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ON MONOCHROMATIC PATHS AND BICOLORED SUBDIGRAPHS IN ARC-COLORED TOURNAMENTS

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Abstract

Consider an arc-colored digraph. A set of vertices N is a kernel by monochromatic paths if all pairs of distinct vertices of N have no monochromatic directed path between them and if for every vertex v not in N there exists $n \in N$ such that there is a monochromatic directed path from v to n.

In this paper we prove different sufficient conditions which imply that an arc-colored tournament has a kernel by monochromatic paths. Our conditions concerns to some subdigraphs of T and its quasimonochromatic and bicolor coloration. We also prove that our conditions are not mutually implied and that they are not implied by those known previously. Besides some open problems are proposed.

Keywords: kernel, kernel by monochromatic paths, tournament.

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1. Introduction

While not every arc-colored digraph has a kernel by monochromatic paths (e.g. a directed cycle colored with three colors), Sands *et al.* [30] proved

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that every 2-colored digraph certaintly does. Later Shen Minggang [26] and Galeana-Sánchez [12] proved the same for arc colored tournaments if certain subdigraphs are colored in a special way: Shen Minggang proved it provided that every directed triangle (that is, a transitive tournament of order 3 or a directed cycle of length 3) is colored with at most two colors and he also proved that this hypothesis is tight when the tournament is colored with at least 5 colors (Galeana Sánchez and Rojas-Monroy proved this for 4 colors, see [17]), meanwhile Galeana-Sánchez proved it for every directed cycle with length at most 4 being a quasimonochromatic cycle (i.e., a cycle such that with at most one exception every arc is colored alike).

In this paper we also propose sufficient conditions for an arc-colored digraph to have a kernel by monochromatic paths and such conditions arise from searching subdigraphs different from cycles and triangles (as in the Shen Minggang conditions and the Galeana conditions) with lucky colorations such that every tournament with those subdigraphs has a kernel by monochromatic paths.

What we found is stated in two sections, one for each new subdigraph in the tournament, $\mathcal{T}_k$ and S_k: the first one is devoted to a condition which asks for the quasimonochromaticity of every $\mathcal{T}_k$ for some $k \geq 4$, and also the at most bicolor coloration of every cycle with length less than k. In the second section we assemble three different conditions related with the S_k subdigraphs: One consists in the bicolor coloration of every cycle of length 3 and 4, and the quasimonochromatic coloration of every S_4; we also prove that if we forbid the existence of a certain subdivision of a 3-colored C_3 and ask for the non polychromatic coloration (more than three colors) of every S_4, every S_5 and every C_3, then the result holds; the last condition asserts that the tournament has a kernel by monochromatic paths whenever every S_k is a non polychromatic subdigraph of the tournament, for some $k \geq 5$, and every cycle with length less than k is non polychromatic as well.

2. Terminology and Notation

We use the standard terminology on digraphs as given in [1]. However we provide most of the necessary definitions and notation for the convenience of the reader.

For a digraph D, the vertex (arc) set is denoted by $V(D)$ ($A(D)$). If $S \subseteq V(D)$ is nonempty then $D[S]$ is the *subdigraph of D induced by S*. An

arc $z_1z_2 \in A(D)$ is called an *asymmetrical arc* (symmetrical) if $z_2z_1 \notin A(D)$ ($z_2z_1 \in A(D)$); the *asymmetrical part of* D (the symmetrical part of D) denoted by $Asym(D)$ ($Sym(D)$) is the spanning subdigraph of D whose arcs are the asymmetrical (symmetrical) arcs of D; D is called an *asymmetrical digraph* if $Asym(D) = D$. A digraph is called *semicomplete* if for every two distinct vertices u and v of D, at least one of the arcs (u, v) or (v, u) is present in D. A semicomplete asymmetrical digraph is called a *tournament*.

An arc $z_1z_2 \in A(D)$ will be called a V_1V_2-*arc* whenever $z_1 \in V_1 \subseteq V(D)$ and $z_2 \in V_2 \subseteq V(D)$. By $[z_1, z_2]_T$ we denote an arc between z_1 and z_2.

For a directed *walk* W we will denote its length by $\ell(W)$. And if $\{z_1, z_2\} \subset V(W)$ then we denote by (z_1, W, z_2) the z_1z_2-*directed walk* contained in W. We will denote by C_n a directed cycle of length n. Throughout the paper all the paths and cycles considered are directed paths and directed cycles.

3. Kernels and Kernels by Monochromatic Paths

3.1. Kernels

The concept of a kernel was first presented in [28] (under the name *solution*) in the context of Game Theory by von Neumann and Morgenstern as an interesting solution for cooperative n-person games with general n, see [1] for more details. Let us repeat the definition of a *kernel* on the context of Graph Theory: A kernel N of D is an independent set of vertices such that for each $z \in V(D) - N$ there exists a zN-arc in D. As the reader can see, not every digraph has a kernel and when a digraph contains a kernel, it may not be the only one. This simple observation compels us to ask for sufficient conditions for the existence of a kernel in a digraph. It is well known that if D is finite, the decision problem of the existence of a kernel in D is NP-complete for a general digraph (see [5] and [25]) and for a planar digraph with indegrees less than or equal to 2, outdegrees less than or equal to 2 and degrees less than or equal to 3 [9].

A digraph D such that every induced subdigraph in D has a kernel is called a *kernel-perfect* digraph (or simply, a *KP-digraph*). The following sufficient conditions for a digraph to be a *KP*-digraph are known:

Theorem 1. *D is a kernel-perfect digraph if one of the following conditions holds:*

(i) *D has no cycles of odd length.*

(ii) *Every directed cycle of odd length in D has at least two symmetric arcs.*

(iii) *$Asym(D)$ is acyclic.*

(iv) *Every directed cycle in D has at least one symmetrical arc.*

These claims were proved respectively by Richardson [29], Duchet [7], Duchet and Meyniel [8], and by Berge and Duchet [2].

There are many applications of this concept in the context of game theory, logic and decision theory (see [1]), as well as several interesting related results (see also [2, 13] and [14, 27, 3]). A selected bibliography can be found in [10], and we also recommend the survey [4].

3.2. Arc colored digraphs and kernel generalizations

A simple variation of the problem first presented by von Neumann and Morgenstern brings an interesting generalization of the concept of kernel (see [6]). In order to present it first let us introduce some notation.

D is an *m-colored digraph* if the arcs of D are colored with m colors. Let D be an m-colored digraph. A directed path (or cycle) is called *monochromatic* if all of its arcs are colored alike and it is called *quasimonochromatic* if with at most one exception all of its arcs are colored alike. A subdigraph H of D is called a *k-colored digraph* if all of its arcs are colored with only k colors, in particular for $k = 2$ we say that H is *bicolor*. We will say that a subdigraph H of D is an *at most k-colored digraph* if all of its arcs are colored with at most k colors, in particular for $k = 2$ we say that H is at most bicolor; H will be called a *polychromatic digraph* if all of its arcs are colored with at least 3 colors.

A set $N \subseteq V(D)$ is said to be a *kernel by monochromatic paths of D* if it satisfies the following conditions: (a) N is an *independent set by monochromatic paths*: for every pair of different vertices u and v in N there is no monochromatic path between them in D; and (b) N is an *absorbing set by monochromatic paths*: for every vertex $x \in V(D) - N$ there is a vertex $n \in N$ such that there is an xn-monochromatic path in D.

The concept of kernel by monochromatic paths is a generalization of the concept of kernel. Another interesting generalization of this concept is that of (k, l)-kernel introduced by M. Kwaśnik in [23] (see also [24, 11, 31, 21] and [34]). Even several results arise around the concept of kernel by monochromatic paths (see [16, 18, 32] and [33]) those results mentioned in

the Introduction are the foundations of this work. For a short historical review of them see [6].

4. Main Results. New Sufficient Conditions

As it was mentioned in the Introduction of this paper, we obtain sufficient conditions for the existence of a kernel by monochromatic paths in an m-colored tournament. Our technique allow us to assure something more powerful: the existence of such kernel in every induced subdigraph of the tournament. We also prove that our conditions are not implied by those known previously.

The spirit of our proofs arises from structural properties of arc colored tournaments (see Lemma 1) and these properties are deduced by working with previous results on kernels (see Theorem 1-iv) on an new digraph associated with our original tournament, its closure:

Definition 1. For an m-colored digraph D, the *closure of* D, $C(D)$, is the multidigraph such that:

$$V(C(D)) = V(D),$$
$$A(C(D)) = A(D) \cup \{uv \mid \text{there is an } uv - \text{monochromatic path in } D\}.$$

Notice that by definition of $C(D)$ it holds that $N \subseteq V(D)$ is a kernel by monochromatic paths of D if and only if $N \subseteq V(C(D))$ is a kernel of $C(D)$. Hence, it is ascertained that the closure of a digraph D relates in a very natural way kernels by monochromatic paths in this digraph with kernels in its closure. Now, notice that if certain properties which imply that D has a kernel also hold in the closure of D, then we can assert that $C(D)$ has a kernel and hence D has a kernel by monochromatic paths (by the definition of $C(D)$). In particular, if the closure of a digraph D satisfies some of the sufficient conditions in Theorem 1, then we get as an immediate application of this theorem that D has a kernel by monochromatic paths. This is an important point to mention because this is not the case with the following results: the sufficient conditions stated in our results hold in tournaments and not in its closure.

We start with the following Lemma which gives us structural properties (i.e., existence and color properties) of certain subdigraphs of an arc colored tournament whose closure is not a KP-digraph and such that every C_3 is a

quasimonochromatic cycle. This Lemma is the heart or our proofs as the reader can see. For an easy reading we separate the proof in several claims written in italic font.

