

## LIST COLORING OF COMPLETE MULTIPARTITE GRAPHS

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### Abstract

The choice number of a graph  $G$  is the smallest integer  $k$  such that for every assignment of a list  $L(v)$  of  $k$  colors to each vertex  $v$  of  $G$ , there is a proper coloring of  $G$  that assigns to each vertex  $v$  a color from  $L(v)$ . We present upper and lower bounds on the choice number of complete multipartite graphs with partite classes of equal sizes and complete  $r$ -partite graphs with  $r - 1$  partite classes of order two.

**Keywords:** list coloring, choice number, complete multipartite graph.

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### 1. INTRODUCTION

All graphs considered here are finite, undirected, without loops and multiple edges. Let  $G$  be a graph with the vertex set  $V(G)$  and the edge set  $E(G)$ . A *list assignment* to the vertices of a graph  $G$  is the assignment of a list  $L(v)$  of colors  $C$  to every vertex  $v \in V(G)$ . A  *$k$ -list assignment* is a list assignment such that  $|L(v)| \geq k$  for every vertex  $v$ . An  *$L$ -coloring* of  $G$  is a function  $f : V(G) \rightarrow C$  such that  $f(v) \in L(v)$  for all  $v \in V(G)$  and  $f(v) \neq f(w)$  for each edge  $vw \in E(G)$ . If  $G$  has an  $L$ -coloring, then  $G$  is said to be  *$L$ -colorable*. If for any  $k$ -list assignment  $L$  there exists an  $L$ -coloring, then  $G$  is  *$k$ -choosable*. The *choice number*  $Ch(G)$  of a graph  $G$  is the minimum integer  $k$  such that  $G$  is  $k$ -choosable.

The study of choice numbers of graphs was initiated by Vizing [7] and by Erdős, Rubin and Taylor [3]. For a survey about the list coloring problem we refer to [6] and [8]. In this paper we focus on the choice numbers of complete multipartite graphs.

## 2. COMPLETE MULTIPARTITE GRAPHS WITH PARTITE CLASSES OF DIFFERENT SIZES

Let  $K_{n_1, n_2, \dots, n_r}$  be the complete  $r$ -partite graph with the partite classes of order  $n_1, n_2, \dots, n_r$ . A well-known result of Erdős, Rubin and Taylor [3] says that the choice number of the complete  $r$ -partite graph  $K_{2, 2, \dots, 2}$  is  $r$ . Gravier and Maffray [4] proved that also  $Ch(K_{3, 3, 2, \dots, 2}) = r$  for  $r \geq 3$ . Enomoto *et al.* [2] showed that  $Ch(K_{5, 2, \dots, 2}) = r + 1$  and the choice number of the complete  $r$ -partite graph  $K_{4, 2, \dots, 2}$  is equal to  $r$  if  $r$  is odd, and  $r + 1$  if  $r$  is even.

Motivated by these results we study the value  $Ch(K_{n, 2, \dots, 2})$  for any positive integer  $n$ . In the proof of Theorem 1 we write  $L(S)$  for the union  $\bigcup_{v \in S} L(v)$  where  $S \subseteq V(G)$ . If  $C$  is a set of colors, then  $L \setminus C$  denotes the list assignment obtained from  $L$  by removing the colors in  $C$  from each  $L(v)$  where  $v \in V(G)$ . First, we show that the graph  $K_{(t+2)(t+3)/2, 2, \dots, 2}$  is  $(r + t)$ -choosable.

**Theorem 1.** *Let  $t$  be a positive integer and let  $G$  be a complete  $r$ -partite graph with one partite class of order  $(t + 2)(t + 3)/2$  and  $r - 1$  partite classes of order two. Then  $Ch(G) \leq r + t$ .*

**Proof.** Let  $V_1$  be the partite class of  $G$  of order  $(t + 2)(t + 3)/2$  and let  $V_i = \{v_i, w_i\}$ ,  $2 \leq i \leq r$ , be the partite classes of order two. Let  $L_1$  be any  $(r + t)$ -list assignment to the vertices of  $G$ . We prove that  $G$  is  $L_1$ -colorable. We distinguish three cases:

*Case 1.*  $t \geq r - 1$ .

