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WIENER INDEX OF THE TENSOR PRODUCT OF A PATH AND A CYCLE

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Abstract

The *Wiener index,* denoted by $W(G)$, of a connected graph G is the sum of all pairwise distances of vertices of the graph, that is, $W(G) = \frac{1}{2}\sum_{u,v\in V(G)} d(u,v)$. In this paper, we obtain the Wiener index of the tensor product of a path and a cycle.

Keywords: tensor product, Wiener index.

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1. Introduction

For two simple graphs G and H their *tensor product,* denoted by $G\times H$, has vertex set $V(G)\times V(H)$ in which (g_1,h_1) and (g_2,h_2) are adjacent whenever g_1g_2 is an edge in G and h_1h_2 is an edge in H. Let G and H be graphs with vertex sets $V(G) = \{x_1,x_2,\ldots,x_m\}$ and $V(H) = \{y_1,y_2,\ldots,y_n\}$. Then $V(G\times H) = V(G)\times V(H)$ and for our convenience, we write $V(G\times H) = \bigcup_{i=1}^{m} X_i$, where $X_i = \{x_i\}\times V(H)$; we may also write $V(G\times H) = \bigcup_{j=1}^{n} Y_j$, where $Y_j = V(G)\times\{y_j\}$. We shall denote the vertices of X_i by $\{x_{i,j}\ |1\le j\le n\}$ and the vertices of Y_j by $\{x_{i,j}|1\le i\le m\}$, where $x_{i,j}$ stands for the vertex (x_i,y_j). We shall call $X_i, 1\le i\le m$, the i-th *layer* of $G\times H$ and $Y_j, 1\le j\le n$, the j-th *column* of $G\times H$; see Figure 1.

For two disjoint subsets A and B of $V(G)$, $E(A, B)$ denotes the set of edges of G having one end in A and other end in B. Let P_r denote a path on r vertices and let C_s denote a cycle on s vertices. For terms not defined here see [1] or [8].

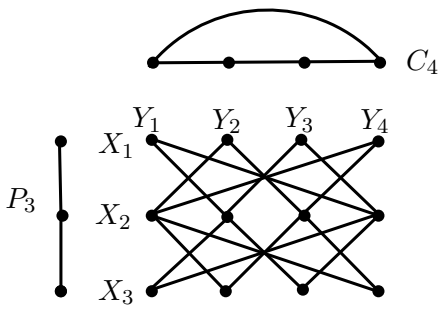

Figure 1. Tensor Product of P_3 and C_4.

The *Wiener index* of a connected graph G, $W(G)$, is defined as $\frac{1}{2}\sum_{u,v\in V(G)} d(u, v)$, where d is the distance function on G. The Wiener index has important applications in chemistry. The graphical invariant $W(G)$ has been studied by many researchers under different names such as distance, transmissions, total status and sum of all distances; see [5, 6, 9]. The chemist Harold Wiener was the first to point out in 1947 that $W(G)$ is well correlated with certain physico-chemical properties of the organic compound.

Besides applications in chemistry, there are many situations in communication, facility location, cryptology, etc., that are effectively modeled by a connected graph G satisfying certain restrictions. Because of cost restraints one is often interested in finding a spanning tree of G that is optimal with respect to one or more properties. Average distance between vertices is frequently one of these properties. Finding a spanning tree T of G that has minimum Wiener index is proved to be important see, [7]. For recent results on Wiener index, see [2, 3, 4, 10]. In this paper, we compute the Wiener index of $P_r \times C_s$.

2. Wiener Index of $P_{2m+1} \times C_{2n+1}$

The following lemma can be seen in [12].

Lemma 1. 1. $W(P_n) = \binom{n+1}{3}$, $n \geq 2$.

2. $W(C_{2n}) = n^3$.
3. $W(C_{2n+1}) = \frac{n(n+1)(2n+1)}{2}$.

It is known that $G \times H$ is connected if G or H is nonbipartite. Hence we consider the tensor product of a path and an odd cycle.

We use the following observations implicitly while finding distances between the vertices of $P_r \times C_s$.