Lemma 1. *Let T be an m-colored tournament. If every $C_3 \subseteq T$ is a quasi-monochromatic cycle and $C(T)$ is not a KP-digraph then there exists a cycle $\gamma = (z_0, z_1, z_2 = 0, 1, 2, \ldots, p = z_0) \subseteq C(T)$ such that the following properties hold:*

(a) $\ell(\gamma) \geq 4$,

(b) $\gamma \subseteq T$,

(c) $(z_0, z_1) \in A(T)$ *with color* a, $(z_1, z_2) \in A(T)$ *with color* b *and there exists a* z_2z_0*-path* $\alpha = (z_2 = 0, 1, 2, \ldots, p = z_0)$ $(p \geq 2)$ *with color* c, $a \neq b$, $b \neq c$, $a \neq c$, *let* a*=red,* b*=blue,* c*=black,*

(d) $(z_2, z_0) \notin A(T)$ *(so* $(z_0, z_2) \in A(T)$*),*

(e) *There is no* z_1z_0*-monochromatic path in* T *and there is no* z_2z_1*-monochromatic path in* T*,*

(f) *Every arc between* z_1 *and an internal vertex in* α *is not black.*

Proof. Proceeding by contradiction, let us suppose that $C(T)$ is not a *KP*-digraph so a well known theorem by Berge and Duchet (see Theorem 1-iv) asserts that there is a cycle $\Gamma \subseteq Asym(C(T))$. Let $\Gamma = (z_0, z_1, \ldots, z_{n-1}, z_n = z_0) \subseteq Asym(C(T))$ be a cycle with minimal length contained in $Asym(C(T))$. Through the following claims we will discover color properties of this cycle and they will allow us to prove the lemma.

1. $\ell(\Gamma) = n \geq 3$.
Recall $\Gamma \subseteq Asym(C(T))$, so $\ell(\Gamma) = n \neq 2$.

2. $\Gamma \subseteq T$.
Suppose that there is an arc $(z_i, z_{i+1}) \in \Gamma - T$. Since T is a tournament we have that $(z_{i+1}, z_i) \in T$ and so $\{(z_i, z_{i+1}, (z_{i+1}, z_i)\} \subseteq Asym(C(T))$, a contradiction.

3. $(z_0, z_1) \in A(T)$ has color a, $(z_1, z_2) \in A(T)$ has color b, $a \neq b$.
Since Γ is not a monochromatic cycle (by the contrary: $(z_0, \Gamma, z_{n-1}) \subseteq Asym(C(T))$ is a monochromatic path, thus $(z_0, z_{n-1}) \in A(C(T))$ and hence $(z_{n-1}, z_0) \in A(Sym(C(T)) \cap \Gamma)$, a contradiction), then there exists two consecutive arcs in Γ colored differently. Say $(z_0, z_1) \in A(\Gamma)$ is red and $(z_1, z_2) \in A(\Gamma)$ is blue.

4. For any $\{z_i, z_j\} \subset V(\Gamma)$ such that $j \notin \{i-1, i+1\}$ it holds that $\{(z_i, z_j), (z_j, z_i)\} \subseteq A(C(T))$.
Let $\{z_i, z_j\} \subset V(\Gamma)$ be such that $j \notin \{i-1, i+1\}$. Since T is a tournament, $(z_i, z_j) \in A(T)$ or $(z_j, z_i) \in A(T)$, without loss of generality let $(z_i, z_j) \in A(T)$. Then $\Gamma' = (z_i, z_j, z_{j+1}, z_{j+2}, \ldots, z_{i-1}, z_i) \subseteq T$ is a cycle with $\ell(\Gamma') < \ell(\Gamma)$. Hence $\Gamma' \not\subseteq Asym(C(T))$ and so $(z_i, z_j) \in A(Sym(C(T))$.

5. $(z_2, z_0) \notin A(T)$.
If $(z_2, z_0) \in A(T)$ then there exists $C_3 = (z_0, z_1, z_2, z_0) \subseteq T$ and it is a quasimonochromatic cycle by hypothesis, so $(z_2, z_0) \in A(T)$ is red or blue. If $(z_2, z_0) \in A(T)$ is red then $(z_2, z_0, z_1) \subseteq T$ is a z_2z_1-monochromatic path and $(z_1, z_2) \in A(Sym(C(T)) \cap \Gamma)$, a contradiction. If $(z_2, z_0) \in A(T)$ is blue, then $(z_1, z_2, z_0) \subseteq T$ is a z_1z_0-monochromatic path and $(z_0, z_1) \in A(Sym(C(T)) \cap \Gamma)$, a contradiction again.

Now, by claims (4) and (5) there exists a z_2z_0-monochromatic path in T with length at least 2. Let $\alpha = (z_2 = 0, 1, 2, \ldots, p = z_0) \subseteq T$ be a z_2z_0-monochromatic path with minimal length ($p \geq 2$).

6. $z_1 \notin V(\alpha)$.
Otherwise $z_1 \in V(\alpha)$ and then (z_2, α, z_1) is a z_2z_1-monochromatic path in T so $(z_1, z_2) \in Sym(C(T))$, contradiction.

7. α is neither red nor blue.
If α is red then $\alpha \cup (z_0, z_1)$ is a z_2z_1-monochromatic path in T and $(z_2, z_1) \in A(Sym(C(T)) \cap \Gamma)$, a contradiction. If α is blue then $(z_1, z_2) \cup \alpha \subseteq T$ is a z_1z_0-monochromatic path in T and $(z_1, z_0) \in A(Sym(C(T)) \cap \Gamma)$, a contradiction again. Let α be black.

Consider $\gamma = (z_0, z_1, z_2) \cup \alpha$. Clearly γ satisfies the first four properties of our Lemma 1. Let us conclude with the following points.

8. There is no z_1z_0-monochromatic path in T and there is no z_2z_1-monochromatic path in T.
Notice that $\{(z_0, z_1), (z_1, z_2)\} \subseteq Asym(C(T))$.

9. Every arc between z_1 and an internal vertex in α is not black.
If there exists i, $1 \leq i \leq p-1$ such that $(i, z_1) \in A(T)$ (*resp.* $(z_1, i) \in A(T)$) is black then $(z_2 = 0, \alpha, i) \cup (i, z_1) \subseteq T$ (*resp.* $(z_1, i) \cup (i, \alpha, z_0)$) is a z_2z_1-monochromatic path in T (*resp. is a* z_1z_0*-monochromatic path*), a contradiction. ■

In order to present the following conditions we must introduce new subdigraphs whose arc coloration in an arc colored tournament T will allow us to

assert the existence of a kernel by monochromatic paths in T and in every induced subdigraph of this digraph. It is important to mention that we decided to write these proofs in separated and enumerated claims (in italic font) in order to make its reading easier. Drawings also can help for a better understand: each arc is marked with the corresponding item in the proof and with prior items that are needed between brackets.

4.1. $\mathcal{T}_k$ subdigraphs

Definition 2. A subdigraph H of D is defined as a $\mathcal{T}_k$ if H consists of a directed path of length $k-1$, $(z_0, z_1, \ldots, z_{k-1})$, and the arc (z_0, z_{k-1}).

Definition 3. Let T be an m-colored tournament. T has the property P_k for some fixed integer $k \geq 4$ if:

(a) Every $\mathcal{T}_k \subseteq T$ is a quasimonochromatic subdigraph of T, and

(b) Every $C_t \subseteq T$ $(t < k)$ is at most bicolor.

Notice that the property P_4 is the corresponding property of the sufficient condition for cycles in the theorem proved by Galeana Sánchez [12] and mentioned in the Introduction.

Theorem 2. *Let T be an m-colored tournament. If T satisfies the property P_k for some integer $k \geq 4$, then $C(T)$ is a KP-digraph.*

Proof. We proceed by contradiction. Suppose that $C(T)$ is not a *KP*-digraph, then by Lemma 1 there exists a cycle $\gamma = (z_0, z_1, \ldots, p = z_0)$ satisfying properties (a) to (f). The following assertions will allow us to obtain a contradiction. First some general assertions:

1. $p \geq k-2$.
If $p < k-2$ then it follows from Lemma 1-c that γ is a 3-colored cycle. But $\ell(\gamma) < k$, a contradiction.

2. $(z_0, z_2) \in A(T)$ (Theorem 1, point 5).

3. For each i such that $0 \leq i \leq p-(k-2)$ it holds that. If $(z_1, i) \in A(T)$ then for every $m \in N$ such that $i+(k-2) \leq i+m(k-2) \leq p$ we have that $(i+m(k-2), z_1) \in A(T)$ whenever m is an odd number and $(z_1, i+m(k-2)) \in A(T)$ whenever m is an even number (this means that if there is an arc from z_1 to some vertex i of α then there is an arc from every vertex in α and separated from i an odd multiple of $k-2$ toward z_1 and there is an arc from z_1 to every vertex in α and separated from i an even multiple of $k-2$).

By induction over m. First let us asume that $(z_1, i) \in A(T)$ for some i, $0 \leq i \leq p-(k-2)$, and suppose by contradiction that $(i+(k-2), z_1) \notin A(T)$, then $(z_1, i+(k-2)) \in A(T)$ (because T is a tournament) and so there exists $\mathcal{T}_k = (z_1, i) \cup (i, \alpha, i+(k-2)) \cup (z_1, i+(k-2)) \subseteq T$ which is not quasimonochromatic by Lemma 1-f ((z_1, i) and $(z_1, i+(k-2))$ are not colored black and $k \geq 4$), contradicting that every $\mathcal{T}_k \subseteq T$ is a quasimonochromatic digraph. By the same way we can conclude that $(z_1, i+2(k-2)) \in A(T)$. Now suppose that for every n such that $i+(k-2) \leq i+n(k-2) \leq p-(k-2)$ we have the following: If n is even then $(z_1, i+n(k-2)) \in A(T)$ and if n is odd then $(i+n(k-2), z_1) \in A(T)$. We will prove the affirmation for $n+1$. If n is even then assume by contradiction that $(z_1, i+(n+1)(k-2)) \in A(T)$. Then there exists $\mathcal{T}_k = (z_1, i+n(k-2)) \cup (i+n(k-2), \alpha, i+(n+1)(k-2)) \cup (z_1, i+(n+1)(k-2)) \subseteq T$ which is not a quasimonochromatic $\mathcal{T}_k$ from Lemma 1-f ($(z_1, i+n(k-2))$ and $(z_1, i+(n+1)(k-2))$ are not colored black, besides $k \geq 4$), a contradiction. If n is odd then assume by contradiction that $(i+(n+1)(k-2), z_1) \in A(T)$. Then there exists $\mathcal{T}_k = (i+n(k-2), \alpha, i+(n+1)(k-2)) \cup (i+(n+1)(k-2), z_1) \cup (i+n(k-2), z_1) \subseteq T$ which is not a quasimonochromatic $\mathcal{T}_k$ ($k \geq 4$ and from Lemma 1-f), a contradiction again.