We can color the vertices of  $V_2, V_3, \dots, V_r$  with  $2r - 2$  different colors. Since  $|L_1(v)| \geq 2r - 1$  for every vertex  $v \in V_1$ , we can color the vertices of  $V_1$  as well.

*Case 2.* There exists a color  $c \in L_1(v_i) \cap L_1(w_i)$  for some  $i \in \{2, 3, \dots, r\}$ .

It is easy to show by induction on  $r$  that  $G$  is  $L_1$ -colorable. The step  $r = 1$  is trivial. For the induction step, assign  $c$  to both  $v_i$  and  $w_i$ , and remove  $c$  from the lists of the remaining vertices. By the induction hypothesis, the remaining vertices can be colored with colors from the revised lists.

*Case 3.*  $t \leq r - 2$  and  $L_1(v_i) \cap L_1(w_i) = \emptyset$  for every  $i \in \{2, 3, \dots, r\}$ .

We prove by contradiction that  $G$  is  $L_1$ -colorable. Assume that  $G$  is not  $L_1$ -colorable. Let  $L$  be an  $(r + t)$ -list assignment such that  $G$  is not  $L$ -colorable. Let  $X_j$ ,  $j = 1, 2, \dots, t$ , be the largest subset of  $V_1 \setminus (\bigcup_{l=1}^{j-1} X_l)$  with  $\bigcap_{v \in X_j} L(v) \neq \emptyset$ . Set  $|X_j| = x_j$  and choose a color  $c_j \in \bigcap_{v \in X_j} L(v)$ . Define  $L' = L \setminus \{c_1, c_2, \dots, c_t\}$  and  $G' = G \setminus (\bigcup_{l=1}^t X_l)$ . Note that  $|L'(v)| = r + t$  for each  $v \in V(G') \cap V_1$  and  $|L'(v_i)|, |L'(w_i)| \geq r$  for any  $i \in \{2, 3, \dots, r\}$ . Since  $G$  is not  $L$ -colorable,  $G'$  is not  $L'$ -colorable. It follows that there exists a set of vertices  $T \subseteq V(G')$  such that  $|L'(T)| < |T|$ , i.e.,  $L'$  does not satisfy Hall's condition. Let  $S$  denote a maximal subset of  $V(G')$  such that  $|L'(S)| < |S|$ . We consider two subcases:

*Case 3a.*  $|S \cap V_i| \leq 1$  for every  $i \in \{2, 3, \dots, r\}$ .

Since  $|L'(v_i)|, |L'(w_i)| \geq r$  and  $|S \setminus V_1| \leq r - 1$ ,  $S \setminus V_1$  can be colored from the list  $L'$ . Further,  $|L'(v)| = r + t$  for  $v \in S \cap V_1$ , therefore we can also color the vertices in  $S \cap V_1$ .

Let  $L'' = L' \setminus L'(S)$ . We show that  $G' \setminus S$  is  $L''$ -colorable. If  $G' \setminus S$  is not  $L''$ -colorable, we have a nonempty subset  $S' \subset V(G') \setminus S$  with  $|L''(S')| < |S'|$ . Then  $|L'(S \cup S')| = |L'(S)| + |L''(S')| < |S| + |S'|$ , which contradicts the maximality of  $S$ .

*Case 3b.* Both  $v_i, w_i \in S$  for some  $i \in \{2, 3, \dots, r\}$ .

Then  $|S| > |L'(S)| \geq |L'(v_i)| + |L'(w_i)| \geq 2(r + t) - t$ . Set  $|S| = 2r + t + 1 + \epsilon$  where  $\epsilon \geq 0$ . Clearly,  $|L'(S)| \leq 2r + t + \epsilon$ . Let  $S_1 = S \cap V_1$ . We have  $|S_1| \geq |S| - (2r - 2) = t + 3 + \epsilon$ . By the maximality of  $X_t$ , every color in  $L'(S)$  appears in the lists of at most  $x_t$  vertices of  $S_1$ . It means that

$$(1) \quad (r + t)|S_1| = \sum_{v \in S_1} |L'(v)| \leq x_t |L'(S)|.$$

It is evident that  $\sum_{l=1}^t x_l + |S_1| \leq |V_1| = (t + 2)(t + 3)/2$ . Hence,  $tx_t + |S_1| \leq (t + 2)(t + 3)/2$ , or equivalently

