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ON TOTAL VERTEX IRREGULARITY STRENGTH OF GRAPHS

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Abstract

Martin Bača *et al.* [2] introduced the problem of determining the total vertex irregularity strengths of graphs. In this paper we discuss how the addition of new edge affect the total vertex irregularity strength.

Keywords: graph labeling, irregularity strength, total assignment, vertex irregular total labeling.

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1. Introduction

In this paper we consider simple undirected graphs. The problem of finding irregularity strength of graphs was proposed by Chartrand *et al.* [3] and has proved to be difficult in general. There are not many graphs for which irregularity strength is known. The readers are advised to refer the papers [1, 4, 5, 6] for survey. Kathiresan *et al.* [7, 8] studied the change in irregularity strength of a graph G by an addition of a new edge from its complement.

Let $G = (V, E)$ be a simple graph. By a total labeling of such a graph we will mean an assignment $f : E \cup V \to Z^+$ to the edges and vertices of G. The weight of a vertex $v \in V$, is defined by $w(v) = f(v) + \sum_{vu \in V} f(vu)$. Moreover, the weighting f is called irregular if for each pair of different vertices their weights are distinct. The total vertex irregularity strength, $tvs(G)$, is the minimum value of the largest label over all such irregular assignments. In [2], Martin Bača *et al.* determined the total vertex irregularity strength of complete graphs, prisms and star graphs. K. Wijaya *et al.* [10, 11] determined the total vertex irregularity strength of wheels, fans, suns, complete bipartite graph and friendship graph. For the updated information about total vertex irregularity strength of trees refer [9].

It is interesting to see that addition of an edge from the complement of the graph to the graph G may increase or decrease the total vertex irregularity strength of the graph G or remains the same. Thus we call it as total positive edge, total negative edge and total stable edge of G respectively. If all the edges of $\overline{G}$ are negative (stable) then we call the graph G as total negative graph (total stable graph). Otherwise we call it as total mixed graph.

In this paper total vertex irregularity strength of disjoint union of t copies of K_3 (i.e., tK_3) and disjoint union of t copies of P_3 (i.e., tP_3) are determined. New characteristics total positive edge, total negative edge and total stable edge are introduced. Further, $K_{1,n}$ and tK_3 are classified as total negative graphs and tP_3 is classified as total mixed graph.

Definition 1. Let $G = (V, E)$ be any graph which is not complete. Let e be any edge of $\overline{G}$, then e is called a *total positive edge* of G, if $tvs(G + e) > tvs(G)$. Analogously we define the *total negative edge* and *total stable edge* of G if $tvs(G + e) < tvs(G)$ and $tvs(G + e) = tvs(G)$ respectively.

Example 2. In P_5, the edge joining the two pendant vertices is a total positive edge of P_5.

Example 3. In P_3, the edge joining the two pendant vertices is a total stable edge P_3.

Definition 4. Let G be any graph which is not complete. If all the edges of $\overline{G}$ are total stable (total negative) edges of G, then G is called a *total stable* (*total negative*) *graph.* Otherwise G is called a *total mixed graph.*

Example 5. In C_5, the addition of any edge from its complement will reduce the total vertex irregularity strength of C_5. Hence, C_5 is a total negative graph.

Example 6. The paw graph $K_{1,3} + e$ is a total stable graph.

2. Mixed and Negative Graphs

Martin Bača *et al.* [2] determined the total vertex irregularity strength of star graph as $tvs(K_{1,n}) = \lceil \frac{n+1}{2} \rceil$. The following theorem proves that star graph is a total negative graph.

Theorem 7. *The star graph $K_{1,n}, n \geq 4$ with n pendant vertices is a total negative graph.*

Proof. Let x be the vertex of degree n and let $v_1, v_2, v_3, \ldots, v_n$ be the pendant vertices of $K_{1,n}$. For any $e \in \overline{K_{1,n}}$, $K_{1,n} + e$ is isomorphic to $K_{1,n} + v_{n-1}v_n$. Define the total labeling $f : E \cup V \to Z^+$ as follows.