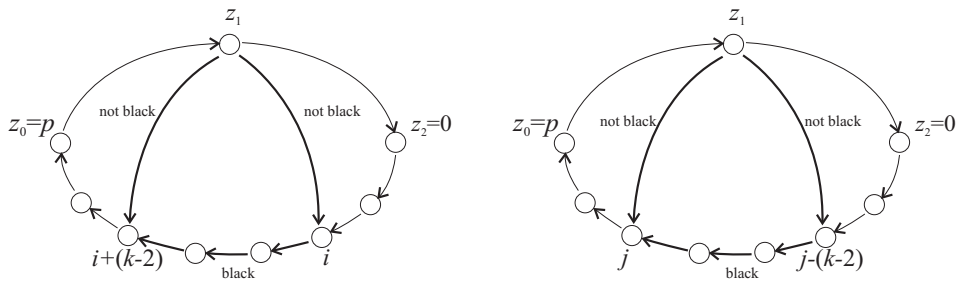

Figure 1. Claims 3 and 4 (from left to right).

4. For each j with $k-2 \leq j \leq p$ it holds that. If $(j, z_1) \in A(T)$ then for every $m \in N$ such that $0 \leq j-m(k-2) \leq j-(k-2)$ we have that $(z_1, j-m(k-2)) \in A(T)$ with m odd, and $(j-m(k-2), z_1) \in A(T)$ with m even.

Let j be such that $k-2 \leq j \leq p$ and $(j, z_1) \in A(T)$ and let m be such that $0 \leq j-m(k-2) \leq j-(k-2)$. If m is odd then suppose by contradiction that $(z_1, j-m(k-2)) \notin A(T)$ so $(j-m(k-2), z_1) \in A(T)$ (T is a tournament) and by the previous point (take $j-m(k-2)$ as i) we have

that $(z_1, j) \in A(T)$, a contradiction. An analogue argument holds when m is even.

5. $(z_1, k-3) \in A(T)$.
If $(k-3, z_1) \in A(T)$ then there exists $C_{k-1} = (z_1, z_2 = 0) \cup (0, \alpha, k-3) \cup (k-3, z_1) \subseteq T$ and it is an at most bicolor subdigraph by hypothesis so $(k-3, z_1) \in A(T)$ is blue (it is not black from Lemma 1-f), then there exists the 3-colored $\mathcal{T}_k = (z_0, z_2 = 0) \cup (0, \alpha, k-3) \cup (k-3, z_1) \cup (z_0, z_1) \subseteq T$ ($\mathcal{T}_k$ contains at least one black arc from α, as $k \geq 4$, and $(z_0, z_1) \in A(T)$ is red from Lemma 1-c), a contradiction.

Now we continue the proof by analyzing the following two cases, depending on the value of k:

Case I. $k = 4$.

6. $(p-1, z_1) \in A(T)$.
If $(z_1, p-1) \in A(T)$ then $C_3 = (z_1, p-1, z_0, z_1) \subseteq T$ is a quasimonochromatic cycle by hypothesis so $(z_1, p-1)$ is red (it is not colored black as a consequence of Lemma 1-f) then $\mathcal{T}_4 = (z_1, p-1, p = z_0, z_2 = 0) \cup (z_1, z_2) \subseteq T$ is not a bicolor subdigraph (Lemma 1-c), a contradiction.

Case I.A. $p = 2m$, for some $m \in N$.

Subcase A1. $p = 2m$, with m odd.
$(p-1, z_1) \in A(T)$ by (6) so it follows from (4) that $(1, z_1) \in A(T)$, contradicting point 5 with $k = 4$. Then this case is impossible.

Subcase A2. $p = 2m$, with m even.
$(z_0, z_1) \in A(T)$ then by (4) with $i = p$ we obtain that $(z_2 = 0, z_1) \in A(T)$, a contradiction.

Case I.B. $p = 2m+1$, for some $m \in N$.

Subcase B1. $p = 2m+1$, with m even.
$(z_1, z_2 = 0) \in A(T)$ then by point (3) we have that $(z_1, p-1) \in A(T)$, a contradiction with point (6).

Subcase B2. $p = 2m+1$, with m odd.

7. $(1, z_0) \in A(T)$.
If $(z_0, 1) \in A(T)$ then there exists the non quasimonochromatic $\mathcal{T}_4 = (z_0, z_1, z_2 = 0, 1) \cup (z_0, 1) \subseteq T$, a contradiction.

8. $(z_2, p-1) \in A(T)$.
If $(p-1, z_2) \in A(T)$ then $\mathcal{T}_4 = (p-1, z_0, z_1, z_2 = 0) \cup (p-1, z_2) \subseteq T$ is a non quasimonochromatic subdigaph, a contradiction.

Then there exists

$$\mathcal{T}_{4_A} = (z_1, 1, z_0, z_2) \cup (z_1, z_2) \subseteq T,$$
$$\mathcal{T}_{4_B} = (z_0, z_2, p-1, z_1) \cup (z_0, z_1) \subseteq T,$$
$$C_3' = (z_0, z_1, 1, z_0) \subseteq T.$$

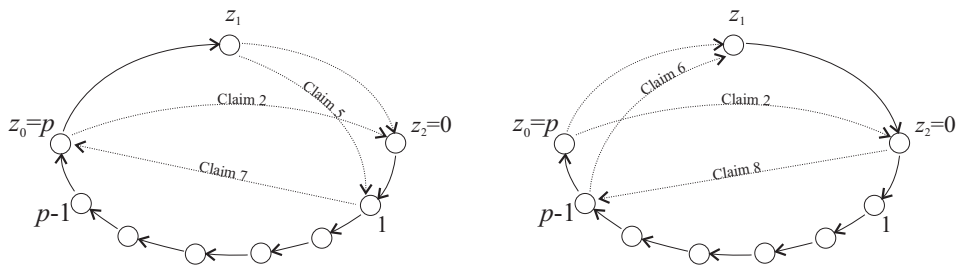

Figure 2. Case I.B2 for $k = 4$. Left: $\mathcal{T}_{4_A}$. Right: $\mathcal{T}_{4_B}$.

9. $color(z_2, p-1) \neq color(p-1, z_1)$ (by the contrary these arcs form a monochromatic z_2z_1-path in T).

10. $color(z_1, 1) \neq color(1, z_0)$ (by the contrary there is a monochromatic z_1z_0-path in T).

11. $(z_1, 1) \in A(T)$ is colored red or $(1, z_0) \in A(T)$ is colored red.
If not then C_3' is a 3-colored cycle, a contradiction.

Subcase B2-a. $(1, z_0) \in A(T)$ is red.

12. $(z_1, 1) \in A(T)$ is blue ($\mathcal{T}_{4_A}$ is a quasimonochromatic one and from point 10).

13. $(z_0, z_2) \in A(T)$ is blue ($\mathcal{T}_{4_A}$ is a quasimonochromatic one and from the previous point).

14. $(p-1, z_1) \in A(\mathcal{T}_{4_B})$ and $(z_2 = 0, p-1) \in A(\mathcal{T}_{4_B})$ are both red or they are both blue.
$(z_0, z_2) \in A(\mathcal{T}_{4_B})$ is blue (previous point) and $\mathcal{T}_{4_B}$ is a quasimonochromatic subdigraph.

Then $(z_2, p-1, z_1) \subseteq T$ is a monochromatic z_2z_1-path, a contradiction.

Subcase B2-b. $(z_1, 1) \in A(T)$ is red.

15. $(1, z_0) \in A(T)$ is blue ($\mathcal{T}_{4_A}$ is a quasimonochromatic one and from point 10).

16. $(z_0, z_2) \in A(T)$ is blue ($\mathcal{T}_{4_A}$ is a quasimonochromatic one and from the previous point).

17. $(p-1, z_1) \in F(\mathcal{T}_{4_B})$ and $(z_2 = 0,\ p-1) \in F(\mathcal{T}_{4_B})$ are both red or they are both blue (by the last point and because $\mathcal{T}_{4_B}$ is a quasimonochromatic subdigraph).

Then $(z_2, p-1, z_1) \subseteq T$ is a monochromatic z_2z_1-path, and a contradiction arises. This contradiction establishes the theorem for $k = 4$.

In what follows we will discuss the case for $k \geq 5$.

Case II. $k \geq 5$.

18. $(z_0 = p, 1) \in A(T)$.
If $(1, p) \in A(T)$ then $C_4 = (1, p = z_0, z_1, z_2 = 0, 1) \subseteq T$ is a 3-colored cycle, a contradiction.

We analyze this case depending on the length of α: if it is equal to $k-2$, less than $k-2$ or greater than $k-2$:

Case II.A. $p > k-2$.
There exists $\mathcal{T}_k = (p, 1) \cup (1, \alpha, k-2) \cup (k-2, z_1) \cup (z_0, z_1) \subseteq T$ ($(k-2, z_1) \in A(T)$ by point (3) with $i = 0$) which is quasimonochromatic by hypothesis and it has al least two black arcs ($k \geq 5$), then $(k-2, z_1) \in A(T)$ is black, a contradiction with Lemma 1-f.