$$(2) \quad x_t \leq [(t + 2)(t + 3)/2 - |S_1|]/t.$$

By (1) and (2), we have  $(r + t)|S_1| \leq [(t + 2)(t + 3)/2 - |S_1|]|L'(S)|/t$ . Since  $|S_1| \geq t + 3 + \epsilon$  and  $|L'(S)| \leq 2r + t + \epsilon$ , we have  $(r + t)(t + 3 + \epsilon) \leq [(t + 2)(t + 3)/2 - (t + 3 + \epsilon)](2r + t + \epsilon)/t$  which yields  $\frac{t^3}{2} + (3 + \epsilon)\frac{t^2}{2} + (r - \frac{1}{2})\epsilon t + (2r + \epsilon)\epsilon \leq 0$ , a contradiction. This finishes the proof.  $\blacksquare$

If  $t = 1$ , then  $Ch(K_{6,2,\dots,2}) \leq r + 1$ . This bound also comes from the result  $Ch(K_{3,3,2,\dots,2}) = r$  of Gravier and Maffray [4], because the complete  $r$ -partite graph  $K_{6,2,\dots,2}$  is a subgraph of the complete  $(r + 1)$ -partite graph  $K_{3,3,2,\dots,2}$ . Since the choice number of the complete  $r$ -partite graph  $K_{5,2,\dots,2}$  is equal to  $r + 1$ , it is clear that  $Ch(K_{6,2,\dots,2}) = r + 1$  as well.

Now we present a lower bound on the choice number of complete  $r$ -partite graphs with  $r - 1$  partite classes of order at most two.

**Theorem 2.** *Let  $s, r, t$  be integers such that  $0 \leq s < r$  and  $t > 0$ . Let  $G$  be a complete  $r$ -partite graph consisting of one partite class of order  $\binom{2t+s}{t}^2$ ,  $r - s - 1$  partite classes of order two, and  $s$  partite classes of order one. Then  $Ch(G) > \lfloor \frac{r+t-1}{2t+s} \rfloor (2t + s)$ .*

**Proof.** Let  $n = \binom{2t+s}{t}^2$  and  $m = \frac{r+t-1}{2t+s}$ . Let  $G$  be a complete  $r$ -partite graph with the partite classes  $V_1, V_i = \{v_i, w_i\}, V_j = \{v_j\}$ , where  $|V_1| = n; i = 2, 3, \dots, r - s$  and  $j = r - s + 1, r - s + 2, \dots, r$ . Let  $A_1, A_2, \dots, A_{2t+s}, B_1, B_2, \dots, B_{2t+s}$  be

disjoint color sets of order  $\lfloor m \rfloor$  such that  $\bigcup_{i=1}^{2t+s} A_i = A$ ,  $\bigcup_{i=1}^{2t+s} B_i = B$ . We define a list assignment  $L$  to  $V(G)$  by the following way:

$$\begin{aligned} L(v_j) &= A, \quad j = 2, 3, \dots, r, \\ L(w_i) &= B, \quad i = 2, 3, \dots, r-s. \end{aligned}$$

The lists of colors given to the vertices of  $V_1$  consist of  $2t+s$  different sets  $A_{x_1}, A_{x_2}, \dots, A_{x_{t+s}}, B_{y_1}, B_{y_2}, \dots, B_{y_t}$ , where  $x_1, x_2, \dots, x_{t+s}, y_1, y_2, \dots, y_t \in \{1, 2, \dots, 2t+s\}$ . Since the number of vertices in  $V_1$  is  $n = \binom{2t+s}{t+s} \binom{2t+s}{t}$ , we are able to assign to any two vertices in  $V_1$  different lists.

