 + \{(x, n^{p+1}(x))\}$ is a Hamiltonian path of $Q^k_a - F_v$ from $u$ to $v$.

**Case 3.2.** $u$ is even and $v$ is odd.

**Case 3.2.1.** $k = 4$. In this case, $\text{Row}(p + 1 : p + 2) = \text{Row}(2 : 3)$. Let $S = \{u, v_{0,3}\}$ and $T = \{v_{1,2}, v_{0,1}\}$. By Lemma 3.10, there exist a $w_{1,2}$-path $P_1$ and a $v_{0,3}v_{0,1}$-path $P_2$ in $\text{Row}(0 : 1) - v_{0,0}$. Moreover, $P_1$ and $P_2$ are two vertex-disjoint $(S,T)$-paths that contain all vertices of $\text{Row}(0 : 1) - v_{0,0}$.

Let $S = \{v_{3,1}, v_{3,3}\}$ and $T = \{v, v_{2,2}\}$. Recall that both $v$ and $v^*$ are odd. By Lemma 3.7, there are two vertex-disjoint $(S,T)$-paths $P_3$ and $P_4$ in $\text{Row}(2 : 3) - v^*$ that contain all vertices of $\text{Row}(2 : 3) - v^*$. Then $\bigcup_{d=1}^4 P_d + \{(v_{0,1}, v_{3,1}), (v_{0,3}, v_{3,3}), (v_{1,2}, v_{2,2})\}$ is a Hamiltonian path of $Q^2_a - F_v$ from $u$ to $v$.

**Case 3.2.2.** $k \geq 6$. If $p = 1$, then $v^* = v_{a,b} \in V(\text{Row}(2 : 3))$ and so $2 \leq a \leq 3$. Recall that $v^* = v_{a,b}$ is in $\text{Row}(\frac{k}{2}, k - 1)$ and $k \geq 6$. Therefore $a \geq \frac{k}{2} \geq 3$. So $a = 3$ and $k = 2 \times 3 = 6$. Let $S = \{v_{3,1}, v_{3,3}\}$ and $T = \{v, v_{2,c}\}(c = 2 \text{ or } 4)$. By Lemma 3.9, there are two vertex-disjoint $(S,T)$-paths $P_1, P_2$ in $\text{Row}(2 : 3) - v^*$ that contain all vertices of $\text{Row}(2 : 3) - v^*$. As $v_{1,c} \in \{v_{1,2}, v_{1,4}\}$ and even $u$ is in $\text{Row}(0 : 1) - v_{0,0}$, Lemma 3.10 and Corollary 3.11 imply that there exist a path $P_3$ from $u$ to $v_{1,c}$ and a path $P_4$ from $v_{0,5}$ to $v_{0,1}$. Moreover, $P_1$ and
$P_2$ are two vertex-disjoint paths in $\text{Row}(0 : 1) - v_{0,0}$ that contain all vertices of $\text{Row}(0 : 1) - v_{0,0}$.

Let $P_5 = C_0^r(v_{4,1}, v_{5,1})$ and $P_6 = C_2^r(v_{4,5}, v_{5,5})$. Clearly, $P_5$ and $P_6$ are vertex-disjoint paths in $\text{Row}(4 : 5)$ that contain all vertices of $\text{Row}(4 : 5)$. Then

$$\bigcup_{d=1}^6 P_d + \{(v_{1,c}, v_{2,c}), (v_{0,5}, v_{5,5}), (v_{0,1}, v_{5,1}), (v_{3,5}, v_{4,5}), (v_{3,1}, v_{4,1})\}$$ is a hamiltonian path of $Q_2^5 - F_v$ from $u$ to $v$.

Suppose that $p \geq 3$. We will choose an odd $u' \in V(\text{Row}(1))$ and construct a $uu'$-path $P_1$ and a $v_{0,k-1}v_{0,1}$-path $P_2$ in $\text{Row}(0 : 1) - v_{0,0}$. Suppose first that $u \in V(\text{Row}(0))$. As $u = v_{0,j}$ is even, we have $u' = v_{1,j}$ is odd. Let $P_1 = uu'$ and $P_2 = C_{j-1}^+(v_{1,0}, v_{0,1}) \cup C_{j+1}^+(v_{1,k-1}, v_{0,k-1}) + \{(v_{1,1}, v_{1,0}), (v_{1,0}, v_{1,k-1})\}$. Then $P_1$ is a path from $u$ to $u'$ and $P_2$ is a path from $v_{0,k-1}$ to $v_{0,1}$. Obviously, $P_1$ and $P_2$ are vertex-disjoint paths that contain all vertices of $\text{Row}(0 : 1) - v_{0,0}$.

