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# SPANNING TREE CONGESTION OF ROOK'S GRAPHS

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**Abstract**

Let $G$ be a connected graph and $T$ be a spanning tree of $G$. For $e \in E(T)$, the *congestion* of $e$ is the number of edges in $G$ joining the two components of $T - e$. The *congestion* of $T$ is the maximum congestion over all edges in $T$. The *spanning tree congestion* of $G$ is the minimum congestion over all its spanning trees. In this paper, we determine the spanning tree congestion of the rook's graph $K_m \,\Box\, K_n$ for any $m$ and $n$.

**Keywords:** spanning tree congestion, Rook's graph.

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## 1. Introduction

For a graph $G$, we denote its vertex set and edge set by $V(G)$ and $E(G)$, respectively. Let $T$ be a spanning tree of a connected graph $G$. The *detour* for an edge $\{u, v\} \in E(G)$ is the unique $u - v$ path in $T$. We define the *congestion* of $e \in E(T)$, denoted by $cng_{G,T}(e)$, as the number of detours that contain $e$. The *congestion of* $T$, denoted by $cng_G(T)$, is the maximum congestion over all edges in $T$. We define the *spanning tree congestion* of $G$, denoted by $stc(G)$, as the minimum congestion over all spanning trees of $G$.

The spanning tree congestion has been studied intensively [4, 5, 8, 9, 12, 10, 16, 15, 17, 18]. Castejón and Ostrovskii [5], and Hruska [8] independently determined the spanning tree congestion of the two-dimensional grid $P_m \square P_n$. Kozawa, Otachi, and Yamazaki [9] determined the spanning tree congestion of the two-dimensional torus $C_m \square C_n$. There are some results for high-dimensional graphs; that is, Cartesian products of three or more graphs. Castejón and Ostrovskii [5] presented asymptotic estimates for the three-dimensional grid $P_n^3 = P_n \square P_n \square P_n$ and the three-dimensional torus $C_n^3 = C_n \square C_n \square C_n$. The spanning tree congestion of the $d$-dimensional hypercube $P_2^d$ was conjectured to be $2^{d-1}$ [8, 9], but it was proven to be $\Theta(2^d \lg d/d)$ by Law [12].

In this paper, we follow the line of studies on the spanning tree congestion of Cartesian product graphs. We investigate the spanning tree congestion of the two-dimensional Hamming graph $K_m \square K_n$, which is also known as the rook's graph. The rest of this paper is organized as follows. In Section 2, we introduce some notions and auxiliary lemmas. In Section 3, we determine the spanning tree congestion of two-dimensional Hamming graphs.

## 2. Preliminaries

Let $G$ be a connected graph. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph induced by $S$. For an edge $e \in E(G)$, we denote by $G - e$ the graph obtained from $G$ by deleting $e$. If $e \in E(G)$ has a vertex of degree one as one of its endpoints, $e$ is called a *leaf edge*, otherwise $e$ is called an *inner edge*. Let $N_G(v)$ denote the neighborhood of $v \in V(G)$ in $G$; that is, $N_G(v) = \{u \mid \{u, v\} \in E(G)\}$. We denote the degree of a vertex $v \in V(G)$ by $deg_G(v)$, and the maximum degree of $G$ by $\Delta(G)$; that is, $deg_G(v) = |N_G(v)|$ and $\Delta(G) = \max_{v \in V(G)} deg_G(v)$. A graph $G$ is *$r$-regular* if $deg_G(v) = r$ for all $v \in V(G)$. For $S \subseteq V(G)$, we denote the edge set of $G[S]$ by $\iota_G(S)$, and the *boundary edge set* by $\theta_G(S)$; that is,

$$\iota_G(S) = \{\{u, v\} \in E(G) \mid u, v \in S\},$$

$$\theta_G(S) = \{\{u, v\} \in E(G) \mid \text{exactly one of } u, v \text{ is in } S\}.$$

We define the functions $\iota$ and $\theta$ also for a positive integer $s \le |V(G)|$ as

$$\iota_G(s) = \max_{S \subseteq V(G),\ |S|=s} |\iota_G(S)|,$$
$$\theta_G(s) = \min_{S \subseteq V(G),\ |S|=s} |\theta_G(S)|.$$

The congestion $cng_{G,T}(e)$ of an edge $e \in E(T)$ satisfies $cng_{G,T}(e) = |\theta_G(L_e)|$, where $L_e$ is the vertex set of one of the two components of $T - e$. We omit the subscripts of the above functions when they are clear from the context. As the next lemma shows, the two functions $\iota$ and $\theta$ can be computed from each other directly.

