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ROMAN BONDAGE IN GRAPHS

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Abstract

A Roman dominating function on a graph G is a function $f: V(G) \to \{0,1,2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The Roman domination number, $\gamma_R(G)$, of G is the minimum weight of a Roman dominating function on G. In this paper, we define the Roman bondage $b_R(G)$ of a graph G with maximum degree at least two to be the minimum cardinality of all sets $E' \subseteq E(G)$ for which $\gamma_R(G - E') > \gamma_R(G)$. We determine the Roman bondage number in several classes of graphs and give some sharp bounds.

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1. Terminology and Introduction

Let $G = (V(G), E(G))$ be a simple graph of order n. We denote the *open neighborhood* of a vertex v of G by $N_G(v)$, or just $N(v)$, and its *closed neighborhood* by $N_G[v] = N[v]$. For a vertex set $S \subseteq V(G)$, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = \bigcup_{v \in S} N[v]$. The *degree* $deg(x)$ of a vertex x denotes the number of neighbors of x in G, and $\Delta(G)$ is the *maximum degree* of G. Also the *eccentricity*, $ecc(x)$, of a vertex x is maximum distance of the vertices of G from x. A set of vertices S in G is a *dominating set*, if $N[S] = V(G)$. The *domination number*, $\gamma(G)$, of G is the minimum cardinality of a dominating set of G. If S is a subset of $V(G)$, then we denote by $G[S]$ the subgraph of G induced by S. For notation and graph theory terminology in general we follow [6].

With K_n we denote the *complete graph* on n vertices and with C_n the *cycle* of length n. For two positive integers m, n, the *complete bipartite graph* $K_{m,n}$ is the graph with partition $V(G) = V_1 \cup V_2$ such that $|V_1| = m$, $|V_2| = n$ and such that $G[V_i]$ has no edge for $i = 1, 2$, and every two vertices belonging to different partition sets are adjacent to each other.

For a graph G, let $f : V(G) \to \{0, 1, 2\}$ be a function, and let $(V_0; V_1; V_2)$ be the ordered partition of $V(G)$ induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$ and for $i = 0, 1, 2$. There is a $1 - 1$ correspondence between the functions $f : V(G) \to \{0, 1, 2\}$ and the ordered partition $(V_0; V_1; V_2)$ of $V(G)$. So we will write $f = (V_0; V_1; V_2)$.

A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* (or just RDF) if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of a Roman dominating function on G. A function $f = (V_0; V_1; V_2)$ is called a γ_R-function (or $\gamma_R(G)$-function when we want to refer f to G), if it is a Roman dominating function and $f(V(G)) = \gamma_R(G)$, [2, 7, 8].

The *bondage number* $b(G)$ of a nonempty graph G is the minimum cardinality among all sets of edges $E' \subseteq E(G)$ for which $\gamma(G - E') > \gamma(G)$.

This concept was introduced by Bauer, Harary, Nieminen and Suffel in [1], and has been further studied for example in [4, 5, 9]). For more information on this topic we refer the reader to the survey article by Dunbar, Haynes, Teschner and Volkmann [3].

In this paper we study bondage by considering a variation based on Roman domination. The *Roman bondage number* $b_R(G)$ of a graph G is the cardinality of a smallest set of edges $E' \subseteq E(G)$ for which $\gamma_R(G - E') > \gamma_R(G)$.

We note that if G is a connected graph on two vertices, then $G \simeq K_2$ and $\gamma_R(G) = 2$. If $e \in E(G)$, then $G - e \simeq \overline{K_2}$ and thus $\gamma_R(G - e) = \gamma_R(G)$. Therefore the Roman bondage number is only defined for a graph G with maximum degree at least two.

We recall that a leaf in a graph G is a vertex of degree one, and a support vertex is the vertex which is adjacent to a leaf.