(i) $f(x) = 1$,

(ii) $f(v_i) = \begin{cases} \lceil \frac{i+1}{2} \rceil & \text{if } 1 \leq i \leq n-2, \\ \lceil \frac{i-1}{2} \rceil & \text{if } i = n-1, n, \end{cases}$

(iii) $f(xv_i) = \lceil \frac{i}{2} \rceil$ if $1 \leq i \leq n-2$,

(iv) If n is odd, $f(xv_i) = \lceil \frac{i-2}{2} \rceil$ if $i = n-1, n$,

(v) If n is even, $f(xv_i) = \lceil \frac{i}{2} \rceil$ if $i = n-1, n$,

(vi) $f(v_{n-1}v_n) = \begin{cases} 2 & \text{if } n \textit{ is odd}, \\ 1 & \text{if } n \textit{ is even}. \end{cases}$

By the above irregular total labeling we get,

$$tvs(K_{1,n} + v_{n-1}v_n) \leq \left\lceil \frac{n}{2} \right\rceil < tvs(K_{1,n}).$$

Thus, $v_{n-1}v_n$ is a total negative edge and hence $K_{1,n}$ is a total negative graph. ■

Theorem 8. *For $t \geq 2$, $tvs(tK_3) = t + 1$.*

Proof. Consider the disjoint union of t copies of K_3. Let v_{i1}, v_{i2} and v_{i3} be the vertices of the ith copy of K_3. Define the total labeling $f : E \cup V \to Z^+$ as follows.

(i) $f(v_{i1}) = f(v_{i2}) = i, \ 1 \leq i \leq t$,

(ii) $f(v_{i3}) = i + 1, \ 1 \leq i \leq t$,

(iii) $f(v_{i1}v_{i2}) = f(v_{i1}v_{i3}) = i, \ 1 \leq i \leq t$,

(iv) $f(v_{i2}v_{i3}) = i + 1, \ 1 \leq i \leq t$.

By the above total labeling the weights of the vertices of tK_3 are $3, 4, 5, \ldots, 3t+2$. Since the maximum degree in tK_3 is 2, $tvs(tK_3) \geq t + 1$. Thus $tvs(tK_3) = t + 1$. ■

Theorem 9. *For $t \geq 2$, tK_3 is a total negative graph.*

Proof. Let v_{i1}, v_{i2} and v_{i3} be the vertices of the ith copy of K_3. Add the edge $v_{(t-1)3}v_{t3}$ to tK_3. Define the total labeling $f : E \cup V \to Z^+$ as follows.

(i) $f(v_{i1}) = f(v_{i2}) = i, \ \ 1 \leq i \leq t-2,$

(ii) $f(v_{i3}) = i+1, \ \ 1 \leq i \leq t-2,$

(iii) $f(v_{i1}v_{i2}) = f(v_{i1}v_{i3}) = i, \ \ 1 \leq i \leq t-2,$

(iv) $f(v_{i2}v_{i3}) = i+1, \ \ 1 \leq i \leq t-2,$

(v) $f(v_{(t-1)1}) = f(v_{(t-1)2}) = f(v_{t1}) = t-1,$

(vi) $f(v_{(t-1)3}) = f(v_{t2}) = f(v_{t3}) = t,$

(vii) $f(v_{(t-1)1}v_{(t-1)2}) = f(v_{(t-1)1}v_{(t-1)3}) = t-1,$

(viii) $f(v_{(t-1)2}v_{(t-1)3}) = f(v_{t1}v_{t2}) = f(v_{t2}v_{t3}) = f(v_{t1}v_{t3}) = t,$

(ix) $f(v_{(t-1)3}v_{t3}) = 2.$

By the above irregular total labeling $tvs(tK_3 + v_{(t-1)3}v_{t3}) \leq t < tvs(tK_3)$. For any $e \in \overline{tK_3}$, $tK_3 + e \cong tK_3 + v_{(t-1)3}v_{t3}$. Thus tK_3 is a total negative graph. ■

Theorem 10. *For $t \geq 2$, $tvs(tP_3) = t+1$.*

Proof. Consider the disjoint union of t copies of P_3. Let v_{i1}, v_{i2} and v_{i3} be the consecutive vertices of the ith copy of P_3. Define the total labeling $f : E \cup V \to Z^+$ as follows.