**Lemma 2.1** [3]. *Let $G$ be $r$-regular and $S \subseteq V(G)$. Then, $2|\iota_G(S)| + |\theta_G(S)| = r|S|$.*

The following lower bound can be derived from a property of the centroid of trees.

**Lemma 2.2** [5, 9]. *For a connected graph $G$,*
$stc(G) \ge \min\left\{\theta(s) \mid \left\lceil (|V(G)| - 1)/\Delta(G) \right\rceil \le s \le \left\lfloor |V(G)|/2 \right\rfloor \right\}$.

The *Cartesian product* of graphs $G$ and $H$, denoted by $G \,\square\, H$, is the graph whose vertex set is $V(G) \times V(H)$ and in which a vertex $(g, h)$ is adjacent to a vertex $(g', h')$ if and only if either $g = g'$ and $\{h, h'\} \in E(H)$, or $h = h'$ and $\{g, g'\} \in E(G)$. It is easy to see that the Cartesian product operation satisfies the associative and commutative laws up to isomorphism. The *dth Cartesian power of a graph $G$*, denoted by $G^d$, is defined as follows: $G^1 = G$ and $G^d = G \square G^{d-1}$ if $d \ge 2$. Obviously, $deg_{G \square H}((g, h)) = deg_G(g) + deg_H(h)$.

Let $[n]$ denote the set $\{0, \ldots, n-1\}$. The *complete graph $K_n$* is the graph whose vertex set is $[n]$, with any two vertices adjacent. The graph $K_n^d = (K_n)^d$ is the *$d$-dimensional Hamming graph.* We call $K_{n_1} \,\square\, K_{n_2} \,\square \cdots \square\, K_{n_d}$ a *generalized $d$-dimensional Hamming graph.* Generalized two-dimensional Hamming graphs are called rook's graphs.

Lindsey [13] solved the edge-isoperimetric problem for generalized $d$-dimensional Hamming graphs. In the *lexicographic order* $\prec_{lex}$, $(a_1, \ldots, a_d) \prec_{lex} (b_1, \ldots, b_d)$ if and only if there exists $i$ ($1 \le i \le d$) such that $a_i < b_i$ and $a_{i'} = b_{i'}$ for each $i' < i$.

**Lemma 2.3** [13]. *Let $p_1 \le p_2 \le \cdots \le p_d$. Then for each $s$, $1 \le s \le \prod_{i=1}^{d} p_i$, the collection of the first $s$ vertices of $K_{p_1} \,\square\, K_{p_2} \,\square \cdots \square\, K_{p_d}$ taken in the lexicographic order $\prec_{lex}$ provides minimum for the function $\theta$.*

## 3. Spanning Tree Congestion of Rook's Graphs

In this section, we determine the spanning tree congestion of generalized two-dimensional Hamming graphs $K_m \,\square\, K_n$. These graphs have several natural characterizations. The *rook's graph* has the vertex set $\{(i, j) \mid i \in [m], j \in [n]\}$ which corresponds to the cells of the $m \times n$ chessboard. A vertex $(i, j)$ in a rook's graph is adjacent to $(i', j')$ if and only if a rook at the cell $(i, j)$ can move to the cell $(i', j')$ (see Figure 1). In other words, $(i, j)$ is adjacent to $(i', j')$ if and only if either $i = i'$ and $j \neq j'$, or $i \neq i'$ and $j = j'$. It is also known that $K_m \,\square\, K_n$ is the line graph of the complete bipartite graph $K_{m,n}$. Line graphs of bipartite graphs are used in the proof of the Strong Perfect Graph Theorem [6]. Several properties of rook's graphs have been studied [1, 2, 7, 11, 14].

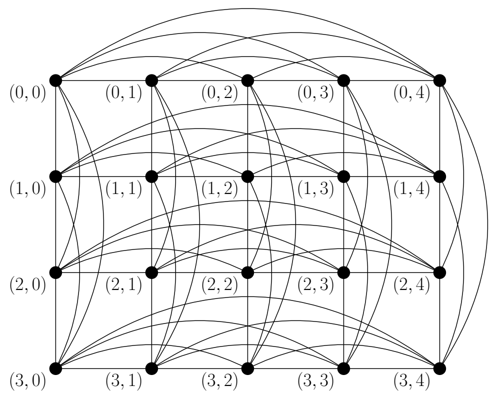



Figure 1. The rook's graph $K_4 \,\square\, K_5$.