2. Upper Bounds

Theorem 1. *If G is a graph, and xyz a path of length 2 in G, then*

$$b_R(G) \leq deg(x) + deg(y) + deg(z) - 3 - |N(x) \cap N(y)|. \tag{1}$$

If x and z are adjacent, then

$$b_R(G) \leq deg(x) + deg(y) + deg(z) - 4 - |N(x) \cap N(y)|. \tag{2}$$

Proof. Let H be the graph obtained from G by removing the edges incident with x, y or z with exception of yz and all edges between y and $N(x) \cap N(y)$. In H, the vertex x is isolated, z is a leaf and y is adjacent to z and all neighbors of y in H, if any, lie in $N_G(x)$.

Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(H)$-function. Then $x \in V_1$ and, without loss of generality, $z \in V_0 \cup V_1$.

If $z \in V_0$, then $y \in V_2$ and therefore $(V_0 \cup \{x\}, V_1 - \{x\}, V_2)$ is a RDF on G of weight less than f, and (1) as well as (2) are proved.

Now assume that $z \in V_1$. If $y \in V_1$, then $(V_0 \cup \{z\}, V_1 - \{y, z\}, V_2 \cup \{y\})$ is also $\gamma_R(H)$-function, and we are in the situation discussed in the previous case. However, if $y \in V_0$, then there exists a vertex $w \in N_G(x) \cap N_G(y)$ such that $w \in V_2$. Since w is a neighbor of x in G, $(V_0 \cup \{x\}, V_1 - \{x\}, V_2)$ is a RDF on G of weight less than f, and again (1) and (2) are proved. ■

Applying Theorem 1 on a path xyz such that one of the vertices x, y or z has minimum degree, we obtain the next result immediately.

Corollary 2. *If G is a connected graph of order $n \geq 3$, then*

$$b_R(G) \leq \delta(G) + 2\Delta(G) - 3.$$

Our next upper bound involves the *edge-connectivity* $\lambda(G)$, which is the fewest number of edges whose removal from a connected graph G creates two components. Since $\lambda(G) \leq \delta(G)$, the next theorem is an extension of Corollary 2.

Observation 3. *If E is an edge cut set in a graph G smaller than $b_R(G)$, then $\gamma_R(G)$ equals the sum of all $\gamma_R(G_i)$ where G_i emerge by removing E.*

Theorem 4. *If G is a connected graph of order $n \geq 3$, then*

$$b_R(G) \leq \lambda(G) + 2\Delta(G) - 3.$$

Proof. Let $\lambda = \lambda(G)$, and let $E = \{e_1, e_2, \ldots, e_\lambda\}$ be a set of edges whose removal disconnects G. Say $e_1 = ab$, and let H_a and H_b denote the components of $G - E$ containing a and b, respectively. By Corollary 2 we may assume that H_a and H_b are non-trivial. Let $a_1 \in V(H_a)$ adjacent to a and $b_1 \in V(H_b)$ adjacent to b, and let F_{a,a_1} and F_{b,b_1} denote the edges of G incident with a or a_1 with exception of aa_1 and b or b_1 with exception of bb_1, respectively. Suppose on the contrary that $b_R(G) > \lambda(G) + 2\Delta(G) - 3$. Noting that $|E| = \lambda < b_R(G)$, we observe that $\gamma_R(G) = \gamma_R(H_a) + \gamma_R(H_b)$. Since

$$|F_{a,a_1} \cup E| \leq deg_G(a) + deg_G(a_1) + \lambda - 3 \leq 2\Delta(G) + \lambda - 3 < b_R(G),$$

we deduce that $\gamma_R(G) = \gamma(H_a - \{a, a_1\}) + 2 + \gamma_R(H_b)$. Similarly, since

$$|F_{b,b_1} \cup E| \leq deg_G(b) + deg_G(b_1) + \lambda - 3 \leq 2\Delta(G) + \lambda - 3 < b_R(G),$$

we deduce that $\gamma_R(G) = \gamma_R(H_b - \{b, b_1\}) + 2 + \gamma_R(H_a)$. Altogether we obtain