(i) $f(v_{i1}) = \begin{cases} 1, & i = 1, \\ 2, & 2 \leq i \leq t, \end{cases}$

(ii) $f(v_{i2}) = \begin{cases} 1, & i = 1, \\ t-1, & 2 \leq i \leq t, \end{cases}$

(iii) $f(v_{i3}) = \begin{cases} 1, & i = 1, \\ i+1, & 2 \leq i \leq t, \end{cases}$

(iv) $f(v_{i1}v_{i2}) = \begin{cases} 1, & i = 1, \\ i+1, & 2 \leq i \leq t, \end{cases}$

(v) $f(v_{i2}v_{i3}) = \begin{cases} 2, & i = 1, \\ t+1, & 2 \leq i \leq t. \end{cases}$

Thus, we get an irregular total labeling with maximum label $t+1$ and hence $tvs(tP_3) \leq t+1$. The optimum weights of the vertices are $2, 3, 4, \ldots, 3t+1$ and since $\triangle(tP_3) = 2$, $tvs(tP_3) \geq t+1$. Hence, $tvs(tP_3) = t+1$. ■

Theorem 11. *For $t \geq 2$, tP_3 is a total mixed graph.*

Proof. Consider the disjoint union of t copies of P_3. Let v_{i1}, v_{i2} and v_{i3} be the consecutive vertices of the ith copy of K_3. For any $e \in \overline{tP_3}$, $tP_3 + e$ is isomorphic to any one of the following graphs.

(i) $tP_3 + v_{(t-1)1}v_{t1}$,

(ii) $tP_3 + v_{(t-1)2}v_{t2}$,

(iii) $tP_3 + v_{(t-1)2}v_{t1}$ and

(iv) $tP_3 + v_{t1}v_{t3}$.

For $t = 2$, assign the total labeling as follows. $f(v_{i1}v_{i2}) = 2, i = 1, 2$ $f(v_{i2}v_{i3}) = i$, $i = 1, 2$, $f(v_{i3}) = 1$, $i = 1, 2$.

If $e = v_{11}v_{21}$, then $f(v_{i1}) = 4 - i$, $i = 1, 2$, $f(v_{i2}) = 1$, $i = 1, 2$, $f(v_{11}v_{21}) = 2$.

If $e = v_{12}v_{22}$, then $f(v_{i1}) = 4 - i$, $i = 1, 2$, $f(v_{i2}) = \begin{cases} 3 & i = 1, \\ 1 & i = 2, \end{cases}$ $f(v_{12}v_{22}) = 2$.

If $e = v_{11}v_{13}$, then $f(v_{i1}) = \begin{cases} 1 & i = 1, \\ 3 & i = 2, \end{cases}$ $f(v_{i2}) = 2$ $i = 1, 2$, $f(v_{11}v_{13}) = 2$.

If $e = v_{11}v_{22}$, then assign label 2 to all the remaining vertices and edge to get irregular total labeling.

Hence, $2P_3$ is a total mixed graph.

Now we discuss the cases when $t > 2$.

Case 1. Define the total labeling for $tP_3 + v_{(t-1)1}v_{t1}$ as follows.
(a) Except for the vertices $v_{(t-1)1}$ and v_{t1} assign the same labels to all the edges and vertices of tP_3 as in theorem 2.4.

(b) Assign the label 1 to the $v_{(t-1)1}$ and v_{t1} and the new edge $v_{(t-1)1}v_{t1}$. Thus, we get a total irregular assignment and hence

$$tvs(tP_3 + v_{(t-1)1}v_{t1}) \leq t + 1.$$

Since, the optimum weights for $tP_3 + v_{(t-1)1}v_{t1}$ are $2, 3, 4, \ldots, 3t+1$ and $\triangle(tP_3 + v_{(t-1)1}v_{t1}) = 2$, it is not possible to obtain the weight $3t + 1$ by using the labels fewer than $t + 1$. Thus,

$$tvs(tP_3 + v_{(t-1)1}v_{t1}) \geq t + 1.$$

Hence, $tvs(tP_3 + v_{(t-1)1}v_{t1}) = t + 1$. $v_{(t-1)1}v_{t1}$ is a total stable edge of tP_3.

