*Discussiones Mathematicae*
*Graph Theory 31 (2011) 763–773*

# ROMAN BONDAGE IN GRAPHS

Nader Jafari Rad [1]

*Department of Mathematics*
*Shahrood University of Technology*
*Shahrood, Iran*
*and*
*School of Mathematics*
*Institute for Research in Fundamental Sciences (IPM)*
*P.O. Box 19395–5746, Tehran, Iran*

**e-mail:** n.jafarirad@shahroodut.ac.ir

and

Lutz Volkmann

*Lehrstuhl II für Mathematik*
*RWTH Aachen University*
*Templergraben 55, D–52056 Aachen, Germany*

**e-mail:** volkm@math2.rwth-aachen.de

**Abstract**

A Roman dominating function on a graph $G$ is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The Roman domination number, $\gamma_R(G)$, of $G$ is the minimum weight of a Roman dominating function on $G$. In this paper, we define the Roman bondage $b_R(G)$ of a graph $G$ with maximum degree at least two to be the minimum cardinality of all sets $E' \subseteq E(G)$ for which $\gamma_R(G - E') > \gamma_R(G)$. We determine the Roman bondage number in several classes of graphs and give some sharp bounds.

[1]The research of first author was in part supported by a grant from IPM (No. 89050040).

**Keywords:** domination, Roman domination, Roman bondage number.

**2010 Mathematics Subject Classification:** 05C69.

## 1. Terminology and Introduction

Let $G = (V(G), E(G))$ be a simple graph of order $n$. We denote the *open neighborhood* of a vertex $v$ of $G$ by $N_G(v)$, or just $N(v)$, and its *closed neighborhood* by $N_G[v] = N[v]$. For a vertex set $S \subseteq V(G)$, $N(S) = \bigcup_{v\in S} N(v)$ and $N[S] = \bigcup_{v\in S} N[v]$. The *degree* $deg(x)$ of a vertex $x$ denotes the number of neighbors of $x$ in $G$, and $\Delta(G)$ is the *maximum degree* of $G$. Also the *eccentricity*, $ecc(x)$, of a vertex $x$ is maximum distance of the vertices of $G$ from $x$. A set of vertices $S$ in $G$ is a *dominating set*, if $N[S] = V(G)$. The *domination number*, $\gamma(G)$, of $G$ is the minimum cardinality of a dominating set of $G$. If $S$ is a subset of $V(G)$, then we denote by $G[S]$ the subgraph of $G$ induced by $S$. For notation and graph theory terminology in general we follow [6].

With $K_n$ we denote the *complete graph* on $n$ vertices and with $C_n$ the *cycle* of length $n$. For two positive integers $m, n$, the *complete bipartite graph* $K_{m,n}$ is the graph with partition $V(G) = V_1 \cup V_2$ such that $|V_1| = m$, $|V_2| = n$ and such that $G[V_i]$ has no edge for $i = 1, 2$, and every two vertices belonging to different partition sets are adjacent to each other.

For a graph $G$, let $f : V(G) \to \{0, 1, 2\}$ be a function, and let $(V_0; V_1; V_2)$ be the ordered partition of $V(G)$ induced by $f$, where $V_i = \{v \in V(G) : f(v) = i\}$ and for $i = 0, 1, 2$. There is a $1 - 1$ correspondence between the functions $f : V(G) \to \{0, 1, 2\}$ and the ordered partition $(V_0; V_1; V_2)$ of $V(G)$. So we will write $f = (V_0; V_1; V_2)$.

A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* (or just RDF) if every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V(G)) = \sum_{u\in V(G)} f(u)$. The *Roman domination number* of a graph $G$, denoted by $\gamma_R(G)$, is the minimum weight of a Roman dominating function on $G$. A function $f = (V_0; V_1; V_2)$ is called a $\gamma_R$-function (or $\gamma_R(G)$-function when we want to refer $f$ to $G$), if it is a Roman dominating function and $f(V(G)) = \gamma_R(G)$, [2, 7, 8].

