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LIST COLORING OF COMPLETE MULTIPARTITE GRAPHS

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Abstract

The choice number of a graph G is the smallest integer k such that for every assignment of a list $L(v)$ of k colors to each vertex v of G, there is a proper coloring of G that assigns to each vertex v a color from $L(v)$. We present upper and lower bounds on the choice number of complete multipartite graphs with partite classes of equal sizes and complete r-partite graphs with $r-1$ partite classes of order two.

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1. Introduction

All graphs considered here are finite, undirected, without loops and multiple edges. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. A *list assignment* to the vertices of a graph G is the assignment of a list $L(v)$ of colors C to every vertex $v \in V(G)$. A *k-list assignment* is a list assignment such that $|L(v)| \geq k$ for every vertex v. An *L-coloring* of G is a function $f : V(G) \to C$ such that $f(v) \in L(v)$ for all $v \in V(G)$ and $f(v) \neq f(w)$ for each edge $vw \in E(G)$. If G has an L-coloring, then G is said to be *L-colorable*. If for any k-list assignment L there exists an L-coloring, then G is *k-choosable*. The *choice number* $Ch(G)$ of a graph G is the minimum integer k such that G is k-choosable.
The study of choice numbers of graphs was initiated by Vizing [7] and by Erdös, Rubin and Taylor [3]. For a survey about the list coloring problem we refer to [6] and [8]. In this paper we focus on the choice numbers of complete multipartite graphs.

2. Complete Multipartite Graphs with Partite Classes of Different Sizes

Let $K_{n_1,n_2,\ldots,n_r}$ be the complete r-partite graph with the partite classes of order $n_1, n_2, \ldots, n_r$. A well-known result of Erdös, Rubin and Taylor [3] says that the choice number of the complete r-partite graph $K_{2,2,\ldots,2}$ is r. Gravier and Maffray [4] proved that also $Ch(K_{3,3,2,\ldots,2}) = r$ for $r \geq 3$. Enomoto *et al.* [2] showed that $Ch(K_{5,2,\ldots,2}) = r+1$ and the choice number of the complete r-partite graph $K_{4,2,\ldots,2}$ is equal to r if r is odd, and $r+1$ if r is even.

Motivated by these results we study the value $Ch(K_{n,2,\ldots,2})$ for any positive integer n. In the proof of Theorem 1 we write $L(S)$ for the union $\bigcup_{v\in S} L(v)$ where $S \subseteq V(G)$. If C is a set of colors, then $L\backslash C$ denotes the list assignment obtained from L by removing the colors in C from each $L(v)$ where $v \in V(G)$. First, we show that the graph $K_{(t+2)(t+3)/2,2,\ldots,2}$ is $(r+t)$-choosable.

Theorem 1. *Let t be a positive integer and let G be a complete r-partite graph with one partite class of order $(t+2)(t+3)/2$ and $r-1$ partite classes of order two. Then $Ch(G) \leq r+t$.*

Proof. Let V_1 be the partite class of G of order $(t+2)(t+3)/2$ and let $V_i = \{v_i, w_i\}, 2 \leq i \leq r$, be the partite classes of order two. Let L_1 be any $(r+t)$-list assignment to the vertices of G. We prove that G is L_1-colorable. We distinguish three cases:

Case 1. $t \geq r-1$.
We can color the vertices of $V_2, V_3, \ldots, V_r$ with $2r-2$ different colors. Since $|L_1(v)| \geq 2r-1$ for every vertex $v \in V_1$, we can color the vertices of V_1 as well.

Case 2. There exists a color $c \in L_1(v_i) \cap L_1(w_i)$ for some $i \in \{2, 3, \ldots, r\}$.
It is easy to show by induction on r that G is L_1-colorable. The step $r = 1$ is trivial. For the induction step, assign c to both v_i and w_i, and remove c from the lists of the remaining vertices. By the induction hypothesis, the remaining vertices can be colored with colors from the revised lists.