Suppose next that $u \in V(\text{Row}(1))$. As $u = v_{1,j}$ is even, we have $u' = v_{1,j-1} \in V(\text{Row}(1))$ is odd, where $1 \leq j \leq k - 1$. Let $P_1 = \text{Row}(1) - \{(v_{1,j-1}, v_{1,j})\}$ and $P_2 = v_{0,k-1}v_{0,k-2}v_{0,k-3} \ldots v_{0,1}$. Then $P_1$ is a path from $u$ to $u'$ and $P_2$ is a path from $v_{0,k-1}$ to $v_{0,1}$. Clearly, $P_1$ and $P_2$ are vertex-disjoint paths that contain all vertices of $\text{Row}(0 : 1) - v_{0,0}$.

Noting that $p$ is odd and $k$ is even, we have both $v_{p+2,1}$ and $v_{p+2,k-1}$ are even. Let $S = \{v_{p+2,1}, v_{p+2,k-1}\}$. As odd $v^*, v \in V(\text{Row}(p + 1 : p + 2))$, Lemma 3.8 implies that there exists a set $T = \{x, v\}$ $(x \in V(\text{Row}(p + 1))$ is even), such that there are two vertex-disjoint $(S, T)$-paths $P_3, P_4$ in $\text{Row}(p + 1 : p + 2) - v^*$ that contain all vertices of $\text{Row}(p + 1 : p + 2) - v^*$.

Note that $x \in V(\text{Row}(p + 1))$ and $u' \in V(\text{Row}(1))$. As $x$ is even and $u'$ is odd, it is easy to see that $e_{u'^v(x), u'^v(x)} = 1$. By Lemma 3.3, there exists a hamiltonian path $P_5$ of $\text{Row}(2 : p)$ from $n^2(u')$ to $n^p(x)$.

We will construct a hamiltonian path of $Q_2^5 - F_v$ from $u$ to $v$ in the following. Noting that $p + 2$ is odd, we consider the following two cases. If $p + 2 = k - 1$, then

$$\bigcup_{d=1}^6 P_d + \{(u', n^2(u')), (v_{0,1}, v_{p+2,1}), (v_{0,k-1}, v_{p+2,k-1}), (n^p(x), x)\}$$ is a hamiltonian path of $Q_2^5 - F_v$ from $u$ to $v$. If $p + 2 \leq k - 3$, let $G_1 = \text{Row}(p + 3 : k - 1) \cap \text{Col}(0 : 1)$ and $G_2 = \text{Row}(p + 3 : k - 1) \cap \text{Col}(2 : k - 1)$. Observe that $G_1$ is isomorphic to $\text{Grid}(k - p - 3, 2)$ and $G_2$ is isomorphic to $\text{Grid}(k - p - 3, 2)$. As $p$ is odd and $k$ is even, we have $e_{v_{p+3,1}, v_{k-1,1}} = e_{v_{p+3,1}, v_{k-1,1}} = 1$. As $v_{p+3,1}$ and $v_{p+3,k-1}$ are corner vertices of $G_1$ and $G_2$, respectively, Lemma 3.2 implies that there are a hamiltonian path $P_6$ of $G_1$ from $v_{p+3,1}$ to $v_{k-1,1}$ and a hamiltonian path $P_7$ of $G_2$ from $v_{p+3,k-1}$ to $v_{k-1,k-1}$. Then

$$\bigcup_{d=1}^6 P_d + \{(u', n^2(u')), (v_{0,1}, v_{k-1,1}), (v_{0,k-1}, v_{k-1,k-1}), (r^p(x), x), (v_{p+2,1}, v_{p+3,1}), (v_{p+2,k-1}, v_{p+3,k-1})\}$$ is a hamiltonian path of $Q_2^5 - F_v$ from $u$ to $v$.

Case 4. $u, v \in V(\text{Row}(2 : p))$. As $u, v \in V(\text{Row}(2 : p))$, it is easy to see that odd $p \geq 3$. Noting that $v^* = v_{u,b} \in V(\text{Row}(p + 1, p + 2))$, we see that $\text{Row}(p + 2)$ exists. Then $k - 1 \geq p + 2 \geq 5$, and so $k \geq 6$. As $e_{u,v} = 1$, by Lemma 3.3, there exists a hamiltonian path $P_1$ of $\text{Row}(2 : p)$ from $u$ to $v$ that contains an edge
(s, t) of Row(2). As \(e_{n^1(s), n^1(t)} = 1\), without loss of generality, we may assume that \(n^1(s)\) is odd and \(n^1(t)\) is even. Let \(n^1(s) = v_{1,m}\) and \(n^1(t) = v_{1,m+1}\).