The *bondage number* $b(G)$ of a nonempty graph $G$ is the minimum cardinality among all sets of edges $E' \subseteq E(G)$ for which $\gamma(G - E') > \gamma(G)$.

This concept was introduced by Bauer, Harary, Nieminen and Suffel in [1], and has been further studied for example in [4, 5, 9]). For more information on this topic we refer the reader to the survey article by Dunbar, Haynes, Teschner and Volkmann [3].

In this paper we study bondage by considering a variation based on Roman domination. The *Roman bondage number* $b_R(G)$ of a graph $G$ is the cardinality of a smallest set of edges $E' \subseteq E(G)$ for which $\gamma_R(G-E') > \gamma_R(G)$.

We note that if $G$ is a connected graph on two vertices, then $G \simeq K_2$ and $\gamma_R(G) = 2$. If $e \in E(G)$, then $G - e \simeq \overline{K_2}$ and thus $\gamma_R(G-e) = \gamma_R(G)$. Therefore the Roman bondage number is only defined for a graph $G$ with maximum degree at least two.

We recall that a leaf in a graph $G$ is a vertex of degree one, and a support vertex is the vertex which is adjacent to a leaf.

## 2. Upper Bounds

**Theorem 1.** *If $G$ is a graph, and $xyz$ a path of length 2 in $G$, then*

$$b_R(G) \leq deg(x) + deg(y) + deg(z) - 3 - |N(x) \cap N(y)|. \tag{1}$$

*If $x$ and $z$ are adjacent, then*

$$b_R(G) \leq deg(x) + deg(y) + deg(z) - 4 - |N(x) \cap N(y)|. \tag{2}$$

***Proof.*** Let $H$ be the graph obtained from $G$ by removing the edges incident with $x$, $y$ or $z$ with exception of $yz$ and all edges between $y$ and $N(x) \cap N(y)$. In $H$, the vertex $x$ is isolated, $z$ is a leaf and $y$ is adjacent to $z$ and all neighbors of $y$ in $H$, if any, lie in $N_G(x)$.

Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(H)$-function. Then $x \in V_1$ and, without loss of generality, $z \in V_0 \cup V_1$.

If $z \in V_0$, then $y \in V_2$ and therefore $(V_0 \cup \{x\}, V_1 - \{x\}, V_2)$ is a RDF on $G$ of weight less than $f$, and (1) as well as (2) are proved.

Now assume that $z \in V_1$. If $y \in V_1$, then $(V_0 \cup \{z\}, V_1 - \{y, z\}, V_2 \cup \{y\})$ is also $\gamma_R(H)$-function, and we are in the situation discussed in the previous case. However, if $y \in V_0$, then there exists a vertex $w \in N_G(x) \cap N_G(y)$ such that $w \in V_2$. Since $w$ is a neighbor of $x$ in $G$, $(V_0 \cup \{x\}, V_1 - \{x\}, V_2)$ is a RDF on $G$ of weight less than $f$, and again (1) and (2) are proved. ■

Applying Theorem 1 on a path $xyz$ such that one of the vertices $x, y$ or $z$ has minimum degree, we obtain the next result immediately.

**Corollary 2.** *If $G$ is a connected graph of order $n \geq 3$, then*

$$b_R(G) \leq \delta(G) + 2\Delta(G) - 3.$$

Our next upper bound involves the *edge-connectivity* $\lambda(G)$, which is the fewest number of edges whose removal from a connected graph $G$ creates two components. Since $\lambda(G) \leq \delta(G)$, the next theorem is an extension of Corollary 2.

**Observation 3.** *If $E$ is an edge cut set in a graph $G$ smaller than $b_R(G)$, then $\gamma_R(G)$ equals the sum of all $\gamma_R(G_i)$ where $G_i$ emerge by removing $E$.*