Since $K_1 \,\square\, K_n \cong K_n$ and $stc(K_n) = n - 1$ [16], we henceforth assume that $2 \leq m \leq n$. We also write $\theta$ for $\theta_{K_m \square K_n}$ and $\iota$ for $\iota_{K_m \square K_n}$. We call the subgraph induced by the vertex set $\{(i, j) \mid j \in [n]\}$ the *row* $i$, and the subgraph induced by the vertex set $\{(i, j) \mid i \in [m]\}$ the *column* $j$. The following lemma is our main tool.

**Lemma 3.1.** *Let $2 \leq m \leq n$, and let $s = qn + r \leq mn$ for nonnegative integers $q$ and $r < n$. Then, $\theta(s) = (m - q)qn + (m + n - 2q - r - 1)r$.*

***Proof.*** Let $S \subseteq V(K_m \,\square\, K_n)$ be the set of the first $s$ vertices taken in the order $\prec_{lex}$. By Lemma 2.3, $|\theta(S)| = \theta(s)$. It is easy to see that $S$ consists of $q$ rows and $r$ vertices included in another row. Let $R$ denote the set of these

$r$ vertices ($R$ may be empty). There are $\binom{n}{2}$ edges in each row, and $n$ edges between each two rows. There are $\binom{r}{2}$ edges in $R$, and $r$ edges between $R$ and any other of the $q$ rows of $S$. Hence, we have $|\iota(S)| = q\binom{n}{2} + \binom{q}{2}n + \binom{r}{2} + qr$. Since $K_m \,\Box\, K_n$ is $(m+n-2)$-regular, we have

$$|\theta(S)| = (m+n-2)(qn+r) - 2|\iota(S)| = (m-q)qn + (m+n-2q-r-1)r,$$

by Lemma 2.1. ∎

### 3.1. Lower bound

Using Lemmas 2.2 and 3.1, we derive a lower bound for $stc(K_m \,\Box\, K_n)$. We divide the range $\lceil (mn-1)/(m+n-2) \rceil \le s \le \lfloor mn/2 \rfloor$, in Lemma 2.2, into two ranges $\lceil (mn-1)/(m+n-2) \rceil \le s \le n$ and $n < s \le \lfloor mn/2 \rfloor$. This is possible since $m, n \ge 2$ implies $\lceil (mn-1)/(m+n-2) \rceil \le n \le \lfloor mn/2 \rfloor$.

**Lemma 3.2.** *If $2 \le m \le n$, then $\theta(s) \ge \min\left\{\theta(n), \theta\left(\left\lceil \frac{mn-1}{m+n-2} \right\rceil\right)\right\}$ for $\left\lceil \frac{mn-1}{m+n-2} \right\rceil \le s \le n$.*

***Proof.*** By Lemma 3.1, $\theta(s) = -s(s-m-n+1)$ for $s \le n$. Since $-s(s-m-n+1)$ is a concave function in $s$, the lemma holds. ∎

**Lemma 3.3.** *If $2 \le m \le n$, then $\theta(s) \ge \theta(n)$ for $n < s \le \lfloor mn/2 \rfloor$.*

***Proof.*** Let $q$ and $r$ be defined as in Lemma 3.1. Clearly, $1 \le q \le m/2$. By Lemma 3.1, we have $\theta(s) = (m-q)qn + (m+n-2q-r-1)r$, and $\theta(n) = (m-1)n$. Since $1 \le q \le m/2$, we have $(m-q)q \ge m-1$, and so, $(m-q)qn \ge (m-1)n$. Since $q \le m/2$ and $r < n$, we have $m+n-2q-r-1 \ge 0$, and so, $(m+n-2q-r-1)r \ge 0$. Therefore, $\theta(s) = (m-q)qn + (m+n-2q-r-1)r \ge (m-1)n = \theta(n)$. ∎

**Corollary 3.4.** *If $2 \le m \le n$, then $stc(K_m \Box K_n) \ge \min\left\{\theta(n), \theta\left(\left\lceil \frac{mn-1}{m+n-2} \right\rceil\right)\right\}$.*

### 3.2. Upper bound

We show upper bounds that meet the lower bound in the above corollary.

**Lemma 3.5.** *If $2 \le m \le n$, then $stc(K_m \,\Box\, K_n) \le \theta(n)$.*

***Proof.*** The spanning tree $T$ is defined as follows (see Figure 2):

1. For each row $i$, construct the star $K_{1,n-1}$ with the center $(i, 0)$;

2. For the column 0, construct the star $K_{1,m-1}$ with the center $(0,0)$;
3. The union of the constructed stars is $T$.