Case 3. $t \leq r-2$ and $L_1(v_i) \cap L_1(w_i) = \emptyset$ for every $i \in \{2, 3, \ldots, r\}$.
We prove by contradiction that G is L_1-colorable. Assume that G is not L_1-colorable. Let L be an $(r+t)$-list assignment such that G is not L-colorable. Let X_j, $j = 1, 2, \ldots, t$, be the largest subset of $V_1\backslash(\bigcup_{l=1}^{j-1} X_l)$ with $\bigcap_{v\in X_j} L(v) \neq \emptyset$. Set $|X_j| = x_j$ and choose a color $c_j \in \bigcap_{v\in X_j} L(v)$. Define $L' = L\backslash\{c_1, c_2, \ldots, c_t\}$ and $G' = G\backslash(\bigcup_{l=1}^{t} X_l)$. Note that $|L'(v)| = r+t$ for each $v \in V(G') \cap V_1$ and $|L'(v_i)|, |L'(w_i)| \geq r$ for any $i \in \{2, 3, \ldots, r\}$. Since G is not L-colorable, G' is not L'-colorable. It follows that there exists a set of vertices $T \subseteq V(G')$ such that $|L'(T)| < |T|$, i.e., L' does not satisfy Hall's condition. Let S denote a maximal subset of $V(G')$ such that $|L'(S)| < |S|$. We consider two subcases:

Case 3a. $|S \cap V_i| \leq 1$ for every $i \in \{2, 3, \ldots, r\}$.
Since $|L'(v_i)|, |L'(w_i)| \geq r$ and $|S \backslash V_1| \leq r-1$, $S \backslash V_1$ can be colored from the list L'. Further, $|L'(v)| = r+t$ for $v \in S \cap V_1$, therefore we can also color the vertices in $S \cap V_1$.

Let $L'' = L' \backslash L'(S)$. We show that $G' \backslash S$ is L''-colorable. If $G' \backslash S$ is not L''-colorable, we have a nonempty subset $S' \subset V(G') \backslash S$ with $|L''(S')| < |S'|$. Then $|L'(S \cup S')| = |L'(S)| + |L''(S')| < |S| + |S'|$, which contradicts the maximality of S.

Case 3b. Both $v_i, w_i \in S$ for some $i \in \{2, 3, \ldots, r\}$.
Then $|S| > |L'(S)| \geq |L'(v_i)| + |L'(w_i)| \geq 2(r+t) - t$. Set $|S| = 2r + t + 1 + \epsilon$ where $\epsilon \geq 0$. Clearly, $|L'(S)| \leq 2r + t + \epsilon$. Let $S_1 = S \cap V_1$. We have $|S_1| \geq |S| - (2r-2) = t + 3 + \epsilon$. By the maximality of X_t, every color in $L'(S)$ appears in the lists of at most x_t vertices of S_1. It means that

$$(r+t)|S_1| = \sum_{v \in S_1} |L'(v)| \leq x_t |L'(S)|. \tag{1}$$

It is evident that $\sum_{l=1}^{t} x_l + |S_1| \leq |V_1| = (t+2)(t+3)/2$. Hence, $tx_t + |S_1| \leq (t+2)(t+3)/2$, or equivalently

$$x_t \leq [(t+2)(t+3)/2 - |S_1|]/t. \tag{2}$$

By (1) and (2), we have $(r+t)|S_1| \leq [(t+2)(t+3)/2 - |S_1|]|L'(S)|/t$. Since $|S_1| \geq t + 3 + \epsilon$ and $|L'(S)| \leq 2r + t + \epsilon$, we have $(r+t)(t+3+\epsilon) \leq [(t+2)(t+3)/2 - (t+3+\epsilon)](2r+t+\epsilon)/t$ which yields $\frac{t^3}{2} + (3+\epsilon)\frac{t^2}{2} + (r-\frac{1}{2})\epsilon t + (2r+\epsilon)\epsilon \leq 0$, a contradiction. This finishes the proof. ■

If $t = 1$, then $Ch(K_{6,2,\ldots,2}) \leq r+1$. This bound also comes from the result $Ch(K_{3,3,2,\ldots,2}) = r$ of Gravier and Maffray [4], because the complete r-partite graph $K_{6,2,\ldots,2}$ is a subgraph of the complete $(r+1)$-partite graph $K_{3,3,2,\ldots,2}$. Since the choice number of the complete r-partite graph $K_{5,2,\ldots,2}$ is equal to $r+1$, it is clear that $Ch(K_{6,2,\ldots,2}) = r+1$ as well.

Now we present a lower bound on the choice number of complete r-partite graphs with $r-1$ partite classes of order at most two.