$$\begin{aligned} 2\gamma_R(G) &= \gamma_R(H_a - \{a, a_1\}) + 2 + \gamma_R(H_b) + \gamma_R(H_b - \{b, b_1\}) + 2 + \gamma_R(H_a) \\ &= \gamma_R(H_a - \{a, a_1\}) + 4 + \gamma_R(H_b - \{b, b_1\}) + \gamma_R(G) \end{aligned}$$

and thus $\gamma_R(G) = \gamma_R(H_a - \{a, a_1\}) + 4 + \gamma_R(H_b - \{b, b_1\})$. This is obviously a contradiction, since

$$\begin{aligned}\gamma_R(G) &\leq \gamma_R(H_a - \{a, a_1\}) + \gamma_R(a_1abb_1) + \gamma_R(H_b - \{b, b_1\}) \\ &\leq \gamma_R(H_a - \{a, a_1\}) + 3 + \gamma_R(H_b - \{b, b_1\}).\end{aligned}$$

■

Observation 5. *If a graph G has a vertex v such that $\gamma_R(G - v) \geq \gamma_R(G)$, then $b_R(G) \leq \Delta(G)$.*

Proof. Let E be the edge set incident with v. It follows that $\gamma_R(G - E) > \gamma_R(G)$, and the result is proved. ■

3. Exact Values of $b_R(G)$

In this section we determine the Roman bondage number for several classes of graphs.

Theorem 6. *If G is a graph of order $n \geq 3$ with exactly $k \geq 1$ vertices of degree $n - 1$, then $b_R(G) = \lceil \frac{k}{2} \rceil$.*

Proof. Since $k \geq 1$, we note that $\gamma_R(G) = 2$. First let $E_1 \subseteq E(G)$ be an arbitrary subset of edges such that $|E_1| < \lceil \frac{k}{2} \rceil$, and let $G' = G - E_1$. It is evident that there is a vertex v in G' such that $deg_G(v) = deg_{G'}(v) = n - 1$, and so $\gamma_R(G') = \gamma_R(G) = 2$. This shows that $b_R(G) \geq \lceil \frac{k}{2} \rceil$.

If $v_1, v_2, \ldots, v_k \in V(G)$ are the vertices of degree $n - 1$, then the subgraph F induced by the vertices $v_1, v_2, \ldots, v_k$ is isomorphic to the complete graph K_k.

If k is even, then let H_1 be the graph obtained from G by removing $\frac{k}{2}$ independent edges from F. Then $\Delta(H_1) = n - 2$ and thus $\gamma_R(H_1) = 3$. This implies $b_R(G) \leq \lceil \frac{k}{2} \rceil$.

If k is odd, then let H_2 be the graph obtained from G by removing $\frac{k-1}{2}$ independent edges from F. Then there exists exactly one vertex, say $v_k \in V(H_2)$ such that $deg_{H_2}(v_k) = n - 1$. Let H_3 be the graph obtained from H_2 by removing an arbitrary edge incident with v_k. Then $\Delta(H_3) = n - 2$ and so $\gamma_R(H_3) = 3$. This implies $b_R(G) \leq \lceil \frac{k}{2} \rceil$.

Combining the obtained inequalities, we deduce that $b_R(G) = \lceil \frac{k}{2} \rceil$, and the proof is complete. ■

