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UPPER BOUNDS ON THE b-CHROMATIC NUMBER AND RESULTS FOR RESTRICTED GRAPH CLASSES

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Abstract

A b-coloring of a graph G by k colors is a proper vertex coloring such that every color class contains a color-dominating vertex, that is, a vertex having neighbors in all other $k-1$ color classes. The b-chromatic number $\chi_b(G)$ is the maximum integer k for which G has a b-coloring by k colors. Moreover, the graph G is called b-continuous if G admits a b-coloring by k colors for all k satisfying $\chi(G) \leq k \leq \chi_b(G)$. In this paper, we establish four general upper bounds on $\chi_b(G)$. We present results on the b-chromatic number and the b-continuity problem for special graphs, in particular for disconnected graphs and graphs with independence number 2. Moreover we determine $\chi_b(G)$ for graphs G with minimum degree $\delta(G) \geq |V(G)| - 3$, graphs G with clique number $\omega(G) \geq |V(G)| - 3$, and graphs G with independence number $\alpha(G) \geq |V(G)| - 2$. We also prove that these graphs are b-continuous.

Keywords: coloring, b-coloring, b-chromatic number, b-continuity.

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1. Introduction

We refer to [3] for terminology and notation not defined here and consider in this paper only finite, simple, and undirected graphs.

The concept of b-colorings and the b-chromatic number were introduced by Irving and Manlove [5] and have already been investigated in several papers (cf. [1, 2, 4, 5, 6, 8, 9]). A *b-coloring* of a graph G by k colors is a proper vertex coloring such that each color class contains a vertex having neighbors in all other $k-1$ color classes. Such a vertex is called a *color-dominating vertex*. The *b-chromatic number* $\chi_b(G)$ is the maximum integer k for which G has a b-coloring by k colors. Irving and Manlove [5] proved that the problem to decide whether $\chi_b(G) \geq K$ for a given graph G and an integer K is $\mathcal{NP}$-complete. So we are interested in the determination of bounds for $\chi_b(G)$ in general and exact values on $\chi_b(G)$ for special graphs or graph classes, respectively.

Since any optimal vertex coloring by $\chi(G)$ colors is a b-coloring, it follows that $\chi_b(G) \geq \chi(G)$ for every graph G. The *b-spectrum* $S_b(G)$ of a graph G is the set of integers k, for which a b-coloring of G by k colors exists. Clearly, $S_b(G) \subseteq \{\chi(G), \ldots, \chi_b(G)\}$. A graph G is called *b-continuous* if $S_b(G) = \{\chi(G), \ldots, \chi_b(G)\}$. Interestingly enough, there exist graphs which are not b-continuous. For instance, a graph that can be obtained from the complete bipartite graph $K_{r,r}$, $r \geq 4$, by removing a perfect matching is not b-continuous since it only has b-colorings by k colors for $k = 2$ and $k = r$. Barth, Cohen, and Faik [1] proved that the problem to decide whether a graph G is b-continuous is $\mathcal{NP}$-complete, even if b-colorings by $\chi(G)$ and by $\chi_b(G)$ colors are part of the input. So it is an intriguing task to find (classes of) graphs which are b-continuous.

In Section 3, we establish four general upper bounds on $\chi_b(G)$ and give sharpness examples. Then we deal with restricted graph classes. In Section 4.1 we consider disconnected graphs and mainly investigate whether there is a connection between $\chi_b(G_1 \cup \cdots \cup G_r)$ and the b-chromatic numbers $\chi_b(G_1), \ldots, \chi_b(G_r)$ of the single components. After this, graphs with independence number 2 are discussed in Section 4.2. We present bounds on the b-chromatic number for these graphs and prove that they are b-continuous. In the last two Sections 4.3 and 4.4 we determine the b-chromatic number for graphs G having $\delta(G) \geq |V(G)|-3$, $\omega(G) \geq |V(G)|-3$, or $\alpha(G) \geq |V(G)|-2$, and we also prove that these graphs are b-continuous.