Case II.B. $p = k-2$.

19. $(z_1, p-1 = k-3) \in A(T)$ is red.
$(z_1, k-3 = p-1) \in A(T)$ (point 5) so there exists $C_3 = (z_1, p-1, z_0, z_1) \subseteq T$ and it is at most bicolor by hypothesis, it follows that $(z_1, p-1) \in A(T)$ is black or red and we conclude that it is colored with red by Lemma 1-f.

20. $(1, z_1) \in A(T)$ and it is colored blue.
First, if $(z_1, 1) \in A(T)$ then there exists $\mathcal{T}_k = (z_1, 1) \cup (1, \alpha, k-2 = z_0) \cup (z_0, z_2) \cup (z_1, z_2) \subseteq T$, which is quasimonochromatic by hypothesis, so $(z_1,\ 1)$is colored with black ($k \geq 5$ then $\mathcal{T}_k$ has at least two black arcs), a contradiction with Lemma 1-f. So we conclude that $(1, z_1) \in A(T)$. Now, $C_3 = (z_1, z_2 = 0, 1, z_1) \subseteq T$ is quasimonochromatic by hypothesis and $(1, z_1) \in A(T)$ is not black by Lemma 1-f, then $(1, z_1) \in A(T)$ is blue.

Then there exists a 3-colored $C_4 \subseteq T$, namely $(z_1, k-3, k-2 = z_0, 1, z_1)$ ($1 \neq k-3$ as $k \geq 5$), and a contradiction arises. This contradiction establishes the theorem. ■

The same technique allow us to easily prove the following Theorems. For the first one consider the next definition.

Definition 4. A subdigraph H of D is called a $(1, 1, t-2)$-subdivision of a 3-colored C_3 with colors 1, 2 and 3, if it is a cycle of length t having a monochromatic path of length $t-2$ colored 1, one arc colored 2 and one arc colored 3.

Theorem 3. *Let T be an m-colored tournament. If every $C_3 \subseteq T$, every $\mathcal{T}_k \subseteq T$ and every $C_k \subseteq T$ is a non polychromatic subdigraph of T for some $k \geq 4$, and T does not contain a $(1, 1, t-2)$-subdivision of a 3-colored C_3 $(t < k)$ then $C(T)$ is a KP-digraph.*

Sketch of the proof: By using Lemma 1, first notice that $p \geq k-2$, then prove that $(k-3, z_0) \in A(T)$ in order to point out the polychromatic $C_k = (z_0, z_1, z_2) \cup (z_2, \alpha, k-3) \cup (k-3, z_0) \subseteq T$.

Theorem 4. *Let T be an m-colored tournament. If every $C_3 \subseteq T$ and every $\mathcal{T}_4 \subseteq T$ is a non polychromatic subdigraph then $C(T)$ is a KP-digraph.*

Sketch of the proof: By using Lemma 1 prove that $p \geq 3$ (use that $(z_1, 1)$) and the existence of the arcs $(p-1, z_1)$ and $(1, z_0)$ in order to prove that there exists $\mathcal{T}_4 = (p-1, z_1, 1, z_0) \cup (p-1, z_0) \subseteq T$. Finally notice that $(z_1, 1, z_0) \subseteq T$ is a $z_1 z_0$-monochromatic path in T (apply Lemma 1-c,f to see that $color(z_1, 1) = color(1, z_0) \neq black$).

In what follows we prove that the conditions in Theorem 2 are tight. We also prove that the condition of the Theorem 2 and the Shen Minggang condition are not mutually implied. Remarks 5 and 6 are important to be considered because they allow us to assure that for $k = 4$ our condition is not the condition of the Shen Minggang's result and also because they justify the theorem for $k > 4$.

Remark 1. In Theorem 2 if we ask only that every $C_3 \subseteq T$ is at most bicolor then the result does not hold, as shows Figure 3 (left).

Proof. The digraph G_5 given in [26] holds that every $C_3 \subseteq G_5$ is at most bicolor, there exists a non quasimonochromatic $\mathcal{T}_4 \subseteq G_5$, namely $(v_2, v_4, v_1, v_3) \cup (v_2, v_3)$, and G_5 does not have a kernel by monochromatic paths. ■

Remark 2. In Theorem 2 if we ask only that every $\mathcal{T}_k \subseteq T$ is a quasi-monochromatic subdigraph then the result does not hold, as shows Figure 3 (center) for $k = 4$.

Proof. The digraph T holds that every $\mathcal{T}_4 \subseteq T$ is a quasimonochromatic subdigraph (there exists only (4, 1, 2, 3)∪(4, 3)), there exists a 3-colored cycle of length 3, namely (3, 2, 1, 3), and T does not have a kernel by monochromatic paths (3 does not absorb 2, 2 does not absorb 3, 3 does not absorb 1 and 4 does not absorb 3). ■

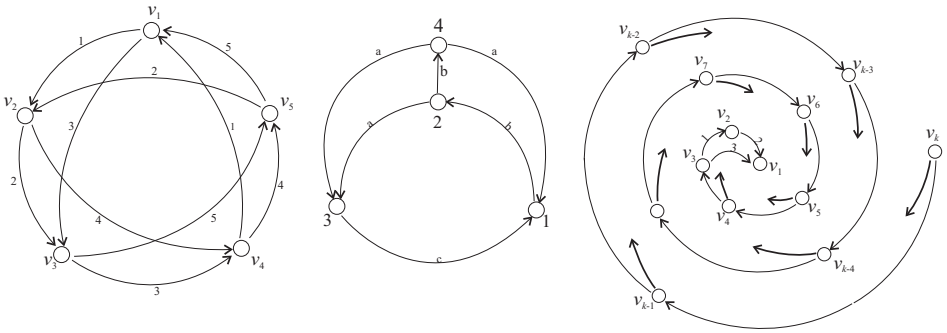

Figure 3. Remarks 1, 2 and 3 (from left to right).

Remark 3. The condition of the Theorem 2 does not imply the Shen Minggang condition.

Proof. Consider a tournament T with $V(T) = \{v_1, v_2, \ldots, v_k\}$ such that for every i, $4 \leq i \leq k$, it holds that $(v_i, v_j) \in A(T)$ for every j such that $j < i$, and it has color 1 (see Figure 3 (right)) and there exists the 3-colored $\mathcal{T}_3 = (v_3, v_2, v_1) \cup (v_3, v_1) \subseteq T$. Notice that every $\mathcal{T}_k \subseteq T$ is a quasimonochromatic subdigraph. ■

Remark 4. The Shen Minggang condition does not imply the condition of the Theorem 2.

Proof. Consider the tournament T in Figure 4 (left) in which every $\mathcal{T}_3 \subseteq T$ and every $C_3 \subseteq T$ is a non polychromatic subdigraph (T is a 2-colored tournament) and $\mathcal{T}_k = (v_1, v_2, \ldots, v_{k-1}) \cup (v_{k-1}, v_k) \subseteq T$ is a non quasimonochromatic one. ■

Remark 5. The condition of every $\mathcal{T}_k \subseteq T$ to be a quasimonochromatic subdigraph does not imply that every $\mathcal{T}_{k-1} \subseteq T$ is also a quasimonochromatic subdigraph.

Proof. Consider the tournament T in Figure 3 (center). The arcs not drown have any direction and they are colored red. Besides $\delta_T^+(v_k) = 0$. Every $\mathcal{T}_k \subseteq T$ is a quasimonochromatic subdigraph of T (if there exists some non quasimonochromatic $\mathcal{T}'_k \subseteq T$ then $\{f_1, f_2\} \subseteq F(\mathcal{T}'_k)$ and $v_k \in V(\mathcal{T}'_k)$ so there exists a v_0v_{k-3}-path P in T such that $v_k \in V(P)$, contradicting $\delta_T^+(v_k) = 0$) but $\mathcal{T}_{k-1} = (v_1, v_2, \ldots, v_{k-2}) \cup (v_{k-2}, v_{k-1}) \subseteq T$ is not a quasimonochromatic one. ■

Remark 6. The Property P_k does not imply the Property P_{k-1}.

Proof. Consider the 3-colored tournament T in Figure 3 (right). The arcs not drown have any direction and they are colored red. Besides $\delta_T^+(v_{k-2}) = 0$ and $\delta_T^-(v_k) = 0$. T holds the Property P_k: there is no polychromatic $C_t \subseteq T_k$ with $t < k$ (if there is some polychromatic $C'_t \subseteq T_k$, $t < k$, then $\{f_1 = (v_{k-1}, v_{k-2}), f_2 = (v_{k-3}, v_{k-2})\} \subseteq A(C'_t)$ and then $\delta_{C_t}^-(v_{k-2}) = 2$, a contradiction) and every $\mathcal{T}_k \subseteq T$ is a non quasimonochromatic one (if not then there is some non quasimonochromatic $\mathcal{T}'_k \subseteq T$) and $\{f_1, f_2)\} \subseteq A(\mathcal{T}'_k)$; as f_1 and f_2 are adjacent arcs there are only two cases, if $f_2 \subseteq \mathcal{T}'_k$ is the last arc in the path of length $k-1$ in $\mathcal{T}'_k$ then it follows from the definition of $\mathcal{T}'_k$ that there is a $v_{k-1}v_{k-3}$-path R in $\mathcal{T}'_k$ such that $v_k \in V(R)$, in other case, if $f_1 \subseteq \mathcal{T}'_k$ is the last arc in the path of length $k-1$ in $\mathcal{T}'_k$ then it follows that there is a $v_{k-3}v_{k-1}$-path Q in $\mathcal{T}'_k$ such that $v_k \in V(Q)$, in both cases a contradiction arises (as $\delta_T^-(v_k) = 0$). But T does not hold Property P_{k-1} as there exists the polychromatic $\mathcal{T}_{k-1} = (v_0 = v_k, v_1, \ldots, v_{k-2}) \cup (v_{k-1}, v_{k-2}) \subseteq T$. ■

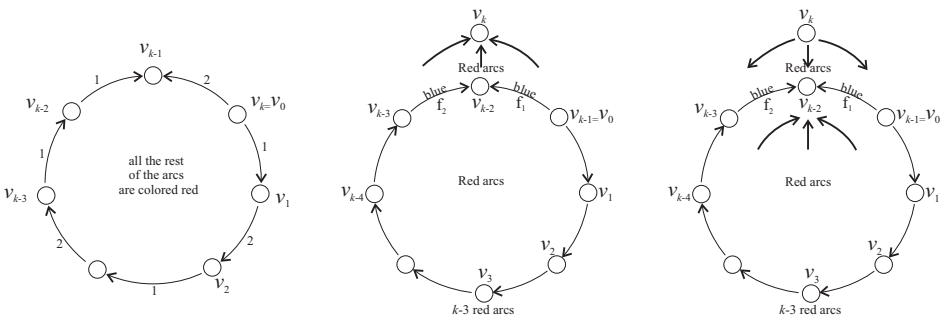

Figure 4. Remarks 4, 5 and 6 (from left to right).