**Theorem 4.** *If $G$ is a connected graph of order $n \geq 3$, then*

$$b_R(G) \leq \lambda(G) + 2\Delta(G) - 3.$$

***Proof.*** Let $\lambda = \lambda(G)$, and let $E = \{e_1, e_2, \ldots, e_\lambda\}$ be a set of edges whose removal disconnects $G$. Say $e_1 = ab$, and let $H_a$ and $H_b$ denote the components of $G - E$ containing $a$ and $b$, respectively. By Corollary 2 we may assume that $H_a$ and $H_b$ are non-trivial. Let $a_1 \in V(H_a)$ adjacent to $a$ and $b_1 \in V(H_b)$ adjacent to $b$, and let $F_{a,a_1}$ and $F_{b,b_1}$ denote the edges of $G$ incident with $a$ or $a_1$ with exception of $aa_1$ and $b$ or $b_1$ with exception of $bb_1$, respectively. Suppose on the contrary that $b_R(G) > \lambda(G) + 2\Delta(G) - 3$. Noting that $|E| = \lambda < b_R(G)$, we observe that $\gamma_R(G) = \gamma_R(H_a) + \gamma_R(H_b)$. Since

$$|F_{a,a_1} \cup E| \leq deg_G(a) + deg_G(a_1) + \lambda - 3 \leq 2\Delta(G) + \lambda - 3 < b_R(G),$$

we deduce that $\gamma_R(G) = \gamma(H_a - \{a, a_1\}) + 2 + \gamma_R(H_b)$. Similarly, since

$$|F_{b,b_1} \cup E| \leq deg_G(b) + deg_G(b_1) + \lambda - 3 \leq 2\Delta(G) + \lambda - 3 < b_R(G),$$

we deduce that $\gamma_R(G) = \gamma_R(H_b - \{b, b_1\}) + 2 + \gamma_R(H_a)$. Altogether we obtain

$$\begin{aligned} 2\gamma_R(G) &= \gamma_R(H_a - \{a, a_1\}) + 2 + \gamma_R(H_b) + \gamma_R(H_b - \{b, b_1\}) + 2 + \gamma_R(H_a) \\ &= \gamma_R(H_a - \{a, a_1\}) + 4 + \gamma_R(H_b - \{b, b_1\}) + \gamma_R(G) \end{aligned}$$

and thus $\gamma_R(G) = \gamma_R(H_a - \{a, a_1\}) + 4 + \gamma_R(H_b - \{b, b_1\})$. This is obviously a contradiction, since

$$\begin{aligned}\gamma_R(G) &\leq \gamma_R(H_a - \{a, a_1\}) + \gamma_R(a_1abb_1) + \gamma_R(H_b - \{b, b_1\})\\ &\leq \gamma_R(H_a - \{a, a_1\}) + 3 + \gamma_R(H_b - \{b, b_1\}).\end{aligned}$$

■

**Observation 5.** *If a graph $G$ has a vertex $v$ such that $\gamma_R(G - v) \geq \gamma_R(G)$, then $b_R(G) \leq \Delta(G)$.*

***Proof.*** Let $E$ be the edge set incident with $v$. It follows that $\gamma_R(G - E) > \gamma_R(G)$, and the result is proved. ■

## 3. Exact Values of $b_R(G)$

In this section we determine the Roman bondage number for several classes of graphs.

**Theorem 6.** *If $G$ is a graph of order $n \geq 3$ with exactly $k \geq 1$ vertices of degree $n - 1$, then $b_R(G) = \lceil \frac{k}{2} \rceil$.*

***Proof.*** Since $k \geq 1$, we note that $\gamma_R(G) = 2$. First let $E_1 \subseteq E(G)$ be an arbitrary subset of edges such that $|E_1| < \lceil \frac{k}{2} \rceil$, and let $G' = G - E_1$. It is evident that there is a vertex $v$ in $G'$ such that $deg_G(v) = deg_{G'}(v) = n - 1$, and so $\gamma_R(G') = \gamma_R(G) = 2$. This shows that $b_R(G) \geq \lceil \frac{k}{2} \rceil$.

If $v_1, v_2, \ldots, v_k \in V(G)$ are the vertices of degree $n - 1$, then the subgraph $F$ induced by the vertices $v_1, v_2, \ldots, v_k$ is isomorphic to the complete graph $K_k$.