Further Notation. Let $G = (V, E)$ be a graph.

Throughout the paper we use several graph invariants of G, which are: the order $n(G) = |V(G)|$, the chromatic number $\chi(G)$, the maximum degree $\Delta(G)$, the minimum degree $\delta(G)$, the independence number $\alpha(G)$, the clique number $\omega(G)$, the clique cover number $\theta(G)$, and the matching number $\nu(G)$. $\overline{G}$ denotes the complement of G.

For a vertex $v \in V$, $N(v)$ or $N_G(v)$ is the set of vertices which are adjacent to v in G. The degree of v is the cardinality of $N_G(v)$ and is denoted by $d(v)$ or $d_G(v)$. For a subset $V' \subseteq V$ we denote the subgraph induced by V' with $G[V']$.

In the following "coloring" means a proper vertex coloring where adjacent vertices receive different colors. Sometimes we write "proper coloring" in order to emphasize that the coloring is proper. For a coloring c and $V' \subseteq V$ we define $c(V') := \{c(v) \mid v \in V'\}$. Moreover, a coloring c of G by k colors shall always be a function $c : V \to \{1, \ldots, k\}$ where all k colors are used.

Let $G_i = (V_i, E_i)$, $i = 1, \ldots, r$, be r disjoint graphs. The *union* $G := G_1 \cup \cdots \cup G_r$ has vertex set $V(G) = \bigcup_{i=1}^r V_i$ and edge set $E(G) = \bigcup_{i=1}^r E_i$. The *join* $G := G_1 + \cdots + G_r$ has vertex set $V(G) = \bigcup_{i=1}^r V_i$ and edge set $E(G) = \bigcup_{i=1}^r E_i \ \cup \ \{\{x, y\} \mid x \in V_i,\, y \in V_j,\, i, j \in \{1, \ldots, r\},\, i \neq j\}$.

2. PRELIMINARIES

It is known that every graph G satisfies

$$\omega(G) \leq \chi(G) \leq \chi_b(G) \leq \Delta(G) + 1. \tag{1}$$

Moreover, Kouider and Mahéo proved the following Nordhaus-Gaddum-type result:

Theorem 1 [8]**.** *For every graph G, $\chi_b(G) + \chi_b(\overline{G}) \leq n(G) + 1$.*

Since $\chi_b(\overline{G}) \geq \omega(\overline{G}) = \alpha(G)$ this implies:

Theorem 2 [8]**.** *For every graph G, $\chi_b(G) \leq n(G) - \alpha(G) + 1$.*

From above we easily deduce:

Proposition 3. $\chi_b(G) \leq \lceil \frac{n(G) - \alpha(G) + \Delta(G) + 1}{2} \rceil$ *for every graph G.*

Definition 4. For $s, t \geq 1$ let $H_1(s,t) = (V, E_1)$ and $H_2(s,t) = (V, E_2)$ be the graphs with $V = X \cup Y \cup Z$, $E_1 = A \cup B \cup C$, and $E_2 = B \cup C$ where:

$$\begin{aligned} X &= \{x_1, \dots, x_s\},\ Y = \{y_1, \dots, y_t\},\ Z = \{z_1, \dots, z_t\}, \\ A &= \big\{\{x_i, x_j\} \mid 1 \leq i < j \leq s\big\}, \\ B &= \big\{\{x_i, y_j\} \mid 1 \leq i \leq s, 1 \leq j \leq t\big\}, \text{ and} \\ C &= \big\{\{y_i, z_j\} \mid 1 \leq i, j \leq t,\ i \neq j\big\}. \end{aligned}$$

Lemma 5. *For $s, t \geq 1$ the graphs $H_1 := H_1(s,t)$ and $H_2 := H_2(s,t)$ satisfy $n(H_1) = n(H_2) = s + 2t$, $\Delta(H_1) = \Delta(H_2) = s + t - 1$, and*