4.2. S_k subdigraphs

Definition 5. A subdigraph H of D is called an S_k subdigraph if H consists of a directed path of length $k-2$, $(z_0, z_1, \ldots, z_{k-2})$, and a path (z_0, z, z_{k-2}), with $z \neq z_i$ for every i such that $0 \leq i \leq k-2$.

Definition 6. Let T be an m-colored tournament. T has the property P if:

(a) Every $S_4 \subseteq T$ is a quasimonochromatic subdigraph of T, and

(b) Every $C_t \subseteq T$ $(t \leq 4)$ is at most bicolor.

Again notice that the property P is the corresponding property of the sufficient condition for cycles and proved by Galeana Sánchez in [12] and mentioned in the Introduction.

Theorem 5. *Let T be an m-colored tournament. If T satisfies the property P, then $C(T)$ is a KP-digraph.*

Proof. We proceed by contradiction. Suppose that $C(T)$ is not a *KP*-digraph, then by Lemma 1 there exists a cycle $\gamma = (z_0, z_1, z_2 = 0, 1, 2, \ldots, p = z_0)$ satisfying properties (a) to (f). The following assertions will allow us to obtain a contradiction. First some general assertions:

1. $p \geq 3$.
From Lemma 1-c we have that $p \geq 2$. If we suppose that $p = 2$ then γ is a 3-colored $C_4 \subseteq T$, a contradiction.

2. $(p-1, z_2) \in A(T)$.
If $(z_2, p-1) \in A(T)$ then $C_4 = (p-1, z_0, z_1, z_2, p-1) \subseteq T$ is a 3-colored cycle, a contradiction.

3. $(z_0, 1) \in A(T)$.
If $(1, z_0) \in A(T)$ then $C_4 = (1, z_0, z_1, z_2, 1) \subseteq T$ is a 3-colored cycle, a contradiction.

4. If $(1, z_1) \in A(T)$ then it is colored blue.
It follows from Lemma 1-f as $C_3 = (1, z_1, z_2, 1) \subseteq T$ is a cycle colored with at most 2 colors.

5. If $(z_1, 1) \in A(T)$ then it is colored red and so $(z_0, z_2) \in A(T)$ is red.
Consequence of Lemma 1-f and the fact that $S_4 = (z_0, z_1, 1) \cup (z_0, z_2, 1) \subseteq T$ is a quasimonochromatic subdigraph of T.

6. If $(z_1, p-1) \in A(T)$ then it is red.

It follows from Lemma 1-f, as $C_3 = (z_1, p-1, z_0, z_1) \subseteq T$ is a cycle colored with at most 2 colors.

7. If $(p-1, z_1) \in A(T)$ then it is colored blue and so $(z_0, z_2) \in A(T)$ is blue: 6.

Consequence of Lemma 1-f and the fact that $S_4 = (p-1, z_1, z_2) \cup (p-1, z_0, z_2) \subseteq T$ is a quasimonochromatic subdigraph of T.

Now we continue the proof by considering several possible cases depending on the direction of some arcs:

Case I. $(p-1, z_1) \in A(T)$ and $(z_1, 1) \in$A(T).
There exists a 3-colored $S_4 = (p-1, z_0, z_2) \cup (p-1, z_1, z_2) \subseteq T$ (as a consequence of points 5 and 7), a contradiction arises.

Case II. $(p-1, z_1) \in A(T)$ and $(1, z_1) \in A(T)$.

Subcase II.A. $p \geq 4$.

8. $(1, p-1) \in A(T)$ and it is blue.
Suppose that $(p-1, 1) \in A(T)$. Then $S_4 = (p-1, 1, z_1) \cup (p-1, z_0, z_1) \subseteq T$ is a 3-colored subdigraph of T (recall 4), a contradiction. So $(1, p-1) \in A(T)$ and then there exists $C_4 = (1, p-1, z_1, z_2, 1) \subseteq T$ which is an at most bicolor cycle by hypothesis, it follows from (7) that $(1, p-1) \in A(T)$ is blue (it is not colored black because of the minimality in the choice of α).

9. $(p-2, z_1) \in A(T)$ and $color(p-2, z_1) = color(p-1, z_2)$=blue.
If $(z_1, p-2) \in A(T)$ then there exists $C_3 = (z_1, p-2, p-1, z_1) \subseteq T$ which is not a polychromatic cycle by hypothesis so $(z_1, p-2) \in A(T)$ is blue (point (6) and Lemma 1-f) and then there exists a 3-colored $C_4 = (z_1, p-2, p-1, z_0, z_1) \subseteq T$, a contradiction. We conclude that $(p-2, z_1) \in A(T)$. As a consequence there exists $S_4 = (p-2, z_1, z_2) \cup (p-2, p-1, z_2) \subseteq T$ (recall 2) and it is a quasimonochromatic subdigraph by hypothesis so the affirmation holds (Lemma 1-f).

10. $(z_0, 1) \in A(T)$ is black (recall 3).
$S_4 = (p-1, z_2, 1) \cup (p-1, z_0, 1) \subseteq T$ is a quasimonochromatic subdigraph and the affirmation holds from the previous point.

11. $(p-2, z_0) \in A(T)$.
By the contrary, if $(z_0, p-2) \in A(T)$ then exists $S_4 = (z_0, p-2, p-1) \cup (z_0, 1, p-1) \subseteq T$ which is a quasimonochromatic subdigraph so $(z_0, p-2) \in A(T)$ is colored black (points 8 and, 10) and $S_4 = (z_0, p-2, z_1) \cup (z_0, 1z_1) \subseteq T$ is not a quasimonochromatic subdigraph (points 4, 9 and 10), a contradiction.

We conclude that $S_4 = (p-2, p-1, z_1) \cup (p-2, z_0, z_1) \subseteq T$ is a 3-colored subdigraph (7, 11), a contradiction that establishes the subcase.

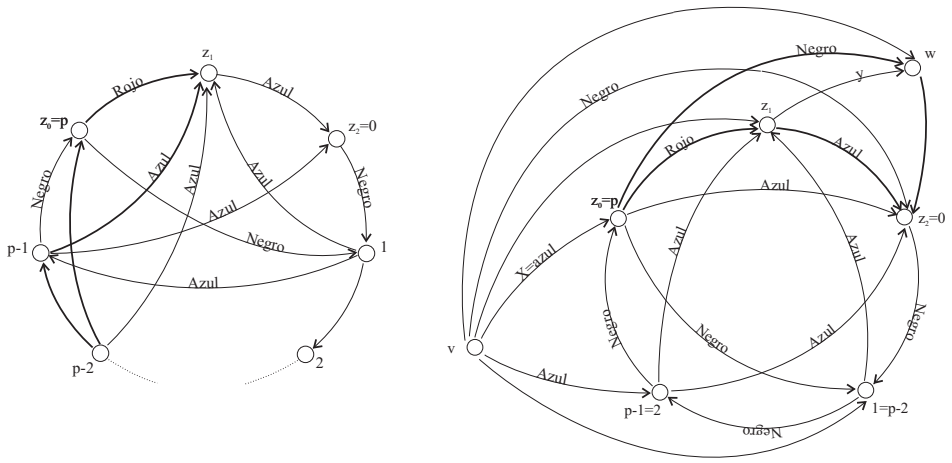

Figure 5. Left: Subcase II.A. Rigth: Subcase II.B.

Subcase II.B. $p = 3$.

12. $(2, z_2) \in A(T)$ is blue.
As $S_4 = (1, 2, z_2) \cup (1, z_1, z_2) \subseteq T$ is a quasimonochromatic one (from point 2 with $p - 1 = 2$, and point 4).

13. $(z_0, 1) \in A(T)$ is black (recall 3).
Consequence of the previous point and noticing that $S_4 = (p-1 = 2, z_0, 1) \cup (p-1 = 2, z_2, 1) \subseteq T)$ (point 2) is a quasimonochromatic one.

14. $(2, z_0) \notin A(Asym(C(T)))$.
As we have the following $z_0 2$-monochromatic path in T: $(z_0 = 3, 1, 2) \subseteq T$ (point 13).

15. There exists some $v \notin \{z_0 = 3, z_1, z_2 = 0, 1, 2\}$ such that $(v, z_0) \in A(T) \cap A(Asym\ (C(T)))$.

From the proof of Theorem 1 we have that there exists some $v \notin \{z_0, z_1, z_2\}$ such that $(v, z_0) \in A(Asym(C(T))$. Now, from points (3) and (14), $v \notin \{1, 2\}$.

Let us say that $(v, z_0) \in A(T)$ is colored x.