If $k$ is even, then let $H_1$ be the graph obtained from $G$ by removing $\frac{k}{2}$ independent edges from $F$. Then $\Delta(H_1) = n-2$ and thus $\gamma_R(H_1) = 3$. This implies $b_R(G) \leq \lceil \frac{k}{2} \rceil$.

If $k$ is odd, then let $H_2$ be the graph obtained from $G$ by removing $\frac{k-1}{2}$ independent edges from $F$. Then there exists exactly one vertex, say $v_k \in V(H_2)$ such that $deg_{H_2}(v_k) = n-1$. Let $H_3$ be the graph obtained from $H_2$ by removing an arbitrary edge incident with $v_k$. Then $\Delta(H_3) = n - 2$ and so $\gamma_R(H_3) = 3$. This implies $b_R(G) \leq \lceil \frac{k}{2} \rceil$.

Combining the obtained inequalities, we deduce that $b_R(G) = \lceil \frac{k}{2} \rceil$, and the proof is complete. ■

**Corollary 7.** *If $n \geq 3$, then $b_R(K_n) = \lceil \frac{n}{2} \rceil$.*

**Lemma 8** [2]**.** *For the classes of paths $P_n$ and cycles $C_n$,*

$$\gamma_R(P_n) = \gamma_R(C_n) = \left\lceil \frac{2n}{3} \right\rceil.$$

**Theorem 9.** *For $n \geq 3$,*

$$b_R(P_n) = \begin{cases} 2 & if\ n \equiv 2 \pmod 3, \\ 1 & otherwise. \end{cases}$$

***Proof.*** Let $P_n = v_1v_2 \dots v_n$. Corollary 2 yields to $b_R(P_n) \leq 2$. First assume that $n = 3k$. Lemma 8 implies that $\gamma_R(P_n) = 2k$ and $\gamma_R(P_n - v_1v_2) = 1 + \gamma_R(P_{n-1}) = 1 + 2k$ and thus $b_R(P_n) = 1$. Next assume that $n = 3k + 1$. According to Lemma 8, we have $\gamma_R(P_n) = 2k + 1$ and $\gamma_R(P_n - v_2v_3) = 2 + \gamma_R(P_{n-2}) = 2 + 2k$ and so $b_R(P_n) = 1$. It remains to assume that $n = 3k + 2$. By Lemma 8, $\gamma_R(P_n) = 2k + 2$. If $e$ is an arbitrary edge of $P_n$, then $P_n - e$ consists of two paths $P_1$ and $P_2$ of order $n_1$ and $n_2$, respectively, such that $n_1 + n_2 = n$ and $\gamma_R(P_n - e) = \gamma_R(P_1) + \gamma_R(P_2)$. Now there are integers $k_1$ and $k_2$ such that $n_1 = 3k_1, n_2 = 3k_2 + 2$ or $n_1 = 3k_1 + 1, n_2 = 3k_2 + 1$ or $n_1 = 3k_1 + 2, n_2 = 3k_2$ and $k_1 + k_2 = k$. In the first case we deduce from Lemma 8 that

$$\begin{aligned} \gamma_R(P_n - e) &= \gamma_R(P_1) + \gamma_R(P_2) \\ &= \left\lceil \frac{6k_1}{3} \right\rceil + \left\lceil \frac{6k_2 + 4}{3} \right\rceil \\ &= 2k_1 + 2k_2 + 2 = 2k + 2 = \gamma_R(P_n). \end{aligned}$$

This implies that $b_R(P_n) \geq 2$ in the first case, and because of $b_R(P_n) \leq 2$ we obtain $b_R(P_n) = 2$. The remaining two cases are similar and are therefore omitted. ■

**Theorem 10.** *For $n \geq 3$,*

$$b_R(C_n) = \begin{cases} 3 & if\ n \equiv 2 \pmod 3, \\ 2 & otherwise. \end{cases}$$

***Proof.*** Let $C_n = v_1v_2 \dots v_nv_1$. Corollary 2 leads to $b_R(C_n) \leq 3$. If $e$ is an arbitrary edge of $C_n$, then $C_n - e = P_n$. Hence Lemma 8 shows that $b_R(C_n) \geq 2$. We distinguish three cases.