Each edge $e$ constructed in the first step is a leaf edge of $T$. Thus $cng(e) = \theta(1)$. If an edge $e$ is constructed in the second step, $cng(e) = \theta(n)$. Since $m, n \geq 2$, $\theta(1) = m + n - 2 \leq (m-1)n = \theta(n)$. Thus the lemma holds. ■

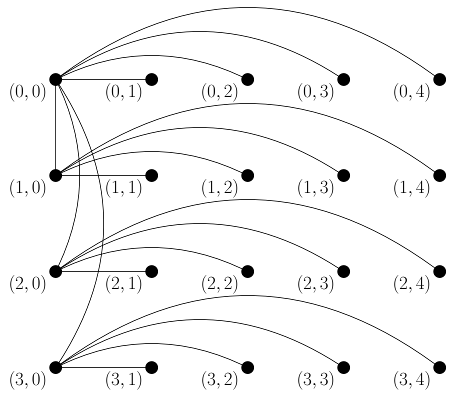



Figure 2. The spanning tree of $K_4 \,\square\, K_5$ in Lemma 3.5.

**Lemma 3.6.** *If $2 \leq m \leq n$, then $stc(K_m \,\square\, K_n) \leq \theta\big(\lceil \frac{mn-1}{m+n-2} \rceil\big)$.*

***Proof.*** Let $x = \lceil \frac{mn-1}{m+n-2} \rceil$. First, we prove the next fact.

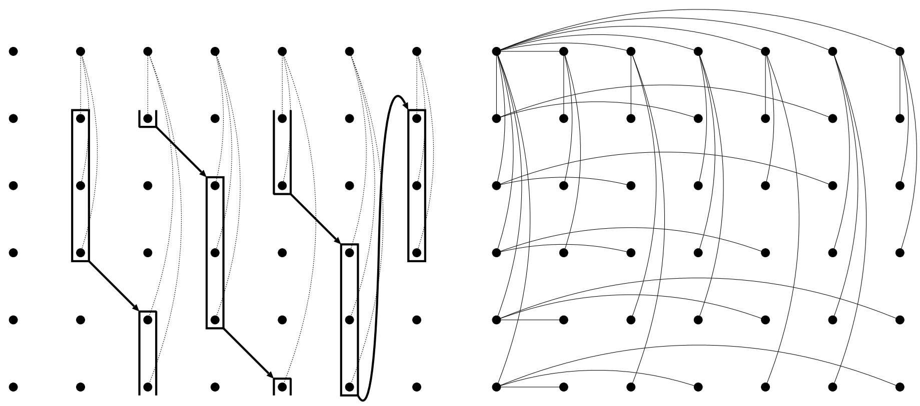
**Proposition 3.7.** *If $2 \leq m \leq n$, then $\theta(s) \leq \theta(x)$ for $s \leq x$.*

***Proof.*** Lemma 3.1 implies $\theta(s) = -s(s - m - n + 1)$ for $s \leq n$. Since this term is monotonously increasing for $s \leq \lceil \frac{m+n-1}{2} \rceil$, the proposition holds provided that $x \leq \lceil \frac{m+n-1}{2} \rceil$. Suppose $x > \lceil \frac{m+n-1}{2} \rceil$; that is, $\lceil \frac{mn-1}{m+n-2} \rceil > \lceil \frac{m+n-1}{2} \rceil$. This implies $\frac{mn-1}{m+n-2} > \frac{m+n-1}{2}$. Simplifying this inequation, we have $(m-1)(m-2) + (n-1)(n-2) < 0$, which contradicts $2 \leq m \leq n$. □

The spanning tree $T$ is constructed as follows (see Figure 3):

1. Construct the star $K_{1,m+n-2}$ with the center $(0,0)$ and the set of the leaves $N_{K_m \square K_n}((0,0)) = \{(i,0) \mid 1 \leq i \leq m-1\} \cup \{(0,j) \mid 1 \leq j \leq n-1\}$;
2. For each column $j$, construct the star $K_{1,x-1}$ with the center $(0,j)$ and the leaves $\{(h(i_j), j), (h(i_j+1), j), \ldots, (h(i_j + x - 2), j)\}$, where $i_j = (j-1)(x-1)$ and $h(i) = (i \bmod m-1) + 1$ (see Figure 3(a));

3. For each row $i$, construct the star with the center $(i, 0)$ whose leaves are the vertices of the row that are not included in any other star constructed in the first and the second steps;
4. The union of the constructed stars is $T$ (see Figure 3(b)).