16. $(v, z_1) \in A(T)$.
By the contrary suppose that $(z_1, v) \in A(T)$ then there exists the non polychromatic $C_3 = (z_1, v, z_0, z_1) \subseteq T$ and so $(z_1, v) \in A(T)$ is red or

it is colored x. If $(z_1, v) \in A(T)$ is red then $(z_0 = 3, z_1, v) \subseteq T$ is a z_0v-monochromatic path, contradicting point (15). In the other way, if $(z_1, v) \in A(T)$ has color x then $(z_1, v, z_0 = 3) \subseteq T$ is a z_1z_0-monochromatic path, a contradiction again (Lemma 1-d).

17. $(v, z_2) \in A(T)$.
If $(z_2, v) \in A(T)$ then there exists $C_4 = (v, z_0, z_1, z_2, v) \subseteq T$ which is not a polychromatic cycle by general hypothesis so we have that (z_2, v) is red or blue and $x \in \{red, blue\}$. If $(z_2, v) \in A(T)$ is blue then $(z_0, z_2, v) \subseteq T$ is a z_0v-monochromatic path ($(z_0, z_2) \in A(T)$ is blue from point (6)), a contradiction. We conclude that $(z_2, v) \in A(T)$ is red. Then there is the 3-colored $S_4 = (z_2, v, z_1) \cup (z_2, 1, z_1) \subseteq T$ (by point 4), a contradiction again.

18. $(v, 2) \in A(T)$.
If $(2, v) \in A(T)$ then there exists $S_4 = (2, v, z_2) \cup (2, z_0, z_2) \subseteq T$ and it a is quasimonochromatic one by general hypothesis so $(2, v) \in A(T)$ is black or blue. It can not be black (by the contrary $(z_0, 1, 2, v) \subseteq T$ is a z_0v-monochromatic path (13) contradicting point 15) so it is colored blue and there exists the 3-colored $S_4 = (2, v, z_1) \cup (2, z_0, z_1) \subseteq T$ (point 16), a contradiction.

19. $(v, z_0) \in A(T)$ and $(v, 2) \in A(T)$ are both colored blue (i.e., $x = blue$). $S_4 = (v, z_0, z_1) \cup (v, 2, z_1) \subseteq T$ is a quasimonochromatic subdigraph so $(v, z_0) \in A(T)$ and $(v, 2) \in A(T)$ are both colored red or blue, in the first case we have that $S_4 = (v, z_0, z_2) \cup (v, 2, z_2) \subseteq T$ is not a quasimonochromatic subdigraph (point 12 and 17), a contradiction. So the affirmation holds.

We can notice now that $(v, z_0, z_1) \subseteq T$ is a bicolor path contained in $Asym(C(T))$ and z_0 holds the properties of z_1 in Lemma 1 so all the results in such Lemma hold; in particular there exists a z_1v-monochromatic path in T with length at least 2 and colored $y \notin \{red, blue\}$ (Lemma 1-c). As a consequence there is some $w \in V(T) - \{z_0, z_1, z_2, v, 1, 2\}$ ($w \notin \{z_0, z_1, z_2, v, 1, 2\}$ because T is a tournament and because the direction of the arcs with an end in z_1.) such that $(z_1, w) \in A(T)$ is colored y. Let us prove the following affirmations:

20. $(v, w) \in A(T)$.
If $(w, v) \in A(T)$ then there exists the 3-colored $C_4 = (w, v, z_0, z_1, w) \subseteq T$, a contradiction.

21. $(z_0, w) \in A(T)$.
By the contrary $(w, z_0) \in A(T)$ and there exists $C_4 = (w, z_0, 1, z_1, w) \subseteq T$

which is not polychromatic by the general hypothesis so $y = black$ (recall that y is neither red nor blue) and the color of (w, z_0) is black or blue (from points 4 and 13). If $(w, z_0) \in A(T)$ is black then $(z_1, w, z_0) \subseteq T$ is a z_1z_0-monochromatic path, a contradiction. We now can assume that $(w, z_0) \in A(T)$ is blue. Then there is the 3-colored $C_3 = (z_1, w, z_0, z_1) \subseteq T$, a contradiction.

22. $(w, z_2) \in A(T)$.
Suppose that $(z_2, w) \in A(T)$ then $S_4 = (z_0, z_2, w) \cup (z_0, z_1, w) \subseteq T$ is a 3-colored subdigraph (from point 7 and because $y \notin \{red, blue\}$), a contradiction.

23. $(z_0, w) \in A(T)$ and $(z_1, w) \in A(T)$ are both colored black.
$S_4 = (2, z_0, w) \cup (2, z_1, w) \subseteq T$ is a quasimonochromatic subdigraph and $y \notin \{red, blue\}$ (point 7).

Then there exists the 3-colored $S_4 = (z_0, w, z_2) \cup (z_0, z_1, z_2) \subseteq T$, a contradiction arises and demonstrates the subcase:

Case III. $(z_1, p-1) \in A(T)$ and $(z_1, 1) \in A(T)$.

Subcase III.A. $p \geq 5$.

24. $(1, p-1) \in A(T)$ and it is colored red.
If $(p-1, 1) \in A(T)$ then there is the 3-colored subdigraph $S_4 = (z_1, p-1, 1) \cup (z_1, z_2, 1) \subseteq T$ (point 6), a contradiction. So $(1, p-1) \in A(T)$ and $C_4 = (z_1, 1, p-1, z_0, z_1) \subseteq T$ holds the property of not being polychromatic by general hypothesis then $(1, p-1) \in A(T)$ is red (from point 5 and it is not black because of the minimality of α).

25. $(z_1, 2) \in A(T)$ and $color(z_1, 2) = color(z_0, 1)$=red.
If $(2, z_1) \in A(T)$ then it is blue ($C_4 = (z_1, z_2 = 0, 1, 2, z_1) \subseteq T$ is an at most bicolor cycle by hypothesis) and there is the 3-colored $C_3 = (2, z_1, 1, 2) \subseteq T$, a contradiction (recall 5). Then $(z_1, 2) \in A(T)$ and there exists $S_4 = (z_0, z_1, 2) \cup (z_0, 1, 2) \subseteq T$ ($(z_0, 1) \in A(T)$ as point (3)) which is a quasimonochromatic one and it follows that $(z_1, 2) \in A(T)$ and $(z_0, 1) \in A(T)$ are both red ($(z_1, 2)$ is not black from Lemma 1-f.

26. $(p-1, z_2) \in A(T)$ is black.
$S_4 = (p-1, z_0, 1) \cup (p-1, z_2, 1) \subseteq T$ is a quasimonochromatic one and from the previous point.

27. $(z_1, p-2) \in A(T)$ and it is colored red.
If $(p-2, z_1) \in A(T)$ then there exists $S_4 = (p-2, z_1, z_2) \cup (p-2, p-1, z_2) \subseteq T$ and it is a quasimonochromatic subdigraph by hypothesis. It follows

that $(p-2, z_1) \in A(T)$ is black (26), contradicting Lemma 1-f. So $(z_1, p-2) \in A(T)$ and it is red ($C_4 = (z_1, p-2, p-1, p = z_0, z_1) \subseteq T$ is not a polychromatic cycle and $(z_1, p-2) \in A(T)$ is not black because of Lemma 1-f).

28. $(1, p-2) \in A(T)$ and it is red.
If $(p-2, 1) \in A(T)$ then $S_4 = (z_1, z_2, 1) \cup (z_1, p-2, 1) \subseteq T$ is a 3-colored subdigraph (27), a contradiction. So $(1, p-2) \in A(T)$ and as $C_4 = (z_0, 1, p-2, p-1, p = z_0) \subseteq T$ is not a polychromatic cycle then $(1, p-2) \in A(T)$ is red (it is not colored black because of the minimality of α).

29. $(z_0, p-2) \in A(T)$ and it is red.
If $(p-2, z_0) \in A(T)$ then there exists $S_4 = (z_1, p-2, z_0) \cup (z_1, p-1, z_0) \subseteq T$ and it is quasimonochromatic by hypothesis. So $(p-2, z_0) \in A(T)$ is red (points 6 and 27) and $(z_1, p-2, z_0) \subseteq T$ is a $z_1 z_0$-monochromatic path, a contradiction. So $(z_0, p-2) \in A(T)$ and as $S_4 = (z_0, z_1, p-1) \cup (z_0, p-2, p-1) \subseteq T$ is a quasimonochromatic subdigraph then we have that $(z_0, p-2) \in A(T)$ is red.

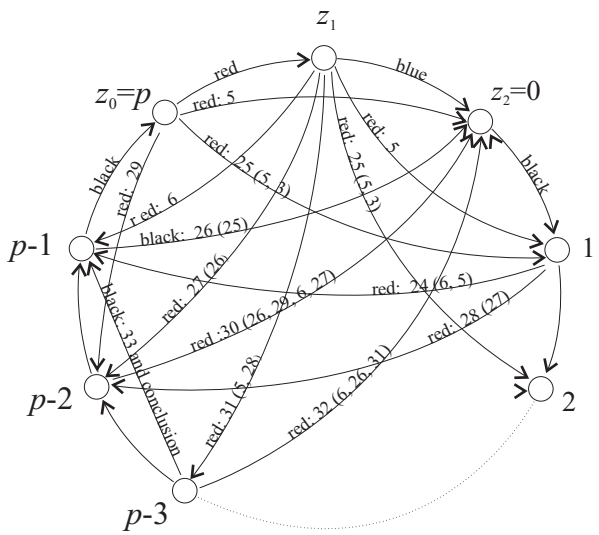

Figure 6. Subcase III.A.

30. $(p-2, z_2) \in A(T)$ and it is red.
If $(z_2, p-2) \in A(T)$ then there is $S_4 = (p-1, z_0, p-2) \cup (p-1, z_2, p-2) \subseteq T$ and it is a quasimonochromatic one by hypothesis, then $(z_2, p-2) \in A(T)$ is colored black (26, 29), contradicting the minimality of α. So $(p-2, z_2) \in$

$A(T)$ and as $S_4 = (z_1, p-2, z_2) \cup (z_1, p-1, z_2) \subseteq T$ is a quasimonochromatic subdigraph then $(p-2, z_2) \in A(T)$ is red (6, 26, 27).