Assume that $n = 3k$. Lemma 8 implies that $\gamma_R(C_n) = 2k$ and $\gamma_R(C_n - \{v_1v_2, v_2v_3\}) = 1 + \gamma_R(P_{3k-1}) = 1 + 2k$ and thus $b_R(C_n) = 2$.

Assume that $n = 3k + 1$. Lemma 8 implies that $\gamma_R(C_n) = 2k + 1$ and $\gamma_R(C_n - \{v_1v_2, v_3v_4\}) = 2 + \gamma_R(P_{3k-1}) = 2 + 2k$ and thus $b_R(C_n) = 2$.

Assume that $n = 3k+2$. By Lemma 8, $\gamma_R(C_n) = 2k+2$. If $e_1$ and $e_2$ are two arbitrary edges of $C_n$, then $C_n - \{e_1, e_2\}$ consists of two paths $P_1$ and $P_2$ of order $n_1$ and $n_2$ such that $n_1 + n_2 = n$ and $\gamma_R(C_n - \{e_1, e_2\}) = \gamma_R(P_1) + \gamma_R(P_2)$. Now there are integers $k_1$ and $k_2$ such that $n_1 = 3k_1, n_2 = 3k_2 + 2$ or $n_1 = 3k_1 + 1, n_2 = 3k_2 + 1$ or $n_1 = 3k_1 + 2, n_2 = 3k_2$ and $k_1 + k_2 = k$. In the second case we deduce from Lemma 8 that

$$\begin{aligned}
\gamma_R(C_n - \{e_1, e_2\}) &= \gamma_R(P_1) + \gamma_R(P_2) \\
&= \left\lceil \frac{6k_1 + 2}{3} \right\rceil + \left\lceil \frac{6k_2 + 2}{3} \right\rceil \\
&= 2k_1 + 1 + 2k_2 + 1 = 2k + 2 = \gamma_R(C_n).
\end{aligned}$$

Because of $b_R(C_n) \le 3$, this leads to $b_R(C_n) = 3$ in this case. The remaining two cases are similar and are therefore omitted. ■

**Theorem 11.** *If $m$ and $n$ are integers such that $1 \le m \le n$ and $n \ge 2$, then $b_R(K_{m,n}) = m$, with exception of the case $m = n = 3$. In addition, $b_R(K_{3,3}) = 4$.*

***Proof.*** Let $G = K_{m,n}$. First notice that if $m = 1$, then $G$ is a star and $\gamma_R(G - e) = 3 > 2 = \gamma_R(G)$ for any edge $e$, and hence $b_R(G) = 1 = m$.

Assume next that $m = 2$. If $n = 2$, then the desired result follows from Theorem 10. If $n \ge 3$, then $\gamma_R(G - e) = \gamma_R(G) = 3$ for any edge $e$. But if $e_1$ and $e_2$ are two edges incident to a vertex of degree two, then $\gamma_R(G - \{e_1, e_2\}) = 4$ and thus $b_R(G) = 2 = m$.