Observation 2. *Let $H = (P_r \times C_s) - E(Y_1, Y_s)$, where Y_i are as defined above; there are two components H_1 and H_2 in H. The vertices in one of the components, say H_1, (resp. H_2) are those (i, j) with i and j are of same (resp. different) parity. By the nature of the graph $P_r \times C_s$, in any shortest path between a pair of distinct vertices of it, consecutive vertices of the path are either in different layers or different columns and hence the length of a shortest path between the vertices is either the number of layers the path visits minus one or number of columns it visits minus one. Further, in any shortest path in $P_r \times C_s$ from $x_{k,1}$, k is odd (resp. even) to a vertex in H_2 (resp. H_1), the path has to use the first edge $x_{k,1}x_{k-1,s}$ or $x_{k,1}x_{k+1,s}$.*

The following observation is helpful in finding a shortest path between a pair of distinct vertices in $P_r \times C_s$.

Observation 3. *A path of length l exists between (u, v) and (x, y) in $G \times H$ only if there exists in G a walk of length l between u and x and a walk of length l between v and y in H.*

The Observation 3 is explained in a different context in [11, p. 273]. In this section, we compute the Wiener index of $P_r \times C_s$, where r and s are odd integers.

Theorem 4. *The Wiener index of $P_{2m+1} \times C_{2n+1}$ is $\frac{2n+1}{3}\big(2m(m+1)\ (m^2 + m + 1) + 3n(n+1)(2m+1)^2\big)$.*

Proof. Let S_{kj} denote the sum of the distances from $x_{k,j}$ to all other vertices of $G = P_{2m+1} \times C_{2n+1}$, that is, $\sum_{v \neq x_{k,j} \in V(G)} d_G(x_{k,j}, v)$. Since there is an automorphism of G which maps $x_{k,i}$ to $x_{k,j}$, $i \neq j$, $S_{kj} = S_{ki}$. Hence, instead of computing S_{rs} for every r and s, it is enough to compute S_{k1} for $k = 1, 2, \ldots, 2m+1$, and then multiply each S_{k1} with number of columns of G to compute $\sum_{u,v \in V(G)} d_G(u, v)$.

For the computation of S_{k1}, for a fixed k, we partition the layers into three sets $\{X_1, X_2, \ldots, X_{k-1}\}$, $\{X_k\}$ and $\{X_{k+1}, X_{k+2}, \ldots, X_{2m+1}\}$ (Note that when $k = 1$ or $2m+1$, the partition consists of only two sets, namely,

$\{X_1\}$, $\{X_2, X_3, \ldots, X_{2m+1}\}$ and $\{X_1, X_2, \ldots, X_{2m}\}$, $\{X_{2m+1}\}$, respectively) and we find distances from $x_{k,1}$ to all the vertices in the layers in the partition separately, that is,

$$\begin{aligned}\sum_{v\in V(G)} d_G(x_{k,1}, v) &= \sum_{\substack{v\in X_i\\ 1\le i\le k-1}} d_G(x_{k,1}, v) + \sum_{v\in X_k} d_G(x_{k,1}, v)\\ &+ \sum_{\substack{v\in X_i\\ k+1\le i\le 2m+1}} d_G(x_{k,1}, v).\end{aligned} \tag{1}$$

First we consider the case n is odd. We divide the proof of the case n is odd into three parts (A), (B) and (C). In (A), we find the distances from $x_{k,1}$ to all the vertices of $\bigcup_{i=1}^{k-1} X_i$, in (B), we find the distances from $x_{k,1}$ to all the vertices of X_k and in (C), we find the distances from $x_{k,1}$ to all the vertices of $\bigcup_{i=k+1}^{2m+1} X_i$.