We show by contradiction that  $G$  cannot be colored from the list  $L$ . Suppose that  $G$  can be colored from  $L$ . We use  $r-1$  different colors of  $A$  to color the vertices  $v_2, v_3, \dots, v_r$  and  $r-s-1$  different colors of  $B$  to color  $w_2, w_3, \dots, w_{r-s}$ . Since  $|A| = |B| = \lfloor m \rfloor(2t+s) \leq r+t-1$ , the number of colors in  $A$  (in  $B$ ) not used to color  $V_2, V_3, \dots, V_r$  is at most  $t$  (at most  $t+s$ ). It follows that there are at most  $2t+s$  sets  $A_{x'_1}, A_{x'_2}, \dots, A_{x'_t}, B_{y'_1}, B_{y'_2}, \dots, B_{y'_{t+s}}$ , where  $x'_1, x'_2, \dots, x'_t, y'_1, y'_2, \dots, y'_{t+s} \in \{1, 2, \dots, 2t+s\}$  containing colors that were not employed in coloring  $V_2, V_3, \dots, V_r$ . Try to color  $V_1$  with these colors. According to the assignment of color sets to the vertices of  $V_1$ , there exists a vertex  $v \in V_1$  having none of the sets  $A_{x'_1}, A_{x'_2}, \dots, A_{x'_t}, B_{y'_1}, B_{y'_2}, \dots, B_{y'_{t+s}}$  in its list, a contradiction. Hence,  $G$  is not  $L$ -colorable. ■

Note that we get the bound  $Ch(K_{\binom{2t}{t}^2, 2, \dots, 2}) \geq r+t$  if  $s=0$  and  $r=pt+1$  for some odd integer  $p$ .

### 3. COMPLETE MULTIPARTITE GRAPHS WITH PARTITE CLASSES OF EQUAL SIZES

Let  $K_{n*r}$  denote the complete multipartite graph with  $r$  partite classes of order  $n$ . The problem is to determine the value of the choice number  $Ch(K_{n*r})$ . If  $n=1$ , then  $K_{n*r}$  is a clique on  $r$  vertices and hence, obviously,  $Ch(K_{1*r}) = r$ . In the previous section we mentioned that  $Ch(K_{2*r}) = r$  as well. Alon [1] established the general bounds  $c_1 r \log n \leq Ch(K_{n*r}) \leq c_2 r \log n$  for every  $r, n \geq 2$ , where  $c_1, c_2$  are two positive constants. Later, Kierstead [5] solved the problem in the case  $n=3$ . He showed that  $Ch(K_{3*r}) = \lceil \frac{4r-1}{3} \rceil$ . Yang [9] studied the value of  $Ch(K_{4*r})$  and obtained the bounds  $\lfloor \frac{3}{2}r \rfloor \leq Ch(K_{4*r}) \leq \lceil \frac{7}{4}r \rceil$ . We present results giving exact bounds on  $Ch(K_{n*r})$  for large  $n$ . In the proof of Theorem 3 we use the following lemma proved in [5].

**Lemma 1.** *A graph  $G$  is  $k$ -choosable if  $G$  is  $L$ -colorable for every  $k$ -list assignment  $L$  such that  $|\bigcup_{v \in V(G)} L(v)| < |V(G)|$ .*

Let us derive an upper bound on the choice number of complete multipartite graphs with partite classes of equal sizes.

**Theorem 3.** *Let  $0 < \alpha < n$  and let  $x_j = \lfloor (\alpha - \frac{\alpha}{n} \sum_{l=1}^{j-1} x_l) \rfloor + 1$ ,  $j = 1, 2, \dots, \lfloor \alpha \rfloor$ . If  $n \leq \sum_{l=1}^{\lfloor \alpha \rfloor} x_l$ , then  $Ch(K_{n*r}) \leq \lceil \alpha r \rceil$ .*

**Proof.** Let  $V_i$ ,  $i = 1, 2, \dots, r$ , be the  $i$ -th partite class of  $K_{n*r}$ . We prove the result by induction on  $r$ . The case  $r = 1$  is trivial. For the induction step consider an  $\lceil \alpha r \rceil$ -list assignment  $L$  to the vertices of  $K_{n*r}$ . We prove that if  $n \leq \sum_{l=1}^{\lfloor \alpha \rfloor} x_l$ , then any partite class  $V_i$  can be colored with at most  $\lfloor \alpha \rfloor$  colors.

Assume that  $n = \sum_{l=1}^{\lfloor \alpha \rfloor} x_l$ . In this paragraph we show by induction on  $j$  ( $j = 1, 2, \dots, \lfloor \alpha \rfloor$ ), that there exists a color  $c_j$  which can be used for coloring  $x_j$  vertices of  $V_i$  that have not been colored by  $c_1, c_2, \dots, c_{j-1}$  yet. Note that  $c_l, c_{l'}$ , where  $l, l' \in \{1, 2, \dots, \lfloor \alpha \rfloor\}$ ,  $l \neq l'$ , do not have to be different.