Finally assume that $m \ge 3$. Let $X$ and $Y$ be the two partite sets with $|X| = m$ and $|Y| = n$. If $E$ is a set of edges with $|E| < m$ and $G_1 = G - E$, then there are two vertices $x \in X$ and $y \in Y$ such that $N_{G_1}(x) = Y$ and $N_{G_1}(y) = X$. It follows that $\gamma_R(G_1) = 4 = \gamma_R(G)$ and thus $b_R(G) \ge m$. However, if we remove all edges incident to a vertex $y \in Y$, then we obtain a graph $G_2$ such that $\gamma_R(G_2) = 5$ when $n \ge 4$. This shows that $b_R(G) = m$ when $n \ge 4$. Finally, let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$ be the partite sets of $K_{3,3}$. Let $E$ be a subset of edges such that $\gamma_R(K_{3,3} - E) > \gamma_R(K_{3,3}) = 4$. Assume that $|E| < 4$, and without loss of generality assume that $|E| = 3$. Let $E = \{e_1, e_2, e_3\}$. If no two edges of $E$ have a common end point, then we may assume, without loss of generality, that $e_i = x_i y_i$ for $i = 1, 2, 3$. Then $\gamma_R(K_{3,3} - E) = 4$ and $(\{x_2, y_2, x_3, y_3\}, \emptyset, \{x_1, y_1\})$ is a $\gamma_R$-function for $K_{3,3} - E$, a contradiction. Thus we assume, without loss of generality, that $e_1 = x_1 y_1$ and $e_2 = x_1 y_2$. If $e_3 = x_1 y_3$, then $\gamma_R(K_{3,3} - E) = 4$, and $(\{y_1, y_2, y_3\}, \{x_1, x_2\}, \{x_3\})$ is a $\gamma_R$-function for $K_{3,3} - E)$, a contradiction. Thus $e_3 \neq x_1 y_3$. Similarly, this case produces a contradiction. We conclude that $b_R(K_{3,3}) \geq 4$. On the other hand $\gamma_R(K_{3,3} - \{x_1 y_2, x_1 y_3, y_1 x_2, y_1 x_3\}) = 5 > \gamma_R(K_{3,3})$. Hence, $b_R(K_{3,3}) = 4$. ■

## 4. Trees and Unicyclic Graphs

**Lemma 12.** *If a graph $G$ has a support vertex $v$ of degree at least three such that all of its neighbors except one is a leaf, then $b_R(G) \leq 2$.*

***Proof.*** Let $N(v) = \{v_1, v_2, \ldots, v_k\}$ such that $deg(v_k) \geq 2$. Applying (1) on the path $v_1 v v_2$ in the case $deg(v) = k = 3$, we obtain $b_R(G) \leq 2$ immediately.

Assume now that $\deg(v) = k \geq 4$. Let $f = (V_0; V_1; V_2)$ be a $\gamma_R$-function of $G - vv_1$. It follows that $v_1 \in V_1$ and, without loss of generality, that $v \in V_2$. Therefore $(V_0 \cup \{v_1\}, V_1 - \{v_1\}; V_2)$ is a RDF on $G$ of weight $\gamma_R(G) - 1$, and thus $b_R(G) = 1$. ■

**Theorem 13.** *For any tree $T$ with at least three vertices, $b_R(T) \leq 3$.*

***Proof.*** If $T$ has a support vertex $v$ of degree at least three such that all of its neighbors except one is a leaf, then $b_R(T) \leq 2$ by Lemma 12. So assume that for any support vertex $v$ either $deg(v) = 2$ or $v$ has at least two neighbors which are no leaves. Let $P = v_1 v_2 \ldots v_k$ be a longest path of $T$. By the assumption, $deg_T(v_2) = 2$. If $deg_T(v_3) \leq 3$, then (1) with the path $v_1 v_2 v_3$ shows that $b_R(T) \leq 3$.

Assume now that $deg_T(v_3) \geq 4$. Suppose to the contrary that $b_R(T) > 3$. So $\gamma_R(T - \{v_2 v_3, v_3 v_4\}) = \gamma_R(T)$. Let $f = (V_0; V_1; V_2)$ be a $\gamma_R$-function on $T - \{v_2 v_3, v_3 v_4\}$. Then $f(v_1) + f(v_2) = 2$. If $v_3 \in V_1$, then

$$((V_0 - \{v_1, v_2\}) \cup \{v_1, v_3\}; V_1 - \{v_3\}; (V_2 - \{v_1, v_2\}) \cup \{v_2\})$$

is a RDF on $T$ of weight less than $\gamma_R(T)$. This contradiction implies that $v_3 \notin V_1$. Similarly, $v_3 \notin V_2$. So $v_3 \in V_0$. We deduce that there is a vertex

$w_1 \in N_{V(T-\{v_2v_3,v_3v_4\})}(v_3) \cap V_2$. If $w_1$ is a leaf, then

$$((V_0 - \{v_1, v_2\}) \cup \{w_1, v_2\}; (V_1 - \{v_1, v_2\}) \cup \{v_1\}; (V_2 - \{v_1, v_2\}) \cup \{v_3\})$$