(A) Initially we find the sum of the distances from $x_{k,1}$ to all the vertices of $\bigcup_{i=1}^{k-1} X_i$. For this, first we assume k is odd (and the case k even will be considered later). We compute $\sum_{v\in X_i} d_G(x_{k,1}, v)$, for a single layer X_i, $1 \le i \le k-1$. $\sum_{v\in X_i} d_G(x_{k,1}, v)$ is given in (2) for i odd and i even separately. If i is odd, then

$$\begin{aligned}\sum_{v\in X_i} d_G(x_{k,1}, v) &= (k-i) + 2(\underbrace{(k-i) + \cdots + (k-i)}_{\frac{k-i}{2}\ times} + (k-i+2)\\ &+ (k-i+4) + \cdots + (n-1)) + 2(2n + (2n-2) + \cdots + (n+1)).\end{aligned}$$

If i is even, then

$$\begin{aligned}\sum_{v\in X_i} d_G(x_{k,1}, v) &= (2n+1) + 2((2n-1) + (2n-3) + \cdots + (n+2))\\ &+ 2(\underbrace{(k-i) + \cdots + (k-i)}_{\frac{k-i+1}{2}\ times} + (k-i+2) + (k-i+4) + \cdots + n).\end{aligned} \tag{2}$$

Explanations for the terms appearing in (2) are as follows.
If i is odd, then $d_G(x_{k,1}, x_{i,1}) = k-i$ and, $d_G(x_{k,1}, x_{i,j}) = k-i$, $j = 3, 5, \ldots, i$ and the respective shortest paths are similar to the one shown in Figure 2. The distances from $x_{k,1}$ to the vertices $x_{i,i+2}, x_{i,i+4}, \ldots, x_{i,n}$ are

$k-i+2, k-i+4, \ldots, n-1$, respectively, and the corresponding shortest paths are similar to the one shown in Figure 3. The distances from $x_{k,1}$ to the vertices $x_{i,2}, x_{i,4}, x_{i,6}, \ldots, x_{i,n+1}$ are $2n, 2n-2, 2n-4, \ldots, n+1$, respectively, and the corresponding shortest paths are similar to the one shown in Figure 4.

If i is even, then $d_G(x_{k,1}, x_{i,1}) = 2n+1$ and, the distances from $x_{k,1}$ to the vertices $x_{i,3}, x_{i,5}, x_{i,7}, \ldots, x_{i,n}$ are $2n-1, 2n-3, 2n-5, \ldots, n+2$, respectively, and the corresponding shortest paths are similar to the one shown in Figure 4. Further, $d_G(x_{k,1}, x_{i,j}) = k-i,\ j = 2, 4, \ldots, i$ and the corresponding shortest paths are similar to the one shown in Figure 2. The distances from $x_{k,1}$ to the vertices $x_{i,i+2}, x_{i,i+4}, \ldots, x_{i,n+1}$ are $k-i+2$, $k-i+4, \ldots, n$, respectively, and the corresponding shortest paths are similar to the one shown in Figure 3.

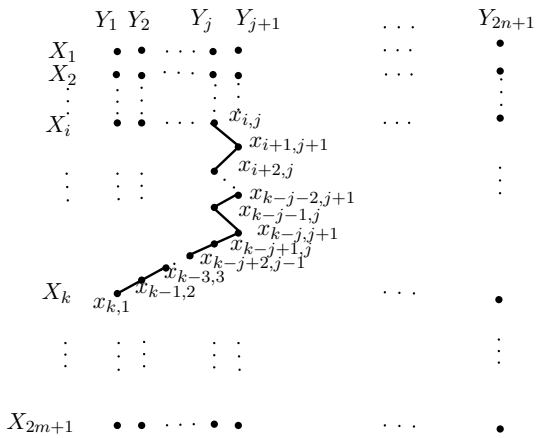

Figure 2. Vertices of $P_{2m+1} \times C_{2n+1}$.

The multiplication factor 2 appears in the sum of (2), except for one term, because $d_G(x_{k,1}, x_{k,j}) = d_G(x_{k,1}, x_{k,2n-j+3})$, $2 \le j \le n+1$. Summing the terms of (2), that is, the summing the distances from $x_{k,1}$ to all the vertices of X_i, gives,

$$\sum_{v \in X_i} d_G(x_{k,1}, v) = \begin{cases} \frac{(k-i)^2}{2} + 2n(n+1) & \text{if } i \text{ is odd,} \\ \frac{(k-i)^2}{2} + 2n(n+1) + \frac{1}{2} & \text{if } i \text{ is even.} \end{cases} \tag{3}$$