Case 2. Define the total labeling for $tP_3 + v_{(t-1)2}v_{t2}$ as follows.
(a) Assign the same labels to all the vertices and edges of tP_3 as in Theorem 2.4 except for the vertices $v_{(t-1)2}$ and v_{t2}.

(b) Assign the label $t - 2$ to the vertices $v_{(t-1)2}$ and v_{t2} and 1 to edge $v_{(t-1)2}v_{t2}$.

Thus we get an irregular total labeling and hence

$$tvs(tP_3 + v_{(t-1)2}v_{t2}) \leq t + 1.$$

There are $2t$ pendant vertices in $tP_3 + v_{(t-1)2}v_{t2}$. The optimum weight for the pendant vertices are $2, 3, 4, \ldots, 2t + 1$. It is not possible to obtain the weight $2t + 1$ by assigning the labels fewer than $t + 1$. Thus,

$$tvs(tP_3 + v_{(t-1)2}v_{t2}) \geq t + 1.$$

Hence, $tvs(tP_3 + v_{(t-1)2}v_{t2}) = t + 1$. $v_{(t-1)2}v_{t2}$ is a total stable edge of tP_3.

Case 3. Define the total labeling for $tP_3 + v_{t1}v_{t3}$ as follows.
(a) Assign the same labels to all the vertices and edges of tP_3 except t^{th} copy of P_3 as in Theorem 2.4.
(b) In the tth copy assign label as follows $f(v_{t1}) = 2$, $f(v_{t2}) = f(v_{t3}) = f(v_{t1}v_{t2}) = t$, $f(v_{t2}v_{t3}) = t + 1$ and $f(v_{t1}v_{t3}) = 1$.

Thus, we get an irregular total labeling and hence

$$tvs(tP_3 + v_{t1}v_{t3}) \leq t + 1.$$

Since the optimum weights for the vertices of $tP_3 + v_{t1}v_{t3}$ are $2, 3, 4, \ldots, 3t + 1$ and $\triangle(tP_3 + v_{t1}v_{t3}) = 2$, $tvs(tP_3 + v_{t1}v_{t3}) \geq t + 1$.

Hence, $tvs(tP_3 + v_{t1}v_{t3}) = t + 1$. Thus $v_{t1}v_{t3}$ is a total stable edge of tP_3.

Case 4. Define the total labeling for $tP_3 + v_{(t-1)2}v_{t1}$ as follows.

(i) $f(v_{i1}) = \begin{cases} 1 & \text{if } 1 \leq i \leq t-1, \\ t & \text{if } i = t, \end{cases}$

(ii) $f(v_{i2}) = \begin{cases} t & \text{if } i \neq t-1, \\ 1 & \text{if } i = t-1, \end{cases}$

(iii) $f(v_{i3}) = t + 1 - i, \quad 1 \leq i \leq t,$

(iv) $f(v_{i1}v_{i2}) = \begin{cases} i & \text{if } 1 \leq i \leq t-2, \\ t & \text{if } i = t-1, t, \end{cases}$

(v) $f(v_{i2}v_{i3}) = \begin{cases} t & \text{if } 1 \leq i \leq t-1, \\ t-1 & \text{if } i = t, \end{cases}$

(vi) $f(v_{(t-1)2}v_{t1}) = t.$

The weights of the vertices of $tP_3 + v_{(t-1)2}v_{t1}$ are $2, 3, 4, \ldots, 3t + 1$, and hence

$$tvs(tP_3 + v_{(t-1)2}v_{t1}) \leq t < tvs(tP_3).$$

Hence, $v_{(t-1)2}v_{t1}$ is a total negative edge. Thus, tP_3 is a total mixed graph. ■

In most of the cases, the joining of any two non adjacent vertices either reduce the total vertex irregularity strength or remains the same. In our experience, we believe that it is not possible to obtain a graph in which each edge of $\overline{G}$ is total positive edge. Thus, we conclude with the following conjecture.

Conjecture. There is no graph G in which all the edges of $\overline{G}$ are total positive edges.

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