(a) Consecutive property of leaves of stars in the second step ($x = 4$).

(b) The union of the stars.

Figure 3. The spanning tree of $K_6 \,\square\, K_7$ in Lemma 3.6.

If an edge $e$ in $T$ is not incident to the vertex $(0, 0)$, then $e$ is a leaf edge, and $e$ has congestion $\theta(1) \le \theta(x)$. Suppose that $e$ is incident to $(0, 0)$. Then, either $e = \{(0, 0), (0, j)\}$ or $e = \{(0, 0), (i, 0)\}$.

*Case* 1. $e = \{(0, 0), (0, j)\}$.
In this case, $cng(e) = |\theta(V(T_j))|$, where $T_j$ is the star in the column $j$ constructed in the second step. Thus $|V(T_j)| = x$ and $V(T_j)$ induces a clique. Hence, $cng(e) = \theta(x)$.

*Case* 2. $e = \{(0, 0), (i, 0)\}$.
In this case, $cng(e) = |\theta(V(T_i))|$, where $T_i$ is the star in the row $i$ constructed in the third step. Thus $V(T_i)$ induces a clique, which implies $|\theta(V(T_i))| = \theta(|V(T_i)|)$. Now, it suffices to show that $|V(T_i)| \le x$. In the second step, $(x-1)(n-1)$ vertices of $\{(i, j) \mid 1 \le i \le m-1, 1 \le j \le n-1\}$ are used. Since the vertices are consecutively taken in the second step, the numbers of the remaining vertices in any two rows can differ by at most one. Thus $T_i$ has at most $\lceil \frac{(m-1)(n-1)-(x-1)(n-1)}{m-1} \rceil = \lceil \frac{(m-x)(n-1)}{m-1} \rceil$ leaves.

Suppose $x-1<\left\lceil\frac{(m-x)(n-1)}{m-1}\right\rceil$, which implies $x-1<\frac{(m-x)(n-1)}{m-1}$ since $x$ is an integer. Then, we have $x<\frac{mn-1}{m+n-2}$, which contradicts $x=\left\lceil\frac{mn-1}{m+n-2}\right\rceil$. ∎

**Corollary 3.8.** *If* $2\leq m\leq n$*, then* $stc(K_m\Box K_n)\leq\min\left\{\theta(n),\theta\left(\left\lceil\frac{mn-1}{m+n-2}\right\rceil\right)\right\}$.

Corollaries 3.4 and 3.8 together imply that $stc(K_m\Box K_n)=\min\left\{\theta(n),\theta\left(\left\lceil\frac{mn-1}{m+n-2}\right\rceil\right)\right\}$. We give the main theorem in a more transparent form.

**Theorem 3.9.** *If* $2\leq m\leq n$*, then*

$$stc(K_m\Box K_n)=\begin{cases}(m-1)n & \text{if } m^2-3m+3<n,\\ \left(m+n-1-\left\lceil\frac{mn-1}{m+n-2}\right\rceil\right)\left\lceil\frac{mn-1}{m+n-2}\right\rceil & \text{otherwise.}\end{cases}$$

***Proof.*** Let $x=\left\lceil\frac{mn-1}{m+n-2}\right\rceil$. By Lemma 3.1, $\theta(s)=(m+n-1-s)s$ for $x\leq s\leq n$. Let $f(s)=-s(s-m-n+1)$. Then $f(s)$ is quadratic with maximum in $s=\frac{m+n-1}{2}$. Thus $f(n)=f(m-1)=\theta(n)$. Since $m\leq n$, it holds that $m-1<\frac{m+n-1}{2}<n$. It is easy to see that $x\leq n$. So, $f(m-1)<f(x)$ if and only if $m-1<x$. Since $m-1$ is an integer, $m-1<\left\lceil\frac{mn-1}{m+n-2}\right\rceil$ if and only if $m-1<\frac{mn-1}{m+n-2}$. Simplifying this inequation, we have $m^2-3m+3<n$. ∎

For readers' convenience, we explicitly state the spanning tree congestion of the square rook's graph $K_n^2=K_n\Box K_n$, which is a direct corollary to Theorem 3.9.

**Corollary 3.10.** *If* $n\geq 2$*, then*

$$stc(K_n^2)=\begin{cases}(3n-4)(n+2)/4 & \text{if } n \text{ is even,}\\ (3n-3)(n+1)/4 & \text{if } n \text{ is odd.}\end{cases}$$

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