Theorem 2. *Let s, r, t be integers such that $0 \leq s < r$ and $t > 0$. Let G be a complete r-partite graph consisting of one partite class of order $\binom{2t+s}{t}^2$, $r-s-1$ partite classes of order two, and s partite classes of order one. Then $Ch(G) > \lfloor \frac{r+t-1}{2t+s} \rfloor (2t+s)$.*

Proof. Let $n = \binom{2t+s}{t}^2$ and $m = \frac{r+t-1}{2t+s}$. Let G be a complete r-partite graph with the partite classes V_1, $V_i = \{v_i, w_i\}, V_j = \{v_j\}$, where $|V_1| = n$; $i = 2, 3, \ldots, r-s$ and $j = r-s+1, r-s+2, \ldots, r$. Let $A_1, A_2, \ldots, A_{2t+s}, B_1, B_2, \ldots, B_{2t+s}$ be disjoint color sets of order $\lfloor m \rfloor$ such that $\bigcup_{i=1}^{2t+s} A_i = A$, $\bigcup_{i=1}^{2t+s} B_i = B$. We define a list assignment L to $V(G)$ by the following way:

$$L(v_j) = A, \; j = 2, 3, \ldots, r,$$
$$L(w_i) = B, \; i = 2, 3, \ldots, r - s.$$

The lists of colors given to the vertices of V_1 consist of $2t + s$ different sets $A_{x_1}, A_{x_2}, \ldots, A_{x_{t+s}}, B_{y_1}, B_{y_2}, \ldots, B_{y_t}$, where $x_1, x_2, \ldots, x_{t+s}$, $y_1, y_2, \ldots, y_t \in \{1, 2, \ldots, 2t + s\}$. Since the number of vertices in V_1 is $n = \binom{2t+s}{t+s}\binom{2t+s}{t}$, we are able to assign to any two vertices in V_1 different lists.

We show by contradiction that G cannot be colored from the list L. Suppose that G can be colored from L. We use $r - 1$ different colors of A to color the vertices $v_2, v_3, \ldots, v_r$ and $r - s - 1$ different colors of B to color $w_2, w_3, \ldots, w_{r-s}$. Since $|A| = |B| = \lfloor m \rfloor (2t + s) \leq r + t - 1$, the number of colors in A (in B) not used to color $V_2, V_3, \ldots, V_r$ is at most t (at most $t + s$). It follows that there are at most $2t + s$ sets $A_{x'_1}, A_{x'_2}, \ldots, A_{x'_t}, B_{y'_1}, B_{y'_2}, \ldots, B_{y'_{t+s}}$, where $x'_1, x'_2, \ldots, x'_t, y'_1, y'_2, \ldots, y'_{t+s} \in \{1, 2, \ldots 2t + s\}$ containing colors that were not employed in coloring V_2, $V_3, \ldots, V_r$. Try to color V_1 with these colors. According to the assignment of color sets to the vertices of V_1, there exists a vertex $v \in V_1$ having none of the sets $A_{x'_1}, A_{x'_2}, \ldots, A_{x'_t}, B_{y'_1}, B_{y'_2}, \ldots, B_{y'_{t+s}}$ in its list, a contradiction. Hence, G is not L-colorable. ■

Note that we get the bound $Ch(K_{\binom{2t}{t}^2, 2, \ldots, 2}) \geq r + t$ if $s = 0$ and $r = pt + 1$ for some odd integer p.

3. Complete Multipartite Graphs with Partite Classes of Equal Sizes

Let K_{n*r} denote the complete multipartite graph with r partite classes of order n. The problem is to determine the value of the choice number $Ch(K_{n*r})$. If $n = 1$, then K_{n*r} is a clique on r vertices and hence, obviously, $Ch(K_{1*r}) = r$. In the previous section we mentioned that $Ch(K_{2*r}) = r$ as well. Alon [1] established the general bounds $c_1 r \log n \leq Ch(K_{n*r}) \leq c_2 r \log n$ for every $r, n \geq 2$, where c_1, c_2 are two positive constants. Later, Kierstead [5] solved the problem in the case $n = 3$. He showed that $Ch(K_{3*r}) = \lceil \frac{4r-1}{3} \rceil$. Yang [9] studied the value of $Ch(K_{4*r})$ and obtained the bounds $\lfloor \frac{3}{2} r \rfloor \leq Ch(K_{4*r}) \leq \lceil \frac{7}{4} r \rceil$. We present results giving exact bounds on $Ch(K_{n*r})$ for large n. In the proof of Theorem 3 we use the following lemma proved in [5].

Lemma 1. *A graph G is k-choosable if G is L-colorable for every k-list assignment L such that $|\bigcup_{v \in V(G)} L(v)| < |V(G)|$.*

Let us derive an upper bound on the choice number of complete multipartite graphs with partite classes of equal sizes.

Theorem 3. *Let* $0 < \alpha < n$ *and let* $x_j = \lfloor(\alpha - \frac{\alpha}{n}\sum_{l=1}^{j-1} x_l)\rfloor + 1$, $j = 1, 2, \ldots, \lfloor\alpha\rfloor$. *If* $n \leq \sum_{l=1}^{\lfloor\alpha\rfloor} x_l$, *then* $Ch(K_{n*r}) \leq \lceil\alpha r\rceil$.