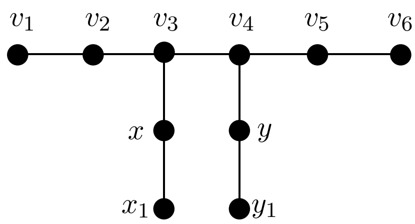
is a RDF on $T$ of weight less than $\gamma_R(T)$, a contradiction. It follows that $w_1$ is a support vertex with $deg_T(w_1) = 2$. Let $u_1$ be a leaf adjacent to $w_1$. By the assumption, $\gamma_R(T - \{v_2v_3, v_3v_4, w_1v_3\}) = \gamma_R(T)$. Let $g$ be a $\gamma_R$-function on $T - \{v_2v_3, v_3v_4, w_1v_3\}$. If $g(v_3) = 1$, then we replace $g(v_3)$ by 0, $g(v_2)$ by 2 and $g(v_1)$ by 0 to obtain a RDF on $T$ of weight less than $\gamma_R(G)$, a contradiction. Similarly, we observe that $g(v_3) \neq 2$. So $g(v_3) = 0$. We deduce that there is a vertex $w_2 \in N_{T-\{v_2v_3,v_3v_4,w_1v_3\}}(v_3)$ such that $g(w_2) = 2$. We can easily see that $w_2$ is a support vertex with $deg_T(w_2) = 2$. Let $u_2$ be the leaf adjacent to $w_2$.

Now we consider the forest $T - \{v_2v_3, v_3w_1, v_3w_2\}$. Our assumption implies that $\gamma_R(T - \{v_2v_3, v_3w_1, v_3w_2\}) = \gamma_R(T)$. Let $h$ be a $\gamma_R$-function on $T - \{v_2v_3, v_3w_1, v_3w_2\}$. Then

$$h(v_1) + h(v_2) = h(w_1) + h(u_1) = h(w_2) + h(u_2) = 2.$$

We replace $g(v_3)$ by 2, $g(v_2), g(w_1), g(w_2)$ by 0, and $g(v_1), g(u_1), g(u_2)$ by 1, to obtain a RDF on $T$ of weight less than $\gamma_R(T)$, a contradiction. Hence $b_R(T) \leq 3$, and the proof is complete. ∎

The following figure shows that the bound of Theorem 13 is sharp. It is a simple matter to verify that $b_R(H) = 3$.




$H$

In the next theorem we give a sharp upper bound for Roman bondage number in unicyclic graphs.

**Theorem 14.** *For any unicyclic graph $G$, $b_R(G) \leq 4$, and this bound is sharp.*

***Proof.*** Let $G$ be a unicyclic graph, and let $C$ be the unique cycle of $G$. If $G = C$, then by Theorem 10, $b_R(G) \leq 3$. Assume that $G \neq C$. Let $v_1 - v_2 - \cdots - v_k$ be the longest path where $v_1$ is a leaf and $\{v_1, v_2, \ldots, v_k\} \cap V(C) = \{v_k\}$. Let $V(C) = \{u_1, u_2, \ldots, u_t\}$, where $u_1 = v_k$ and $N_C(v_k) = \{u_2, u_t\}$. If $b_R(G) \leq 2$, then we have done. So suppose that $b_R(G) \geq 3$. First assume that $k \geq 4$. By Lemma 12, $deg(v_2) = 2$. If $deg(v_3) \leq 4$, then (1) with the path $v_1v_2v_3$ shows that $b_R(G) \leq 4$. So we assume that $deg(v_3) \geq 5$. Let $A$ be the set of all leaves of $G$ at distance 2 from $v_3$ except the leaves adjacent to $v_4$. Let $e_1, e_2, e_3$ be three edges incident with $v_3$ with $\{e_1, e_2, e_3\} \cap \{v_2v_3, v_3v_4\} = \emptyset$. We show that $\gamma_R(G - \{v_2v_3, e_1, e_2, e_3\}) > \gamma_R(G)$. Suppose to the contrary that $\gamma_R(G - \{v_2v_3, e_1, e_2, e_3\}) = \gamma_R(G)$. Let $f$ be a $\gamma_R$-function for $G - \{v_2v_3, e_1, e_2, e_3\}$. It follows that $g : V(G) \longrightarrow \{0, 1, 2\}$ defined by $g(v_3) = 2$, $g(x) = 0$ if $x \in N(v_3)$, $g(x) = 1$ if $x \in A$, and $g(x) = f(x)$ if $x \notin N[V_3] \cup A$, is a RDF for $G$ with weight less than $\gamma_R(G)$. This contradiction implies that $\gamma_R(G - \{v_2v_3, e_1, e_2, e_3\}) > \gamma_R(G)$, and so $b_R(G) \leq 4$.