If  $j = 1$ , we have  $x_1 = \lfloor \alpha \rfloor + 1$ . Since  $\sum_{v \in V_i} |L(v)| = \lceil \alpha r \rceil n$  and by Lemma 1,  $|\bigcup_{v \in V(K_{n*r})} L(v)| < rn$ , there exists a color  $c_1$  which appears in the lists of at least  $\lfloor \alpha \rfloor + 1$  vertices of  $V_i$ . Color these vertices with  $c_1$ . Suppose  $j \geq 2$ . We can color  $\sum_{l=1}^{j-1} x_l$  vertices with  $c_1, c_2, \dots, c_{j-1}$ . The sum of the numbers of colors in the lists of the remaining  $n - \sum_{l=1}^{j-1} x_l$  vertices of  $V_i$  is  $(n - \sum_{l=1}^{j-1} x_l) \lceil \alpha r \rceil$ . Since  $|\bigcup_{v \in V_i} L(v)| < rn$ , there is a color  $c_j$  that appears in the lists of other  $\lfloor (n - \sum_{l=1}^{j-1} x_l) \frac{\alpha}{n} \rfloor + 1 = x_j$  vertices. Hence, we can color these vertices with  $c_j$ . It follows that it is possible to color  $n = \sum_{l=1}^{\lfloor \alpha \rfloor} x_l$  vertices of  $V_i$  with at most  $\lfloor \alpha \rfloor$  different colors.

Clearly, if  $n < \sum_{l=1}^{\lfloor \alpha \rfloor} x_l$ , all the vertices of  $V_i$  can be colored with at most  $\lfloor \alpha \rfloor$  colors too. Let us remove the colors that were employed in coloring  $V_i$  from the lists given to the vertices in  $V(K_{n*r}) \setminus V_i$ . We have at least  $\lceil \alpha r \rceil - \lfloor \alpha \rfloor$  colors. Since  $\lceil \alpha r \rceil - \lfloor \alpha \rfloor \geq \lceil \alpha(r-1) \rceil$ , by applying the induction hypothesis,  $r-1$  partite classes can be colored with  $\lceil \alpha(r-1) \rceil$  colors, i.e., there exists a proper coloring of the vertices in  $V(K_{n*r}) \setminus V_i$  with colors from the revised lists. ■

Unfortunately, the result presented in Theorem 3 cannot be bounded from above by  $cr \log n$ , where  $c$  is a constant. Theorem 3, for example, yields the upper bounds  $Ch(K_{5*r}) \leq \lceil \frac{5}{2}r \rceil$ ,  $Ch(K_{15*r}) \leq 5r$ ,  $Ch(K_{40*r}) \leq 10r$ ,  $Ch(K_{75*r}) \leq 15r$  and  $Ch(K_{121*r}) \leq 20r$ . One can check that  $10r \approx 6.24r \log 40$ ,  $15r \approx 8r \log 75$  and  $20r \approx 9.6r \log 121$ .

The following result gives a lower bound on  $Ch(K_{n*r})$ .

**Theorem 4.** *Let  $x, t, r, n$  be integers such that  $x, t, r \geq 2$ ,  $x \geq t$  and  $n = \binom{x}{x-t+1}$ . Then  $Ch(K_{n*r}) > (x-t+1) \lfloor \frac{tr-1}{x} \rfloor$ .*

**Proof.** Let  $x, t, r \geq 2$ ,  $x \geq t$ ,  $n = \binom{x}{x-t+1}$  and let  $k = (x-t+1) \lfloor \frac{tr-1}{x} \rfloor$ . We show that there exists a  $k$ -list assignment  $L$  of  $K_{n*r}$  such that  $K_{n*r}$  is not  $L$ -colorable.

Let  $V_i$ ,  $i = 1, 2, \dots, r$ , be the  $i$ -th partite class of  $K_{n*r}$ . Let  $A_1, A_2, \dots, A_x$  be a family of disjoint color sets such that  $|A_j| = |A_1|$  or  $|A_j| = |A_1| + 1$ ,  $j = 2, 3, \dots, x$ , and  $|\bigcup_{j=1}^x A_j| = tr - 1$ . Obviously,  $|A_j| \geq \lfloor \frac{tr-1}{x} \rfloor$  for any  $j \in \{1, 2, \dots, x\}$ .