Proof. Let V_i, $i = 1, 2, \ldots, r$, be the i-th partite class of K_{n*r}. We prove the result by induction on r. The case $r = 1$ is trivial. For the induction step consider an $\lceil\alpha r\rceil$-list assignment L to the vertices of K_{n*r}. We prove that if $n \leq \sum_{l=1}^{\lfloor\alpha\rfloor} x_l$, then any partite class V_i can be colored with at most $\lfloor\alpha\rfloor$ colors.

Assume that $n = \sum_{l=1}^{\lfloor\alpha\rfloor} x_l$. In this paragraph we show by induction on j ($j = 1, 2, \ldots, \lfloor\alpha\rfloor$), that there exists a color c_j which can be used for coloring x_j vertices of V_i that have not been colored by $c_1, c_2, \ldots, c_{j-1}$ yet. Note that $c_l, c_{l'}'$, where $l, l' \in \{1, 2, \ldots, \lfloor\alpha\rfloor\}$, $l \neq l'$, do not have to be different.

If $j = 1$, we have $x_1 = \lfloor\alpha\rfloor + 1$. Since $\sum_{v\in V_i} |L(v)| = \lceil\alpha r\rceil n$ and by Lemma 1, $|\bigcup_{v\in V(K_{n*r})} L(v)| < rn$, there exists a color c_1 which appears in the lists of at least $\lfloor\alpha\rfloor + 1$ vertices of V_i. Color these vertices with c_1. Suppose $j \geq 2$. We can color $\sum_{l=1}^{j-1} x_l$ vertices with $c_1, c_2, \ldots, c_{j-1}$. The sum of the numbers of colors in the lists of the remaining $n - \sum_{l=1}^{j-1} x_l$ vertices of V_i is $(n - \sum_{l=1}^{j-1} x_l)\lceil\alpha r\rceil$. Since $|\bigcup_{v\in V_i} L(v)| < rn$, there is a color c_j that appears in the lists of other $\lfloor(n - \sum_{l=1}^{j-1} x_l)\frac{\alpha}{n}\rfloor + 1 = x_j$ vertices. Hence, we can color these vertices with c_j. It follows that it is possible to color $n = \sum_{l=1}^{\lfloor\alpha\rfloor} x_l$ vertices of V_i with at most $\lfloor\alpha\rfloor$ different colors.

Clearly, if $n < \sum_{l=1}^{\lfloor\alpha\rfloor} x_l$, all the vertices of V_i can be colored with at most $\lfloor\alpha\rfloor$ colors too. Let us remove the colors that were employed in coloring V_i from the lists given to the vertices in $V(K_{n*r})\backslash V_i$. We have at least $\lceil\alpha r\rceil - \lfloor\alpha\rfloor$ colors. Since $\lceil\alpha r\rceil - \lfloor\alpha\rfloor \geq \lceil\alpha(r-1)\rceil$, by applying the induction hypothesis, $r-1$ partite classes can be colored with $\lceil\alpha(r-1)\rceil$ colors, i.e., there exists a proper coloring of the vertices in $V(K_{n*r})\backslash V_i$ with colors from the revised lists. ■

Unfortunately, the result presented in Theorem 3 cannot be bounded from above by $cr\log n$, where c is a constant. Theorem 3, for example, yields the upper bounds $Ch(K_{5*r}) \leq \lceil\frac{5}{2}r\rceil$, $Ch(K_{15*r}) \leq 5r$, $Ch(K_{40*r}) \leq 10r$, $Ch(K_{75*r}) \leq 15r$ and $Ch(K_{121*r}) \leq 20r$. One can check that $10r \approx 6.24r \log 40$, $15r \approx 8r \log 75$ and $20r \approx 9.6r \log 121$.

The following result gives a lower bound on $Ch(K_{n*r})$.

Theorem 4. *Let* x, t, r, n *be integers such that* $x, t, r \geq 2$, $x \geq t$ *and* $n = \binom{x}{x-t+1}$. *Then* $Ch(K_{n*r}) > (x - t + 1)\lfloor\frac{tr-1}{x}\rfloor$.

Proof. Let $x, t, r \geq 2$, $x \geq t$, $n = \binom{x}{x-t+1}$ and let $k = (x-t+1)\lfloor\frac{tr-1}{x}\rfloor$. We show that there exists a k-list assignment L of K_{n*r} such that K_{n*r} is not L-colorable.