If \(m = 0\), then \(n^1(s) = v_{1,0}\) and \(n^1(t) = v_{1,1}\). Let \(P_2 = v_{1,0}v_{1,k-1}v_{0,k-1}\) and \(P_3 = C_{k-2}^-(v_{1,1}, v_{0,1})\). If \(m \neq 0\), let \(P_2 = v_{1,m}v_{0,m}v_{0,m+1} \ldots v_{0,k-1}\), \(P_3^1 = v_{1,m+1}v_{m+2}v_{m+3} \ldots v_{1,k-1}v_{1,0}v_{1,1}\) and \(P_3 = P_3^1 \cup C_{m-1}^+(v_{1,1}, v_{0,1})\). Then \(P_2\) is a path from \(n^1(s)\) to \(v_{0,k-1}\) and \(P_3\) is a path from \(n^1(t)\) to \(v_{0,1}\). Obviously, \(P_2\) and \(P_3\) are vertex-disjoint paths in \(\text{Row}(0 : 1) - v_{0,0}\) that contain all vertices of \(\text{Row}(0 : 1) - v_{0,0}\).

As \(v_{k-1,1}, v_{k-1,k-1} \in V(\text{Row}(k - 1))\) are even and \(v^* \in V(\text{Row}(p + 1 : p + 2))\) is odd, Lemma 3.4 and Lemma 3.5(ii) imply that there is a hamiltonian path \(P_4\) of \(\text{Row}(p + 1 : k - 1) - v^*\) from \(v_{k-1,1}\) to \(v_{k-1,k-1}\). Then \(\bigcup_{d=1}^4 P_d - \{(s, t)\} + \{(s, n^1(s)), (t, n^1(t)), (v_{0,1}, v_{k-1,1}), (v_{0,k-1}, v_{k-1,k-1})\}\) is a hamiltonian path of \(Q_k^2 - F_v\) from \(u\) to \(v\).

Case 5. \(u \in V(\text{Row}(2 : p))\) and \(v \in V(\text{Row}(p + 3 : k - 1))\). As \(u \in V(\text{Row}(2 : p))\), it is easy to see that odd \(p \geq 3\). Noting that \(v \in V(\text{Row}(p + 3 : k - 1))\), we have \(k - 1 \geq p + 3\) and so \(k \geq p + 4 \geq 7\). As \(k\) is even, we have \(k \geq 8\). Recall that \(v = v_{i',j'}\). If \(i'\) is odd, let \(q = i' - 1\). If \(i'\) is even, let \(q = i'\). Clearly, \(q \geq p + 3\) is even and \(v \in V(\text{Row}(q : q + 1))\). Now we consider the following two cases.

Case 5.1. \(v \in V(\text{Row}(q))\). As \(e_{u,v} = 1\), without loss of generality, we assume that \(u\) is even and \(v\) is odd. Choose an odd \(w \in V(\text{Row}(p))\). Then \(e_{u,w} = 1\).

By Lemma 3.3, there is a hamiltonian path \(P_1\) of \(\text{Row}(2 : p)\) from \(u\) to \(w\) that contains an edge \((s, t)\) of \(\text{Row}(2)\). Similar to Case 4, there exist an \(n^1(s)v_{0,k-1}\)-path \(P_2\) and an \(n^1(t)v_{0,1}\)-path \(P_3\) in \(\text{Row}(0 : 1) - v_{0,0}\). Moreover, \(P_2\) and \(P_3\) are vertex-disjoint paths that contain all vertices of \(\text{Row}(0 : 1) - v_{0,0}\).

As \(v_{k-1,1}, v_{k-1,k-1} \in V(\text{Row}(k - 1))\) are even and \(v \in V(\text{Row}(q))\) is odd, Lemma 3.4 and Lemma 3.5(ii) imply that there is a hamiltonian path \(P_4\) of \(\text{Row}(q : k - 1) - v\) from \(v_{k-1,1}\) to \(v_{k-1,k-1}\). As both \(w\) and \(v\) are odd, we have both \(n^p+1(w)\) and \(n^q(v)\) are even. Note that the odd \(v^*\) is in \(\text{Row}(p + 1 : p + 2)\).