Hence,

$$\sum_{\substack{v\in X_i\\ 1\le i\le k-1}} d_G(x_{k,1},v) = \sum_{1\le i\le k-1}\left(\frac{(k-i)^2}{2}+2n(n+1)\right)+\underbrace{\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2}}_{\frac{k-1}{2}\ \ times}$$

$$= \frac{k(k-1)(2k-1)}{12}+2n(n+1)(k-1)+\frac{k-1}{4}. \tag{4}$$

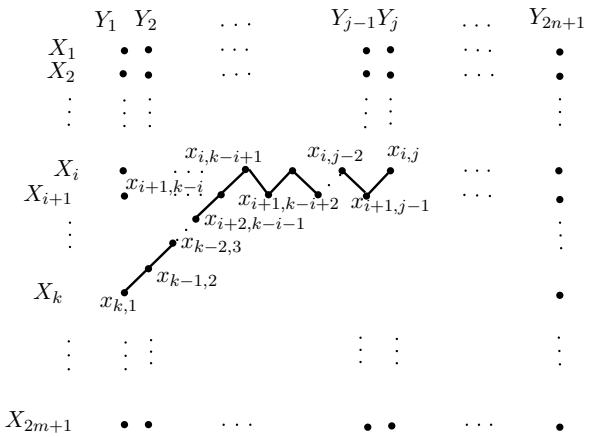

Figure 3. Vertices of $P_{2m+1}\times C_{2n+1}$.

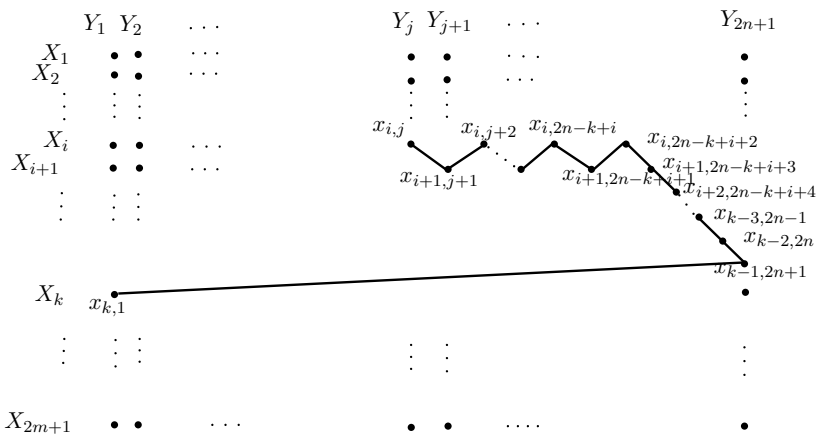

Figure 4. Vertices of $P_{2m+1}\times C_{2n+1}$.

Next we assume that k is even. If i is odd, then this case is similar to the case k odd and i even considered above further, if i is even, then this case is similar to the case k and i are odd discussed above. Consequently, the following equation is same as Equation (3) except for the parity of i. Therefore, for a fixed i,

$$\sum_{v\in X_i} d_G(x_{k,1},v) = \begin{cases} \frac{(k-i)^2}{2} + 2n(n+1) + \frac{1}{2}, & \text{if } i \text{ is odd,} \\ \frac{(k-i)^2}{2} + 2n(n+1), & \text{if } i \text{ is even.} \end{cases} \tag{5}$$

Hence,

$$\begin{aligned} \sum_{\substack{v\in X_i \\ 1\le i\le k-1}} d_G(x_{k,1},v) &= \sum_{1\le i\le k-1}\left(\frac{(k-i)^2}{2} + 2n(n+1)\right) + \underbrace{\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2}}_{\frac{k}{2}\ \ times} \\ &= \frac{k(k-1)(2k-1)}{12} + 2n(n+1)(k-1) + \frac{k}{4}. \end{aligned} \tag{6}$$

Thus the Equations (4) and (6) give the sum of the distances from $u = x_{k,1}$ to all the vertices of $\bigcup_{i=1}^{k-1} X_i$.