Corollary 7. *If $n \geq 3$, then $b_R(K_n) = \lceil \frac{n}{2} \rceil$.*

Lemma 8 [2]**.** *For the classes of paths P_n and cycles C_n,*

$$\gamma_R(P_n) = \gamma_R(C_n) = \left\lceil \frac{2n}{3} \right\rceil .$$

Theorem 9. *For $n \geq 3$,*

$$b_R(P_n) = \begin{cases} 2 & \text{if } n \equiv 2 \pmod 3, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Let $P_n = v_1 v_2 \ldots v_n$. Corollary 2 yields to $b_R(P_n) \leq 2$. First assume that $n = 3k$. Lemma 8 implies that $\gamma_R(P_n) = 2k$ and $\gamma_R(P_n - v_1v_2) = 1 + \gamma_R(P_{n-1}) = 1 + 2k$ and thus $b_R(P_n) = 1$. Next assume that $n = 3k + 1$. According to Lemma 8, we have $\gamma_R(P_n) = 2k + 1$ and $\gamma_R(P_n - v_2v_3) = 2 + \gamma_R(P_{n-2}) = 2 + 2k$ and so $b_R(P_n) = 1$. It remains to assume that $n = 3k + 2$. By Lemma 8, $\gamma_R(P_n) = 2k + 2$. If e is an arbitrary edge of P_n, then $P_n - e$ consists of two paths P_1 and P_2 of order n_1 and n_2, respectively, such that $n_1 + n_2 = n$ and $\gamma_R(P_n - e) = \gamma_R(P_1) + \gamma_R(P_2)$. Now there are integers k_1 and k_2 such that $n_1 = 3k_1, n_2 = 3k_2 + 2$ or $n_1 = 3k_1 + 1, n_2 = 3k_2 + 1$ or $n_1 = 3k_1 + 2, n_2 = 3k_2$ and $k_1 + k_2 = k$. In the first case we deduce from Lemma 8 that

$$\begin{aligned} \gamma_R(P_n - e) &= \gamma_R(P_1) + \gamma_R(P_2) \\ &= \left\lceil \frac{6k_1}{3} \right\rceil + \left\lceil \frac{6k_2 + 4}{3} \right\rceil \\ &= 2k_1 + 2k_2 + 2 = 2k + 2 = \gamma_R(P_n). \end{aligned}$$

This implies that $b_R(P_n) \geq 2$ in the first case, and because of $b_R(P_n) \leq 2$ we obtain $b_R(P_n) = 2$. The remaining two cases are similar and are therefore omitted. ■

Theorem 10. *For $n \geq 3$,*

$$b_R(C_n) = \begin{cases} 3 & \text{if } n \equiv 2 \pmod 3, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. Let $C_n = v_1 v_2 \ldots v_n v_1$. Corollary 2 leads to $b_R(C_n) \leq 3$. If e is an arbitrary edge of C_n, then $C_n - e = P_n$. Hence Lemma 8 shows that $b_R(C_n) \geq 2$. We distinguish three cases.

Assume that $n = 3k$. Lemma 8 implies that $\gamma_R(C_n) = 2k$ and $\gamma_R(C_n - \{v_1v_2, v_2v_3\}) = 1 + \gamma_R(P_{3k-1}) = 1 + 2k$ and thus $b_R(C_n) = 2$.

Assume that $n = 3k + 1$. Lemma 8 implies that $\gamma_R(C_n) = 2k + 1$ and $\gamma_R(C_n - \{v_1v_2, v_3v_4\}) = 2 + \gamma_R(P_{3k-1}) = 2 + 2k$ and thus $b_R(C_n) = 2$.

Assume that $n = 3k+2$. By Lemma 8, $\gamma_R(C_n) = 2k+2$. If e_1 and e_2 are two arbitrary edges of C_n, then $C_n - \{e_1, e_2\}$ consists of two paths P_1 and P_2 of order n_1 and n_2 such that $n_1 + n_2 = n$ and $\gamma_R(C_n - \{e_1, e_2\}) = \gamma_R(P_1) + \gamma_R(P_2)$. Now there are integers k_1 and k_2 such that $n_1 = 3k_1, n_2 = 3k_2 + 2$ or $n_1 = 3k_1 + 1, n_2 = 3k_2 + 1$ or $n_1 = 3k_1 + 2, n_2 = 3k_2$ and $k_1 + k_2 = k$. In the second case we deduce from Lemma 8 that

$$\begin{aligned}\gamma_R(C_n - \{e_1, e_2\}) &= \gamma_R(P_1) + \gamma_R(P_2)\\ &= \left\lceil \frac{6k_1 + 2}{3} \right\rceil + \left\lceil \frac{6k_2 + 2}{3} \right\rceil\\ &= 2k_1 + 1 + 2k_2 + 1 = 2k + 2 = \gamma_R(C_n).\end{aligned}$$

Because of $b_R(C_n) \leq 3$, this leads to $b_R(C_n) = 3$ in this case. The remaining two cases are similar and are therefore omitted. ■

Theorem 11. *If m and n are integers such that $1 \leq m \leq n$ and $n \geq 2$, then $b_R(K_{m,n}) = m$, with exception of the case $m = n = 3$. In addition, $b_R(K_{3,3}) = 4$.*

Proof. Let $G = K_{m,n}$. First notice that if $m = 1$, then G is a star and $\gamma_R(G - e) = 3 > 2 = \gamma_R(G)$ for any edge e, and hence $b_R(G) = 1 = m$.