Define a list assignment  $L$  as follows: Let the lists given to the  $n$  vertices of every partite class  $V_i$  consist of  $x - t + 1$  different sets  $A_{y_1}, A_{y_2}, \dots, A_{y_{x-t+1}}$ ,  $y_1, y_2, \dots, y_{x-t+1} \in \{1, 2, \dots, x\}$ , where any two vertices in the same part have different lists. Note that  $|L(v)| \geq (x - t + 1) \lfloor \frac{tr-1}{x} \rfloor$  for each vertex  $v \in V(K_{n*r})$ . Then for any partite class  $V_i$  and any  $t - 1$  colors  $a_j \in A_{y'_j}$ ,  $j = 1, 2, \dots, t - 1$ ;  $y'_j \in \{1, 2, \dots, x\}$  there is a vertex  $v \in V_i$  having none of the sets  $A_{y'_j}$  in its list. Therefore, in any coloring from these lists, we must use at least  $t$  colors on each partite class. Since the number of colors in  $\bigcup_{j=1}^x A_j$  is less than  $tr$ ,  $K_{n*r}$  is not  $L$ -colorable. ■

Theorem 4 says that if, for instance  $t = 2$ , then  $n = x$  and  $Ch(K_{n*r}) > (n - 1) \lfloor \frac{2r-1}{n} \rfloor$ . In particular, for  $n = 5$  we have  $Ch(K_{5*r}) > 4 \lfloor \frac{2r-1}{5} \rfloor$ . If  $t = 3$ , then  $Ch(K_{n*r}) > (x - 2) \lfloor \frac{3r-1}{x} \rfloor$ . For example, in the case  $x = 6$  we get  $Ch(K_{15*r}) > 4 \lfloor \frac{3r-1}{6} \rfloor = 4 \lfloor \frac{r-1}{2} \rfloor$ .

Finally, we present a corollary of Theorem 4 which yields a lower bound in the form  $cr \log n$ .

**Corollary 1.** *Let  $r \geq 2$  and  $n = \binom{x}{\lceil x/2 \rceil}$  where  $x \geq 5$ . Then*

$$Ch(K_{n*r}) > \lfloor \frac{r}{2} \rfloor \lceil \frac{\log_{2.1} n}{2} \rceil.$$

**Proof.** For  $x, t, r \geq 2$ ,  $x \geq t$  and  $n = \binom{x}{x-t+1}$ , we have  $Ch(K_{n*r}) > (x - t + 1) \lfloor \frac{tr-1}{x} \rfloor$ . Let  $t = \lfloor \frac{x}{2} \rfloor + 1$ . Then  $Ch(K_{n*r}) > \lceil \frac{x}{2} \rceil \lfloor \frac{\lfloor x/2 \rfloor r + r - 1}{x} \rfloor \geq \lceil \frac{x}{2} \rceil \lfloor \frac{r}{2} \rfloor$ . It is well-known that  $\frac{x^x}{e^{x-1}} \leq x! \leq \frac{(x+1)^{x+1}}{e^x}$  for any  $x$ . For  $x \geq 5$ , the following inequalities also hold:  $\frac{2x^x}{e^{x-1}} < x! < \frac{6x^{x+1}}{5e^x}$ . Then  $n = \frac{x!}{\lfloor x/2 \rfloor! \lceil x/2 \rceil!} < \frac{6x^{x+1}/(5e^x)}{4 \lfloor x/2 \rfloor! \lceil x/2 \rceil! e^{x-2}} \leq \frac{3x^{x+1}}{10 \lfloor x/2 \rfloor! x e^2} \leq \frac{3x^x x 2^x}{10(x-1)^x e^2}$ . Since  $x 2^x < 7.6(2.1)^x$  for any  $x$  (note that  $7.5(2.1)^x < x 2^x$  for  $19 \leq x \leq 22$ ) and  $(\frac{x}{x-1})^x < 3.1$  for any  $x \geq 5$ , we have  $n < \frac{7.068(2.1)^x}{e^2} < (2.1)^x$ . Consequently,  $\log_{2.1} n < x$ , hence  $Ch(K_{n*r}) > \lfloor \frac{r}{2} \rfloor \lceil \frac{\log_{2.1} n}{2} \rceil$  for any  $n = \binom{x}{\lceil x/2 \rceil}$  where  $x \geq 5$ . ■

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