(B) Next we find the sum of the distances from $u = x_{k,1}$ to all other vertices of X_k.

$$\begin{aligned} \sum_{v=x_{k,j}\in X_k} d_G(x_{k,1},v) &= 2\left(2+4+\cdots+(n-1)\right) \\ &\quad + 2\left(2n+(2n-2)+\cdots+(n+1)\right). \end{aligned} \tag{7}$$

Explanations for the terms appearing in (7) are described below. $d_G(x_{k,1},x_{k,j}) = j-1$, $j = 3,5,\ldots,n$ and the corresponding shortest paths are similar to the one shown in Figure 5. Further, $d_G(x_{k,1},x_{k,j}) = 2n-j+2$, $j = 2,4,\ldots,n+1$, the corresponding shortest paths are similar to the one shown in Figure 6. The multiplication factor 2 appears in the Equation (7) because $d_G(x_{k,1},x_{k,j}) = d_G(x_{k,1},x_{k,2n-j+3})$, $2 \le j \le n+1$. Hence

$$\begin{aligned} \sum_{v=x_{k,j}\in X_k} d_G(x_{k,1},v) &= 2(2+4+6+\cdots+2n) \\ &= 2n(n+1). \end{aligned} \tag{8}$$

Thus we have obtained the sum of the distances from $x_{k,1}$ to all other vertices of X_k.

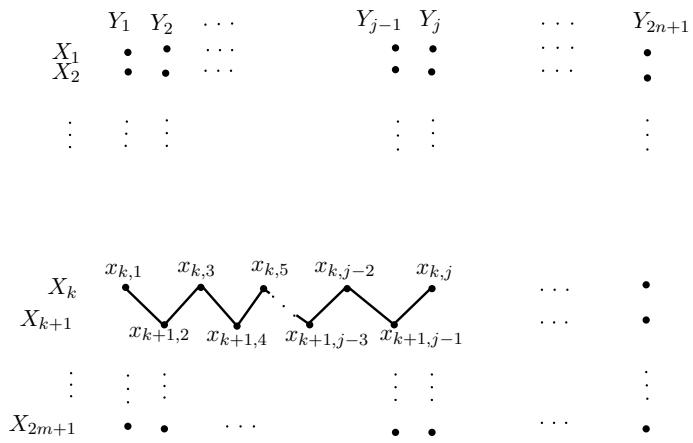

Figure 5. Vertices of $P_{2m+1} \times C_{2n+1}$.

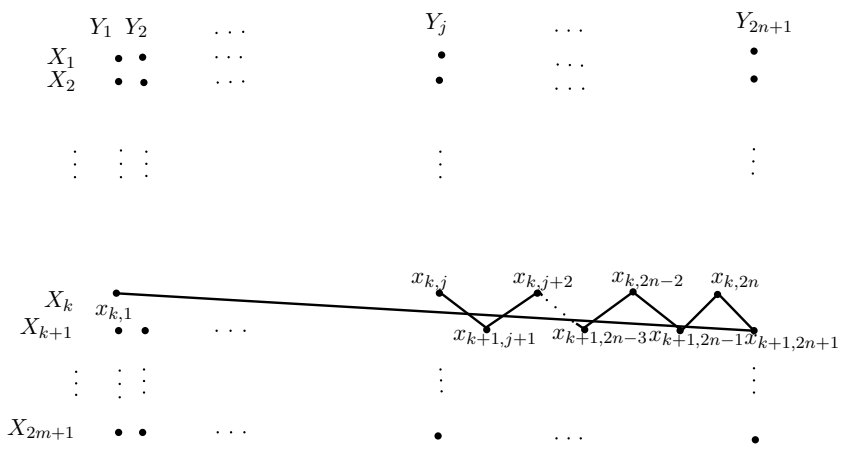

Figure 6. Vertices of $P_{2m+1} \times C_{2n+1}$.