Assume next that $m = 2$. If $n = 2$, then the desired result follows from Theorem 10. If $n \geq 3$, then $\gamma_R(G - e) = \gamma_R(G) = 3$ for any edge e. But if e_1 and e_2 are two edges incident to a vertex of degree two, then $\gamma_R(G - \{e_1, e_2\}) = 4$ and thus $b_R(G) = 2 = m$.

Finally assume that $m \geq 3$. Let X and Y be the two partite sets with $|X| = m$ and $|Y| = n$. If E is a set of edges with $|E| < m$ and $G_1 = G - E$, then there are two vertices $x \in X$ and $y \in Y$ such that $N_{G_1}(x) = Y$ and $N_{G_1}(y) = X$. It follows that $\gamma_R(G_1) = 4 = \gamma_R(G)$ and thus $b_R(G) \geq m$. However, if we remove all edges incident to a vertex $y \in Y$, then we obtain a graph G_2 such that $\gamma_R(G_2) = 5$ when $n \geq 4$. This shows that $b_R(G) = m$ when $n \geq 4$. Finally, let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$ be the partite sets of $K_{3,3}$. Let E be a subset of edges such that $\gamma_R(K_{3,3} - E) > \gamma_R(K_{3,3}) = 4$. Assume that $|E| < 4$, and without loss of generality assume that $|E| = 3$. Let $E = \{e_1, e_2, e_3\}$. If no two edges of E have a common end point, then we may assume, without loss of generality, that $e_i = x_iy_i$ for $i = 1, 2, 3$. Then $\gamma_R(K_{3,3} - E) = 4$ and $(\{x_2, y_2, x_3, y_3\}, \emptyset, \{x_1, y_1\})$ is a γ_R-function for $K_{3,3} - E$, a contradiction. Thus we assume, without loss of generality, that $e_1 = x_1y_1$ and $e_2 = x_1y_2$. If $e_3 = x_1y_3$, then $\gamma_R(K_{3,3} - E) = 4$, and $(\{y_1, y_2, y_3\}, \{x_1, x_2\}, \{x_3\})$ is a γ_R-function for $K_{3,3} - E)$, a contradiction. Thus $e_3 \neq x_1y_3$. Similarly, this case produces a contradiction. We conclude that $b_R(K_{3,3}) \geq 4$. On the other hand $\gamma_R(K_{3,3} - \{x_1y_2, x_1y_3, y_1x_2, y_1x_3\}) = 5 > \gamma_R(K_{3,3})$. Hence, $b_R(K_{3,3}) = 4$. ∎

4. Trees and Unicyclic Graphs

Lemma 12. *If a graph G has a support vertex v of degree at least three such that all of its neighbors except one is a leaf, then $b_R(G) \leq 2$.*

Proof. Let $N(v) = \{v_1, v_2, \ldots, v_k\}$ such that $deg(v_k) \geq 2$. Applying (1) on the path v_1vv_2 in the case $deg(v) = k = 3$, we obtain $b_R(G) \leq 2$ immediately.