Now suppose that $k \leq 3$. For $k = 2$, it is straightforward to verify that if $deg(v_2) \geq 4$, then $\gamma_R(G - \{v_1v_2, u_1u_t, u_1u_2\}) > \gamma_R(G)$. Suppose that $deg(v_2) = 3$. As an immediately result $deg(u_i) \leq 3$ for $i = 1, 2, \ldots, t$. Again we can easily see that for $deg(u_2) = 2$, $\gamma_R(G - \{v_1v_2, v_2u_t, u_2u_3\}) > \gamma_R(G)$, and for $deg(u_2) = 3$, $\gamma_R(G - \{v_2u_2, v_2u_t, u_2u_3\}) > \gamma_R(G)$. Thus $b_R(G) \leq 3$. It remains to suppose that $k = 3$. By Lemma 12, $deg(v_2) = 2$. If $deg(v_3) \leq 4$, then (1) with the path $v_1v_2v_3$ shows that $b_R(G) \leq 4$. So suppose that $deg(v_3) \geq 5$. This time $\gamma_R(G - \{v_2v_3, v_3x, v_3y\}) > \gamma_R(G)$, where $\{x, y\} \cap \{u_2, u_t, v_2\} = \emptyset$. We deduce that $b_R(G) \leq 3$.

To see the sharpness, let $G$ be a graph obtained from any cycle $C_n$ on $n \geq 3$ vertices by identifying every vertex of $C_n$ with the central vertex of a path $P_5$. It is straightforward to verify that $\gamma_R(G) = 4n$, and $b_R(G) = 4$. ∎

We close the paper with the following problem.

**Problem 15.** Determine the trees $T$ with $\gamma_R(T) = 1$, $\gamma_R(T) = 2$ and $\gamma_R(T) = 3$.

**Acknowledgement**

We would like to thank the referees for their careful review of our manuscript and some helpful suggestions.

## References


[1] D. Bauer, F. Harary, J. Nieminen and C.L. Suffel, *Domination alteration sets in graphs*, Discrete Math. **47** (1983) 153–161.

[2] E.J. Cockayne, P.M. Dreyer Jr., S.M. Hedetniemi and S.T. Hedetniemi, *Roman domination in graphs*, Discrete Math. **278** (2004) 11–22.

[3] J.E. Dunbar, T.W. Haynes, U. Teschner and L. Volkmann, *Bondage, insensitivity, and reinforcement*, in: T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.), Domination in Graphs: Advanced Topics (Marcel Dekker, New York, 1998) 471–489.

[4] J.F. Fink, M.S. Jacobson, L.F. Kinch and J. Roberts, *The bondage number of a graph*, Discrete Math. **86** (1990) 47–57.

[5] B.L. Hartnell and D.F. Rall, *Bounds on the bondage number of a graph*, Discrete Math. **128** (1994) 173-177.

[6] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, New York, 1998).

[7] C.S. ReVelle and K.E. Rosing, *Defendens imperium romanum: a classical problem in military strategy*, Amer Math. Monthly **107** (2000) 585–594.

[8] I. Stewart, *Defend the Roman Empire!*, Sci. Amer. **281** (1999) 136–139.

[9] U. Teschner, *New results about the bondage number of a graph*, Discrete Math. **171** (1997) 249–259.

[10] D.B. West, Introduction to Graph Theory, (2nd edition) (Prentice Hall, USA, 2001).



Received 14 June 2010
Revised 23 November 2010
Accepted 23 November 2010