(C) Finally, we find the sum of the distances from $u = x_{k,1}$ to all the vertices of $\bigcup_{i=k+1}^{2m+1} X_i$. For this, it is enough to replace $k - i$ by $i - k$, in the argument given in (A). Hence, if k is odd, then for a fixed i,

$$\sum_{v \in X_i} d_G(x_{k,1}, v) = \begin{cases} \frac{(i-k)^2}{2} + 2n(n+1) & \text{if } i \text{ is odd,} \\ \frac{(i-k)^2}{2} + 2n(n+1) + \frac{1}{2} & \text{if } i \text{ is even.} \end{cases} \tag{9}$$

Hence,

$$\sum_{\substack{v\in X_i\\ k+1\le i\le 2m+1}} d_G(x_{k,1},v) = \sum_{k+1\le i\le 2m+1}\left(\frac{(i-k)^2}{2}+2n(n+1)\right)+\underbrace{\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2}}_{\frac{2m-k+1}{2}\ \ times}$$

$$= \left(\frac{(2m-k+1)(2m-k+2)(4m-2k+3)}{12}+2n(n+1)(2m-k+1)\right) + \frac{(2m-k+1)}{4}. \tag{10}$$

If k is even, then for a fixed i,

$$\sum_{v\in X_i} d_G(x_{k,1},v) = \begin{cases} \frac{(i-k)^2}{2}+2n(n+1)+\frac{1}{2} & \text{if } i \text{ is odd,}\\ \frac{(i-k)^2}{2}+2n(n+1) & \text{if } i \text{ is even.}\end{cases} \tag{11}$$

Hence,

$$\sum_{\substack{v\in X_i\\ k+1\le i\le 2m+1}} d_G(x_{k,1},v) = \sum_{k+1\le i\le 2m+1}\left(\frac{(i-k)^2}{2}+2n(n+1)\right)+\underbrace{\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2}}_{\frac{2m-k+2}{2}\ \ times}$$

$$= \frac{(2m-k+1)(2m-k+2)(4m-2k+3)}{12}+2n(n+1)(2m-k+1) + \frac{(2m-k+2)}{4}. \tag{12}$$

From (4), (8) and (10), when k is odd,

$$\begin{aligned}&\sum_{v\in V(G)} d_G(x_{k,1},v) = \\ &= \frac{k(k-1)(2k-1)}{12}+2n(n+1)(k-1)+\frac{k-1}{4}+2n(n+1)\\ &+\frac{(2m-k+1)(2m-k+2)(4m-2k+3)}{12}\\ &+2n(n+1)(2m-k+1)+\frac{(2m-k+1)}{4}.\end{aligned} \tag{13}$$

From (6), (8) and (12), when k is even,

$$\begin{aligned}
&\sum_{v\in V(G)} d_G(x_{k,1},v) = \\
&= \frac{k(k-1)(2k-1)}{12} + 2n(n+1)(k-1) + \frac{k}{4} + 2n(n+1) \\
&+ \frac{(2m-k+1)(2m-k+2)(4m-2k+3)}{12} \\
&+ 2n(n+1)(2m-k+1) + \frac{2m-k+2}{4}.
\end{aligned} \tag{14}$$

Combining (13) and (14) and summing over $k = 1, 2, \ldots, 2m+1$, we get

(15)

$$\begin{aligned}
&\sum_{k=1}^{2m+1} \sum_{v\in V(G)} d_G(x_{k,1},v) = \\
&= \sum_{k=1}^{2m+1} \left(\frac{k(k-1)(2k-1)}{12} + \frac{(2m-k+1)(2m-k+2)(4m-2k+3)}{12} \right) \\
&+ \sum_{k=1}^{2m+1} \Big(2n(n+1)(k-1) + 2n(n+1) + 2n(n+1)(2m-k+1) \Big) \\
&+ \underbrace{\frac{m}{2} + \frac{m}{2} + \cdots + \frac{m}{2}}_{(m+1)\ \ times} + \underbrace{\frac{m+1}{2} + \frac{m+1}{2} + \cdots + \frac{m+1}{2}}_{m\ \ times} \\
&= \frac{m}{6}(m+1)(2m+1)^2 + \frac{m}{6}(m+1)(2m+1)^2 + 2n(n+1)m(2m+1) \\
&+ 2n(n+1)(2m+1) + 2n(n+1)m(2m+1) + \frac{m(m+1)}{2} \\
&+ \frac{m(m+1)}{2} \\
&= \frac{2}{3}(2m(m+1)(m^2+m+1) + 3n(n+1)(2m+1)^2).
\end{aligned}$$