Assume now that $\deg(v) = k \geq 4$. Let $f = (V_0; V_1; V_2)$ be a γ_R-function of $G - vv_1$. It follows that $v_1 \in V_1$ and, without loss of generality, that $v \in V_2$. Therefore $(V_0 \cup \{v_1\}, V_1 - \{v_1\}; V_2)$ is a RDF on G of weight $\gamma_R(G) - 1$, and thus $b_R(G) = 1$. ∎

Theorem 13. *For any tree T with at least three vertices, $b_R(T) \leq 3$.*

Proof. If T has a support vertex v of degree at least three such that all of its neighbors except one is a leaf, then $b_R(T) \leq 2$ by Lemma 12. So assume that for any support vertex v either $deg(v) = 2$ or v has at least two neighbors which are no leaves. Let $P = v_1v_2 \ldots v_k$ be a longest path of T. By the assumption, $deg_T(v_2) = 2$. If $deg_T(v_3) \leq 3$, then (1) with the path $v_1v_2v_3$ shows that $b_R(T) \leq 3$.

Assume now that $deg_T(v_3) \geq 4$. Suppose to the contrary that $b_R(T) > 3$. So $\gamma_R(T - \{v_2v_3, v_3v_4\}) = \gamma_R(T)$. Let $f = (V_0; V_1; V_2)$ be a γ_R-function on $T - \{v_2v_3, v_3v_4\}$. Then $f(v_1) + f(v_2) = 2$. If $v_3 \in V_1$, then

$$((V_0 - \{v_1, v_2\}) \cup \{v_1, v_3\}; V_1 - \{v_3\}; (V_2 - \{v_1, v_2\}) \cup \{v_2\})$$

is a RDF on T of weight less than $\gamma_R(T)$. This contradiction implies that $v_3 \notin V_1$. Similarly, $v_3 \notin V_2$. So $v_3 \in V_0$. We deduce that there is a vertex

$w_1 \in N_{V(T-\{v_2v_3,v_3v_4\})}(v_3) \cap V_2$. If w_1 is a leaf, then

$$((V_0 - \{v_1, v_2\}) \cup \{w_1, v_2\}; (V_1 - \{v_1, v_2\}) \cup \{v_1\}; (V_2 - \{v_1, v_2\}) \cup \{v_3\})$$

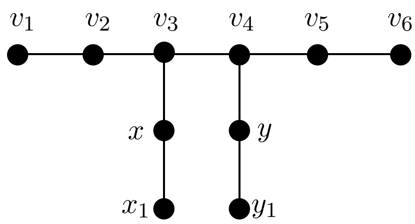
is a RDF on T of weight less than $\gamma_R(T)$, a contradiction. It follows that w_1 is a support vertex with $deg_T(w_1) = 2$. Let u_1 be a leaf adjacent to w_1. By the assumption, $\gamma_R(T - \{v_2v_3, v_3v_4, w_1v_3\}) = \gamma_R(T)$. Let g be a γ_R-function on $T - \{v_2v_3, v_3v_4, w_1v_3\}$. If $g(v_3) = 1$, then we replace $g(v_3)$ by 0, $g(v_2)$ by 2 and $g(v_1)$ by 0 to obtain a RDF on T of weight less than $\gamma_R(G)$, a contradiction. Similarly, we observe that $g(v_3) \neq 2$. So $g(v_3) = 0$. We deduce that there is a vertex $w_2 \in N_{T-\{v_2v_3,v_3v_4,w_1v_3\}}(v_3)$ such that $g(w_2) = 2$. We can easily see that w_2 is a support vertex with $deg_T(w_2) = 2$. Let u_2 be the leaf adjacent to w_2.

Now we consider the forest $T - \{v_2v_3, v_3w_1, v_3w_2\}$. Our assumption implies that $\gamma_R(T - \{v_2v_3, v_3w_1, v_3w_2\}) = \gamma_R(T)$. Let h be a γ_R-function on $T - \{v_2v_3, v_3w_1, v_3w_2\}$. Then

$$h(v_1) + h(v_2) = h(w_1) + h(u_1) = h(w_2) + h(u_2) = 2.$$

We replace $g(v_3)$ by 2, $g(v_2), g(w_1), g(w_2)$ by 0, and $g(v_1), g(u_1), g(u_2)$ by 1, to obtain a RDF on T of weight less than $\gamma_R(T)$, a contradiction. Hence $b_R(T) \leq 3$, and the proof is complete. ■

The following figure shows that the bound of Theorem 13 is sharp. It is a simple matter to verify that $b_R(H) = 3$.