31. $(z_1, p-3) \in A(T)$ and it is red.
If $(p-3, z_1) \in A(T)$ then, as $S_4 = (p-3, p-2, p-1) \cup (p-3, z_1, p-1) \subseteq T$ is a quasimonochromatic subdigraph then $(p-3, z_1) \in A(T)$ is black, contradicting Lemma 1. So $(z_1, p-3) \in A(T)$ and then there exists $S_4 = (z_1, p-3, p-2) \cup (z_1, 1, p-2) \subseteq T$ which is a quasimonochromatic subdigraph and as a consequence $(z_1, p-3) \in A(T)$ is red (5, 28).

32. $(p-3, z_2) \in A(T)$ and it is red.
If $(z_2, p-3) \in A(T)$ then as there exists the quasimonochromatic $S_4 = (z_2, p-3, p-2) \cup (z_2, 1, p-2) \subseteq T$ then $(z_2, p-3) \in A(T)$ is black, contradicting the minimality of α. So $(p-3, z_2) \in A(T)$ and there exists $S_4 = (z_1, p-3, z_2) \cup (z_1, p-1, z_2) \subseteq T$ and it is quasimonochromatic by hypothesis. It follows that $(p-3, z_2) \in A(T)$ is red (6, 26, 31).

33. $(p-3, p-1) \in A(T)$.
If $(p-1, p-3) \in A(T)$ then it is red (there exists the quasimonochromatic $S_4 = (p-1, p-3, z_2) \cup (p-1, z_0, z_2) \subseteq T$ and from points 5 and 32) and $S_4 = (p-1, z_0, p-2) \cup (p-1, p-3, p-2) \subseteq T$ is not a quasimonochromatic subdigraph (29), a contradiction.

Then $(p-3, p-1) \in A(T)$ is black (there exists $S_4 = (p-3, p-1, z_2) \cup (p-3, p-2, z_2) \subseteq T$, it is a quasimonochromatic subdigraph and if follows from 26, 30 and 33), contradicting the choice of α. We conclude the Subcase III.A.

Subcase III.B. $p = 3$.

34. $(z_0, 1) \in A(T)$ is red (recall point 3).
$S_4 = (z_0, z_1, p-1 = 2) \cup (z_0, 1, p-1 = 2) \subseteq T$ is a quasimonochromatic subdigraph, $(z_1, 2)$ can not be black because of Lemma 1-f and finally the affirmation holds from point 6.

35. $(p-1 = 2, z_2) \in A(T)$ is black (recall 2).
This follows from point (34) and because of the quasimonochromaticity of $S_4 = (p-1 = 2, z_2, 1) \cup (p-1, p = z_0, 1) \subseteq T$.

36. $(z_2, 1) \notin A(Asym(C(T)))$.
$(1, 2, z_2) \subseteq T$ is a $1z_2$-monochromatic path.

37. There exists some $v \notin \{z_0 = 3, z_1, z_2 = 0, 1, 2\}$ such that $(z_2, v) \in A(T) \cap A(Asym(C(T)))$.
From the proof of Lemma 1 we have that there exists some $v \notin \{z_0, z_1, z_2\}$ such that $(z_2, v) \in A(Asym(C(T))$ $((z_1, z_2) \in A(Asym(C(T))$ so $v \neq z_1$

and if $v = z_0$ then $\gamma = (z_0, z_1, z_2, v = z_0) \subseteq T$ is an at most bicolor cycle by hypothesis and it follows that $(z_2, v) \in A(T)$ is colored red or blue, and then there is a z_2z_1-red path or there is a z_1z_0-blue path respectively, a contradiction in both cases. Now, from points (2) and (36) $v \notin \{1, 2\}$.

Let us say that $(z_2, v) \in A(T)$ is colored x.

38. $(z_1, v) \in A(T)$.
By the contrary suppose that $(v, z_1) \in A(T)$ then there exists the non polychromatic $C_3 = (z_1, z_2, v, z_1) \subseteq T$ and so $(v, z_1) \in A(T)$ is blue or it is colored x. If $(v, z_1) \in A(T)$ is blue then $(v, z_1, z_2) \subseteq T$ is a vz_2-monochromatic path, contradicting point (37). In the other way if $(v, z_1) \in A(T)$ is colored x then $(z_2, v, z_1) \subseteq T$ is a z_2z_1-monochromatic path, a contradiction again (Lemma 1-d).

39. $(z_0, v) \in A(T)$.
If $(v, z_0) \in A(T)$ then there exists $C_4 = (z_0, z_1, z_2, v, z_0) \subseteq T$ and it is a not polychromatic cycle by general hypothesis then we have that $(v, z_0) \in A(T)$ is red or blue. If $(v, z_0) \in A(T)$ is red then $(v, z_0, z_2) \subseteq T$ is a vz_2-monochromatic path ($(z_0, z_2) \in A(T)$ is red from point 5), a contradiction. We conclude that $(v, z_0) \in A(T)$ is blue. Then there is the 3-colored $S_4 = (z_1, v, z_0) \cup (z_1, 2, z_0) \subseteq T$ (point 16), a contradiction again.

40. $(1, v) \in A(T)$.
If $(v, 1) \in A(T)$ then there exists $S_4 = (z_1, v, 1) \cup (z_1, z_2, 1) \subseteq T$ and it is a quasimonochromatic one by general hypothesis so $(v, 1) \in A(T)$ is black or blue. It can not be black (by the contrary $(v, 1, p - 1 = 2, z_2) \subseteq T$ is a vz_2-monochromatic path contradicting point (37)) so it is colored blue and there exists the 3-colored $S_4 = (z_0, z_2, 1) \cup (z_0, v, 1) \subseteq T$ (point 5), a contradiction.

41. $(2, v) \in A(T)$.
If $(v, 2) \in A(T)$ then there exists the non polychromatic $C_4 = (v, 2, z_0, z_2, v) \subseteq T$ so $(v, 2) \in A(T)$ is black or red and $(z_2, v) \in A(T)$ is also black or red (i.e., $x \in \{red, black\}$). $(v, 2) \in A(T)$ can not be black (by the contrary $(v, p - 1 = 2, z_2) \subseteq T$ is a vz_2-monochromatic path contradicting point (37)) so $(v, 2) \in A(T)$ is red. Even more, $x = color(z_2, v)$ is also red (there exists the quasimonochromatic $S_4 = (z_1, z_2, v) \cup (z_1, 1, v) \subseteq T$ (5)). Now we have that $(z_0, v) \in A(T)$ is black ($S_4 = (2, z_0, v) \cup (2, z_2 = 0, v) \subseteq T$ is a quasimonochromatic subdigraph (35)) and then there exists the non-quasimonochromatic $S_4 = (z_0, v, 2) \cup (z_0, 1, 2) \subseteq T$ ($(z_0, 1,) \in A(T)$, point (34)), a contradiction.

42. $(z_2, v) \in A(T)$ and $(1, v) \in A(T)$ are both colored red (i.e., x =red).
$S_4 = (z_1, z_2, v) \cup (z_1, 1, v) \subseteq T$ and $S_4 = (z_0, z_2, v) \cup (z_0, 1, v) \subseteq T$ are both quasimonochromatic subdigraphs (point 5 and 34).

Notice that $(z_1, z_2, v) \subseteq T$ a is a bicolor path contained in $Asym(C(T))$ and z_2 holds the properties of z_1 in Lemma 1 so all the results in such Lemma holds, in particular there exists a vz_1-monochromatic path in T with length at least 2 and colored $y \notin \{red, blue\}$ (Lemma 1-c). As a consequence there is some $w \in V(T) - \{z_0, z_1, z_2, v, 1, 2\}$ such that $(w, z_1) \in A(T)$ is colored y ($w \notin \{z_0, z_1, z_2, v, 1, 2\}$ because T is a tournament and because the direction of the arcs with one end in z_1, besides, $y \notin \{red, blue\}$). See the following affirmations:

43. $(w, z_2) \in A(T)$.
If $(z_2, w) \in A(T)$ then y=black and $(z_2, w) \in A(T)$ is red or black. ($C_4 = (w, z_1, 2, z_2, w) \subseteq T$ is a non polychromatic subdigraph and from points 6 and 35). But if $(z_2, w) \in A(T)$ is black then $(z_2, w, z_1) \subseteq T$ is a z_2z_1-monochromatic path, contradicting that $(z_1, z_2) \in A(Asym(C(T)))$. So $(z_2, w) \in A(T)$ is red and $C_3 = (z_1, z_2, w, z_1) \subseteq T$ is a 3-colored cycle, a contradiction again.

44. $(z_0, w) \in A(T)$.
If $(w, z_0) \in A(T)$ then it is colored red ($S_4 = (w, z_0, 1) \cup (w, z_1, 1) \subseteq T$ is a quasimonochromatic one and because $y \neq red$). So there exists the 3-colored $S_4 = (w, z_0, z_2) \cup (w, z_1, z_2) \subseteq T$ (from points 5 and 34), a contradiction.

45. $(w, z_1) \in A(T)$ and $(w, z_2) \in A(T)$ are both colored black.
$S_4 = (w, z_1, 1) \cup (w, z_2, 1) \subseteq T$ is a quasimonochromatic subdigraph, $y \notin \{red, blue\}$ and point 5.

We conclude this case noticing that there exists the 3-colored $S_4 = (z_0, w, z_2) \cup (z_0, z_1, z_2) \subseteq T$, a contradiction (recall 43 and 44)

Case III.C. $p = 4$.

46. $(1, 3) \in A(T)$ and it is colored red.
If $(3, 1) \in A(T)$ then $S_4 = (z_1, 3, 1) \cup (z_1, z_2, 1) \subseteq T$ is a 3-colored subdigraph (recall point 6). Then we have that $(1, 3) \in A(T)$ and exists $C_4 = (z_0, z_1, 1, 3, z_0) \subseteq T$ which is a non-polychromatic cycle, so $(1, 3) \in A(T)$ is red (from point 5 and notice that it is not colored black because the choice of α in Lemma 1).