As there is an automorphism of G which maps $x_{k,1}$ to $x_{k,j}$, $j \neq 1$, the sum of the distances from $x_{k,1}$ to all the vertices of G is same as the sum of the

distances from $x_{k,j},\ 2 \le j \le 2n+1$, to all the vertices of G, we have

$$\begin{aligned}
&\sum_{u,v\in V(G)} d_G(u,v) = \\
&= (2n+1)\left(\sum_{k=1}^{2m+1}\sum_{v\in V(G)} d_G(x_{k,1},v)\right) \\
&= (2n+1)\left(\frac{2}{3}(2m(m+1)(m^2+m+1)+3n(n+1)(2m+1)^2)\right) \\
&= \frac{2(2n+1)}{3}\left(2m(m+1)(m^2+m+1)+3n(n+1)(2m+1)^2\right).
\end{aligned} \tag{16}$$

The proof is similar when n is even and in this case also $\sum_{u,v\in V(G)} d_G(u,v)$ is found to be the same as (16); we omit the details. Hence, irrespective of the parity of n, we have

$$\sum_{u,v\in V(G)} d_G(u,v) = \frac{2(2n+1)}{3}\left(2m(m+1)(m^2+m+1)+3n(n+1)(2m+1)^2\right).$$

Hence,

$$\begin{aligned}
W(P_{2m+1}\times C_{2n+1}) &= \frac{1}{2}\sum_{u,v\in V(G)} d_G(u,v) \\
&= \frac{(2n+1)}{3}\left(2m(m+1)(m^2+m+1)+3n(n+1)(2m+1)^2\right). \qquad \blacksquare
\end{aligned}$$

3. Wiener Index of $P_{2m}\times C_{2n+1}$.

In this section, we compute the Wiener index of the tensor product of an even length path and an odd cycle.

Theorem 5. *The Wiener index of $P_{2m}\times C_{2n+1}$ is $\frac{m^2(2n+1)}{3}((2m^2+1)+12n(n+1))$.*

Proof. As in the proof of the previous theorem, we consider (A), (B) and (C). (A) and (B) are the same as in the previous theorem and hence, for (A) and (B) we consider the sum as in the proof of the previous theorem. Also, in (C), the summation varies from $k+1$ to $2m$ instead of $k+1$ to $2m+1$.

Hence we consider the sum corresponding to (C) of the above theorem. Therefore, if k is odd, then for a fixed i,

$$\sum_{v\in X_i} d_G(x_{k,1},v) = \begin{cases} \frac{(i-k)^2}{2} + 2n(n+1) & \text{if } i \text{ is odd,} \\ \frac{(i-k)^2}{2} + 2n(n+1) + \frac{1}{2} & \text{if } i \text{ is even.} \end{cases} \tag{17}$$

Hence,

$$\begin{aligned} &\sum_{\substack{v\in X_i \\ k+1\le i\le 2m}} d_G(x_{k,1},v) = \\ &= \sum_{k+1\le i\le 2m} \left(\frac{(i-k)^2}{2} + 2n(n+1)\right) + \underbrace{\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2}}_{\frac{2m-k+1}{2}\ \ times} \\ &= \frac{(2m-k)(2m-k+1)(4m-2k+1)}{12} + 2n(n+1)(2m-k) \\ &+ \frac{(2m-k+1)}{4}. \end{aligned} \tag{18}$$

If k is even, then for a fixed i,

$$\sum_{v\in X_i} d_G(x_{k,1},v) = \begin{cases} \frac{(i-k)^2}{2} + 2n(n+1) + \frac{1}{2} & \text{if } i \text{ is odd,} \\ \frac{(i-k)^2}{2} + 2n(n+1) & \text{if } i \text{ is even.} \end{cases} \tag{19}$$