Let V_i, $i = 1, 2, \ldots, r$, be the i-th partite class of K_{n*r}. Let $A_1, A_2, \ldots, A_x$ be a family of disjoint color sets such that $|A_j| = |A_1|$ or $|A_j| = |A_1|+1$, $j = 2, 3, \ldots, x$, and $|\bigcup_{j=1}^{x} A_j| = tr - 1$. Obviously, $|A_j| \geq \lfloor \frac{tr-1}{x} \rfloor$ for any $j \in \{1, 2, \ldots, x\}$.

Define a list assignment L as follows: Let the lists given to the n vertices of every partite class V_i consist of $x - t + 1$ different sets $A_{y_1}, A_{y_2}, \ldots, A_{y_{x-t+1}}$, $y_1, y_2, \ldots, y_{x-t+1} \in \{1, 2, \ldots, x\}$, where any two vertices in the same part have different lists. Note that $|L(v)| \geq (x - t + 1)\lfloor \frac{tr-1}{x} \rfloor$ for each vertex $v \in V(K_{n*r})$. Then for any partite class V_i and any $t - 1$ colors $a_j \in A_{y'_j}$, $j = 1, 2, \ldots, t - 1$; $y'_j \in \{1, 2, \ldots, x\}$ there is a vertex $v \in V_i$ having none of the sets $A_{y'_j}$ in its list. Therefore, in any coloring from these lists, we must use at least t colors on each partite class. Since the number of colors in $\bigcup_{j=1}^{x} A_j$ is less than tr, K_{n*r} is not L-colorable. ■

Theorem 4 says that if, for instance $t = 2$, then $n = x$ and $Ch(K_{n*r}) > (n - 1)\lfloor \frac{2r-1}{n} \rfloor$. In particular, for $n = 5$ we have $Ch(K_{5*r}) > 4\lfloor \frac{2r-1}{5} \rfloor$. If $t = 3$, then $Ch(K_{n*r}) > (x - 2)\lfloor \frac{3r-1}{x} \rfloor$. For example, in the case $x = 6$ we get $Ch(K_{15*r}) > 4\lfloor \frac{3r-1}{6} \rfloor = 4\lfloor \frac{r-1}{2} \rfloor$.

Finally, we present a corollary of Theorem 4 which yields a lower bound in the form $cr \log n$.

Corollary 1. *Let $r \geq 2$ and $n = \binom{x}{\lceil x/2 \rceil}$ where $x \geq 5$. Then*

$$Ch(K_{n*r}) > \lfloor \tfrac{r}{2} \rfloor \lceil \tfrac{\log_{2.1} n}{2} \rceil.$$

Proof. For $x, t, r \geq 2$, $x \geq t$ and $n = \binom{x}{x-t+1}$, we have $Ch(K_{n*r}) > (x - t + 1)\lfloor \frac{tr-1}{x} \rfloor$. Let $t = \lfloor \frac{x}{2} \rfloor + 1$. Then $Ch(K_{n*r}) > \lceil \frac{x}{2} \rceil \lfloor \frac{\lfloor x/2 \rfloor r + r - 1}{x} \rfloor \geq \lceil \frac{x}{2} \rceil \lfloor \frac{r}{2} \rfloor$. It is well-known that $\frac{x^x}{e^{x-1}} \leq x! \leq \frac{(x+1)^{x+1}}{e^x}$ for any x. For $x \geq 5$, the following inequalities also hold: $\frac{2x^x}{e^{x-1}} < x! < \frac{6x^{x+1}}{5e^x}$. Then $n = \frac{x!}{\lfloor x/2 \rfloor ! \lceil x/2 \rceil !} < \frac{6x^{x+1}/(5e^x)}{4\lfloor x/2 \rfloor^{\lfloor x/2 \rfloor} \lceil x/2 \rceil^{\lceil x/2 \rceil}/e^{x-2}} \leq \frac{3x^{x+1}}{10\lfloor x/2 \rfloor^x e^2} \leq \frac{3x^x x 2^x}{10(x-1)^x e^2}$. Since $x2^x < 7.6(2.1)^x$ for any x (note that $7.5(2.1)^x < x2^x$ for $19 \leq x \leq 22$) and $(\frac{x}{x-1})^x < 3.1$ for any $x \geq 5$, we have $n < \frac{7.068(2.1)^x}{e^2} < (2.1)^x$. Consequently, $\log_{2.1} n < x$, hence $Ch(K_{n*r}) > \lfloor \frac{r}{2} \rfloor \lceil \frac{\log_{2.1} n}{2} \rceil$ for any $n = \binom{x}{\lceil x/2 \rceil}$ where $x \geq 5$. ■

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