H

In the next theorem we give a sharp upper bound for Roman bondage number in unicyclic graphs.

Theorem 14. *For any unicyclic graph G, $b_R(G) \leq 4$, and this bound is sharp.*

Proof. Let G be a unicyclic graph, and let C be the unique cycle of G. If $G = C$, then by Theorem 10, $b_R(G) \leq 3$. Assume that $G \neq C$. Let $v_1 - v_2 - \cdots - v_k$ be the longest path where v_1 is a leaf and $\{v_1, v_2, \ldots, v_k\} \cap V(C) = \{v_k\}$. Let $V(C) = \{u_1, u_2, \ldots, u_t\}$, where $u_1 = v_k$ and $N_C(v_k) = \{u_2, u_t\}$. If $b_R(G) \leq 2$, then we have done. So suppose that $b_R(G) \geq 3$. First assume that $k \geq 4$. By Lemma 12, $deg(v_2) = 2$. If $deg(v_3) \leq 4$, then (1) with the path $v_1v_2v_3$ shows that $b_R(G) \leq 4$. So we assume that $deg(v_3) \geq 5$. Let A be the set of all leaves of G at distance 2 from v_3 except the leaves adjacent to v_4. Let e_1, e_2, e_3 be three edges incident with v_3 with $\{e_1, e_2, e_3\} \cap \{v_2v_3, v_3v_4\} = \emptyset$. We show that $\gamma_R(G - \{v_2v_3, e_1, e_2, e_3\}) > \gamma_R(G)$. Suppose to the contrary that $\gamma_R(G - \{v_2v_3, e_1, e_2, e_3\}) = \gamma_R(G)$. Let f be a γ_R-function for $G - \{v_2v_3, e_1, e_2, e_3\}$. It follows that $g : V(G) \longrightarrow \{0, 1, 2\}$ defined by $g(v_3) = 2$, $g(x) = 0$ if $x \in N(v_3)$, $g(x) = 1$ if $x \in A$, and $g(x) = f(x)$ if $x \notin N[V_3] \cup A$, is a RDF for G with weight less than $\gamma_R(G)$. This contradiction implies that $\gamma_R(G - \{v_2v_3, e_1, e_2, e_3\}) > \gamma_R(G)$, and so $b_R(G) \leq 4$.

Now suppose that $k \leq 3$. For $k = 2$, it is straightforward to verify that if $deg(v_2) \geq 4$, then $\gamma_R(G - \{v_1v_2, u_1u_t, u_1u_2\}) > \gamma_R(G)$. Suppose that $deg(v_2) = 3$. As an immediately result $deg(u_i) \leq 3$ for $i = 1, 2, \ldots, t$. Again we can easily see that for $deg(u_2) = 2$, $\gamma_R(G - \{v_1v_2, v_2u_t, u_2u_3\}) > \gamma_R(G)$, and for $deg(u_2) = 3$, $\gamma_R(G - \{v_2u_2, v_2u_t, u_2u_3\}) > \gamma_R(G)$. Thus $b_R(G) \leq 3$. It remains to suppose that $k = 3$. By Lemma 12, $deg(v_2) = 2$. If $deg(v_3) \leq 4$, then (1) with the path $v_1v_2v_3$ shows that $b_R(G) \leq 4$. So suppose that $deg(v_3) \geq 5$. This time $\gamma_R(G - \{v_2v_3, v_3x, v_3y\}) > \gamma_R(G)$, where $\{x, y\} \cap \{u_2, u_t, v_2\} = \emptyset$. We deduce that $b_R(G) \leq 3$.

To see the sharpness, let G be a graph obtained from any cycle C_n on $n \geq 3$ vertices by identifying every vertex of C_n with the central vertex of a path P_5. It is straightforward to verify that $\gamma_R(G) = 4n$, and $b_R(G) = 4$. ■

We close the paper with the following problem.

Problem 15. Determine the trees T with $\gamma_R(T) = 1$, $\gamma_R(T) = 2$ and $\gamma_R(T) = 3$.

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