Hence,

$$\begin{aligned} &\sum_{\substack{v\in X_i \\ k+1\le i\le 2m}} d_G(x_{k,1},v) = \\ &= \sum_{k+1\le i\le 2m} \left(\frac{(i-k)^2}{2} + 2n(n+1)\right) + \underbrace{\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2}}_{\frac{2m-k}{2}\ \ times} \\ &= \frac{(2m-k)(2m-k+1)(4m-2k+1)}{12} + 2n(n+1)(2m-k) \\ &+ \frac{(2m-k)}{4}. \end{aligned} \tag{20}$$

From (4), (8) and (18), when k is odd,

$$\sum_{v\in V(G)} d_G(x_{k,1},v) =$$

$$= \frac{k(k-1)(2k-1)}{12} + 2n(n+1)(k-1) + \frac{k-1}{4} + 2n(n+1) \tag{21}$$

$$+ \frac{(2m-k)(2m-k+1)(4m-2k+1)}{12} + 2n(n+1)(2m-k) + \frac{m}{2}$$

(where (4) and (8) are taken from the proof of Theorem 4).

From (6), (8) and (20), when k is even,

$$\sum_{v\in V(G)} d_G(x_{k,1},v) =$$

$$= \frac{k(k-1)(2k-1)}{12} + 2n(n+1)(k-1) + \frac{k}{4} + 2n(n+1) \tag{22}$$

$$+ \frac{(2m-k)(2m-k+1)(4m-2k+1)}{12} + 2n(n+1)(2m-k) + \frac{m}{2}$$

(where (4) and (8) are taken from the proof of Theorem 4).
Combining (21) and (22) and summing over $k = 1, 2, \ldots, 2m$, we get

$$\sum_{k=1}^{2m}\sum_{v\in V(G)} d_G(x_{k,1},v) =$$

$$= \sum_{k=1}^{2m}\left(\frac{k(k-1)(2k-1)}{12} + \frac{(2m-k)(2m-k+1)(4m-2k+1)}{12}\right)$$

$$+ \sum_{k=1}^{2m}(2n(n+1)(k-1) + 2n(n+1) + 2n(n+1)(2m-k))$$

$$+ \underbrace{\frac{m}{2} + \frac{m}{2} + \cdots + \frac{m}{2}}_{m \quad times} + \underbrace{\frac{m}{2} + \frac{m}{2} + \cdots + \frac{m}{2}}_{m \quad times} \tag{23}$$

$$= \frac{m^2}{6}(2m+1)(2m-1) + \frac{m^2}{6}(2m+1)(2m-1)$$

$$+ 2n(n+1)m(2m-1) + 4n(n+1)m + 2n(n+1)m(2m-1)$$

$$+ \frac{m^2}{2} + \frac{m^2}{2} = \frac{2m^2}{3}\left((2m^2+1) + 12n(n+1)\right).$$

As there is an automorphism of G which maps $x_{k,1}$ to $x_{k,j}$, $j \neq 1$, the sum of the distances from $x_{k,1}$ to all the vertices of G is same as the sum of the distances from $x_{k,j}$, $2 \leq j \leq 2n+1$, to all the vertices of G and hence we have

$$\begin{aligned}
\sum_{u,v \in V(G)} d_G(u,v) &= (2n+1)\left(\sum_{k=1}^{2m} \sum_{v \in V(G)} d_G(x_{k,1}, v)\right) \\
&= (2n+1)\left(\frac{2m^2}{3}\left((2m^2+1) + 12n(n+1)\right)\right) \qquad (24) \\
&= \frac{2m^2(2n+1)}{3}\left((2m^2+1) + 12n(n+1)\right).
\end{aligned}$$

The proof is similar when n is even and in this case also $\sum_{u,v \in V(G)} d_G(u,v)$ is found to be the same as (24); we omit the details. Hence, irrespective of the parity of n, we have

$$\sum_{u,v \in V(G)} d_G(u,v) = \frac{2m^2(2n+1)}{3}\left((2m^2+1) + 12n(n+1)\right).$$

Hence,

$$\begin{aligned}
W(P_{2m} \times C_{2n+1}) &= \frac{1}{2} \sum_{u,v \in V(G)} d_G(u,v) \\
&= \frac{m^2(2n+1)}{3}\left((2m^2+1) + 12n(n+1)\right).
\end{aligned}$$

■

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