47. $(z_1, 2) \in A(T)$.
If $(2, z_1) \in A(T)$ then there exists the non-polychromatic cycle $C_4 = (z_1, z_2, 1, 2, z_1) \subseteq T$ and it follows from Lemma 1-f that $(2, z_1) \in A(T)$ is blue. Then

$C_3 = (2, z_1, 1, 2) \subseteq T$ is a 3-colored cycle, a contradiction (recall point 5).

48. $(z_0, 1) \in A(T)$ and $(z_1, 2) \in A(T)$ are both red (recall points 2 and 3). If follows from the existence of the quasimonochromatic $S_4 = (z_0, z_1, 2) \cup (z_0, 1, 2) \subseteq T$ and from Lemma 1-f.

49. $(3, z_2) \in A(T)$ is black.
$S_4 = (p-1 = 3, z_0, 1) \cup (p-1 = 3, z_2, 1) \subseteq T$ is a quasimonochromatic subdigraph and from the previous point.

50. $(2, z_2) \in A(T)$ and it is red.
If $(z_2, 2) \in A(T)$ then there exists the 3-colored $S_4 = (z_1, z_2, 2) \cup (z_1, 1, 2) \subseteq T$ (point 5), a contradiction. Then $(2, z_2) \in A(T)$ and it is colored red because $S_4 = (z_1, 2, z_2) \cup (z_1, p-1 = 3, z_2) \subseteq T$ is a quasimonochromatic subdigraph (points 6, 48 and 49).

Then there exists the non-quasimonochromatic $S_4 = (1, 2, z_2) \cup (1, p-1 = 3, z_2) \subseteq T$ (points 46, 49 and 50). This contradiction establishes the last subcase.

Case IV. $(z_1, p-1) \in A(T)$ and $(1, z_1) \in A(T)$.
There is the 3-colored cycle $C_4 = (z_1, p-1, z_2, 1, z_1) \subseteq T$ (points 2, 4 and 6), a contradiction. And the Theorem is proved. ■

The same technique allow us to prove the following Theorems. In the first one notice a less restrictive coloration than in the previous theorem, but the subdigraphs S_5 and (1, 1, 2)-subdivision of a 3-colored C_3 are also involved.

Definition 7. Let T be an m-colored tournament. T has the property Q if:

(a) Every $S_4 \subseteq T$ and every $S_5 \subseteq T$ are non-polychromatic subdigraphs,

(b) Every $C_3 \subseteq T$ is a non-polychromatic cycle

(c) There is no $(1, 1, 2)$-subdivision of a 3-colored C_3.

Theorem 6. *Let T be an m-colored tournament. If T satisfies the property Q, then $C(T)$ is a KP-digraph.*

Sketch of the proof: Consider the cycle of Lemma 1 and prove that $p \geq 3$ as well as the following assertions: If $(1, z_1) \in A(T)$ then it is blue, if $(p-1, z_1) \in A(T)$ then it is colored blue and if $(z_1, 1) \in A(T)$ then it is colored red, if $(z_1, p-1) \in A(T)$ then it is red. Finally prove that the following cases lead us a contradiction: If $(1, z_1) \in A(T)$ then there exists the 3-colored $S_5 = (p-1, p, z_1) \cup (p-1, z_2, 1, z_1) \subseteq T$; if $(z_1, 1) \in A(T)$ and

$(z_1, p-1) \in A(T)$ then $(z_1, 1, z_0) \subseteq T$ is a $z_1 z_0$-red path and if $(z_1, 1) \in A(T)$ and $(p-1, z_1) \in A(T)$ then there is the 3-colored $S_5 = (p-1, z_1, 1) \cup (p-1, p = z_0, z_2 = 0, 1) \subseteq T$.

Definition 8. Let T be a m-colored tournament. T has the property R_k for some $k \geq 5$ if:

(a) Every $S_k \subseteq T$ is a non-polychromatic subdigraph of T,

(b) Every $C_t \subseteq T$ $(t \leq k)$ is a non-polychromatic cycle.

Theorem 7. *Let T be a m-colored tournament. If T satisfies the property R_k for some integer $k \geq 5$, then $C(T)$ is a KP-digraph.*

Sketch of the proof: Use Lemma 1. Prove first that $p > k-2$ and also that for every i such that $1 \leq i \leq k-3$ and for every j such that $p-1 \geq j \geq p-(k-3)$ it holds that $(z_0, i) \in A(T)$ and $(j, z_2) \in A(T)$, as well as the existence of the arc $(p-1, 1) \in A(T)$. Consider two general cases, if $p \geq 2(k-3)$ and if $k-2 < p < 2(k-3)$. Both cases lead us to a contradiction by analyzing the two possible directions of the arc between $k-3$ and z_1 (in the first case if $(k-3, z_1) \in A(T)$ the reader can prove that $(k-3, z_1) \in A(T)$ is colored blue, that $(z_1, k-4) \in A(T)$ and it is colored blue, also if $p > 2(k-3)$ then $(k-3, p-(k-3)) \in A(T)$, and that $(z_0, p-(k-3)) \in A(T)$ in order to prove that there exists the 3-colored $S_k = (z_0, p-(k-3), z_2) \cup (z_2, \alpha, k-4) \cup (z_0, z_1, k-4) \subseteq T$, on the other hand, if $(z_1, k-3) \in A(T)$ it can be proved that $(z_1, k-3) \in A(T)$ is red as well as the arc $(z_1, p-(k-3)) \in A(T)$ and those arcs will help the reader to find the 3-colored $S_k = (z_1, p-(k-3) \cup (p-(k-3), \alpha, p-1) \cup (p-1, 1) \cup (z_1, z_2, 1)) \subseteq T$; in the second case if $(z_1, k-3) \in A(T)$ notice that $(z_1, k-3) \in A(T)$ is red as well as the arcs $(z_1, p-(k-3)) \in A(T)$ and $(z_1, p-(k-4)) \in A(T)$ to prove that $S_k = (z_1, p-(k-4)) \cup (p-(k-4), \alpha, z_0) \cup (z_0, 1) \cup (z_1, z_2, 1) \subseteq T$ is a 3-colored subdigraph, in the other hand, if $(k-3, z_1) \in A(T)$ then notice that $S_k = (p-1, z_2) \cup (z_2, \alpha, k-4) \cup (k-4, z_1) \cup (p-1, z_0, z_1) \subseteq T$ ($(p-1, z_2) \in A(T)$ is a 3-colored subdigraph using that $(k-3, z_1) \in A(T)$ is colored blue as well as the arc $(k-4, z_1) \in A(T)$).

The following two remarks guarantee that the conditions of Theorems 5 and 6 which involve the S_4 colorations, do not implicate each other.

Remark 7. Property P does not imply Property Q.

Proof. Consider the tournament T in Figure 7 (left). The arcs which were not drown have any direction. Every $S_4 \subseteq T$ is a quasimonochromatic subdigraph: if there exists a non-quasimonochromatic $S_4' \subseteq T$ then

$\{(v_5, v_4), (v_5, v_3)\} \subseteq A(S'_4)$, a contradiction ($v_1$ is the only vertex adjacent from v_4 and v_1 is not adjacent from v_3). Besides every C_4 is a non-polychromatic cycle: if there exists a polychromatic cycle $C'_4 \subseteq T$ then $\{(v_5, v_4), (v_5, v_3)\} \subseteq A(C'_4)$, contradicting that for e7ery vertex $v \in C'_4$ we have that $\delta^+_{C_4}(v) = 2$. But $S_5 = (v_5, v_3, v_2, v_1) \cup (v_5, v_4, v_1) \subseteq T$ is a polychromatic subdigraph. ■

Remark 8. Property Q does not imply Property P.

Proof. See the tournament T in Figure 7 (right) in which every $S_4 \subseteq T$ and every $S_4 \subseteq T$ is not a polychromatic subdigraph (T is a 2-colored tournament) and there exists a non-quasi-monochromatic $S_4 \subseteq T_5$, namely $S_4 = (v_3, v_2, v_1) \cup (v_3, v_4, v_1) \subseteq T_5$ ■

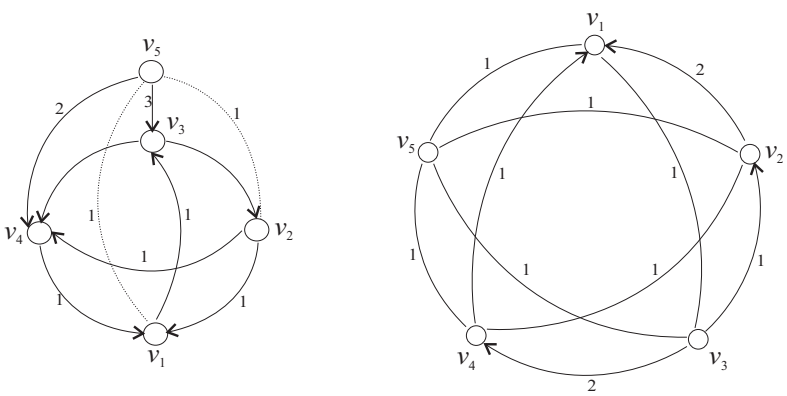

Figure 7. Remarks 7 and 8 (from left to right).

5. Open Problems

We still ask ourselves if an m-colored tournament has a kernel by monochromatic paths whenever one of the following statements holds. Unfortunately the technique used in the previous theorems do not allow us to answer:

(1) There is no 3-colored C_3 and every $\mathcal{T}_4$ is an at most 3-colored subdigraph.

(2) Every $C_t \subseteq T$ and every $S_t \subseteq T$ ($t \leq 4$) are subdigraphs colored with at most 2 colors.

(3) The tournament does not contain some 3-colored C_3 and every $S_4 \subseteq T$ is a quasimonochromatic subdigraph.

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