By Lemma 3.4 and Lemma 3.5(iii), there is a hamiltonian path \(P_5\) of \(\text{Row}(p + 1 : q - 1) - v^*\) from \(n^p+1(w)\) to \(n^q(v)\).

Then \(\bigcup_{d=1}^5 P_d - \{(s, t)\} + \{(s, n^1(s)), (t, n^1(t)), (v_{0,1}, v_{k-1,1}), (v_{0,k-1}, v_{k-1,k-1}), (w, n^p+1(w)), (v, n^q(v))\}\) is a hamiltonian path of \(Q_k^2 - F_v\) from \(u\) to \(v\).

Case 5.2. \(v \in V(\text{Row}(q + 1))\). As \(e_{u,v} = 1\), without loss of generality, we assume that \(u\) is odd and \(v\) is even. Choose an even \(w \neq v\) in \(\text{Row}(q + 1)\). As \(e_{w,v} = 0\) and \(e_{w,v^*} = 1\), Lemma 3.5(ii) implies that there is a hamiltonian path \(P_1\) of \(\text{Row}(p + 1 : q + 1) - v^*\) from \(w\) to \(v\). Choose an odd \(x \in V(\text{Row}(1))\) and an odd \(y \in V(\text{Row}(0))\). Then \(n^2(x)\) is even. Noting that \(u\) is odd, we have \(e_{u,n^2(x)} = 1\).

By Lemma 3.3, there is a hamiltonian path \(P_2\) of \(\text{Row}(2 : p)\) from \(u\) to \(n^2(x)\). Note that \(u^*\) is even and both \(x\) and \(y\) are odd. By Lemma 3.4, there is a hamiltonian path \(P_3\) of \(\text{Row}(0 : 1) - u^*\) from \(x\) to \(y\).
We will construct a hamiltonian path of $Q^k_2 - F_v$ from $u$ to $v$ in the following. Noting that $q + 1$ is odd, we consider the following two cases. Suppose first that $q + 1 = k - 1$. As $w$ is even, we have $n^{0}(w)$ is odd. Let $y = n^{0}(w)$. Then $P_1 \cup P_2 \cup P_3 + \{(w, n^0(w)), (x, n^2(x))\}$ is a hamiltonian path of $Q^k_2 - F_v$ from $u$ to $v$. Suppose next that $q + 1 \leq k - 3$. As $w$ is even and $y$ is odd, we have $n^{q+2}(w)$ is odd and $n^{k-1}(y)$ is even. By Lemma 3.3, there exists a hamiltonian path $P_4$ of $Row(q + 2 : k - 1)$ from $n^{q+2}(w)$ to $n^{k-1}(y)$. Then $\bigcup_{d=1}^{4} P_d + \{(y, n^{k-1}(y)), (x, n^2(x)), (w, n^{q+2}(w))\}$ is a hamiltonian path of $Q^k_2 - F_v$ from $u$ to $v$. The proof of this theorem is complete.

Given an even $k \geq 4$, let $F_v$ be the set of faulty vertices of a $Q^k_2$. Recall that $f_{v}^{\text{max}} = \max\{|F_v \cap X|, |F_v \cap Y|\}$, where $X$ be the set of even vertices and $Y$ be the set of odd vertices of the $Q^k_2$. The following result is a direct consequence of Theorem 1.1 and 3.12.

**Corollary 3.13.** Let $k \geq 4$ be even and let $f_v$ be the number of faulty vertices and $f_e$ be the number of faulty edges in $Q^k_2$ with $0 \leq f_v + f_e \leq 2$. Given any two healthy vertices $u$ and $v$ of $Q^k_2$, then there is a path from $u$ to $v$ of length $k^2 - 2f_{v}^{\text{max}} - 1$ if $e_{u,v} = 1$.

4. Conclusions

In this paper, we investigate the problem of embedding hamiltonian paths into faulty $k$-ary 2-cubes, where $k \geq 4$ is even. For any two healthy vertices $u, v$ with $e_{u,v} = 1$, we proved that the faulty $k$-ary $n$-cube admits a path of length $k^2 - 2f_{v}^{\text{max}} - 1$ if $f_v + f_e \leq 2$. The above result show that the fault-tolerant capability of the $k$-ary 2-cube is nice in terms of the path embeddings. The work will help engineers to develop corresponding applications on the distributed-memory parallel system that employs the $k$-ary 2-cube as the interconnection network.

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