(a) *$\alpha(H_1) = t + 1$, $\omega(H_1) = \chi(H_1) = s + 1$, $\chi_b(H_1) = s + t$, and if $t \geq 3$, then H_1 has no b-coloring by $s + 2$ colors,*

(b) *$\alpha(H_2) = s + t$, $\omega(H_2) = \chi(H_2) = 2$, $\chi_b(H_2) = t + 1$, and if $t \geq 3$, then H_2 has no b-coloring by 3 colors.*

Proof. Clearly, $n(H_1) = n(H_2) = s + 2t$ and $\Delta(H_1) = \Delta(H_2) = s + t - 1$.

(a) It can easily be verified that $\omega(H_1) = s + 1$, and $\alpha(H_1) = t + 1$. The coloring $c_\chi : V \to \{1, \dots, s+1\}$ defined by $c_\chi(x_i) = i$ for $i = 1, \dots, s$ and $c_\chi(y_j) = s + 1$, $c_\chi(z_j) = 1$ for $j = 1, \dots, t$, is a proper coloring of H_1 by $\omega(H_1)$ colors. Hence, $\chi(H_1) = \omega(H_1)$.

Moreover, the b-chromatic number satisfies $\chi_b(G) \leq \Delta(H_1) + 1 = s + t$ (by (1)). We notice that this bound is attained since the coloring $c_b : V \to \{1, \dots, s+t\}$ defined by $c_b(x_i) = i$ for $i = 1, \dots, s$ and $c_b(y_j) = c_b(z_j) = j + s$ for $j = 1, \dots, t$ is a proper b-coloring of H_1.

Let $t \geq 3$ and suppose that there exists a b-coloring c of H_1 by $s + 2$ colors.

Since the vertices from X induce a clique we may assume w.l.o.g. that $c(x_i) = i$ for $i = 1, \dots, s$. This (along with $s \geq 1$) implies $c(y_j) \in \{s+1, s+2\}$ and $1 \notin c(N(z_j)) \subseteq \{s+1, s+2\}$ for $j \in \{1, \dots, t\}$. Hence, the color-dominating vertices of color $s + 1$ and $s + 2$, respectively, are not in Z but in Y. W.l.o.g. let y_1, y_2 be color-dominating vertices with $c(y_1) = s + 1$ and $c(y_2) = s + 2$. The former implies $c(z_2) = s + 2$ and the latter implies $c(z_1) = s + 1$. Since $N(y_3) = \{x_1, \dots, x_s, z_1, z_2\}$ it follows that $c(N(y_3)) = \{1, \dots, s + 2\}$. So there is no color left for y_3 and therefore c cannot be a proper vertex coloring by $s + 2$ colors, a contradiction.

(b) Since H_2 is bipartite and $X \cup Z$ is a maximum independent set of H_2 we deduce that $\omega(H_2) = \chi(H_2) = 2$, and $\alpha(H_2) = s + t$.

The b-chromatic number satisfies $\chi_b(H_2) \leq n(H_2) - \alpha(H_2) + 1 = t + 1$ (compare Theorem 2). This bound is attained since the coloring $c_b : V \to \{1, \ldots, t+1\}$ defined by $c_b(x_i) = 1$ for $i = 1, \ldots, s$ and $c_b(y_j) = c_b(z_j) = j+1$ for $j = 1, \ldots, t$ is a proper b-coloring of H_2.

Let $t \geq 3$ and suppose that there exists a b-coloring c of H_2 by 3 colors.

Suppose that there exists an integer $j \in \{1, \ldots, t\}$ such that $c(y_j) \neq c(z_j)$. W.l.o.g. let $j = 1$, $c(y_1) = 1$ and $c(z_1) = 2$. Thus, $1 \notin c(X \cup Z)$ and $2 \notin c(Y)$. Since X, Y, and Z are independent sets it follows that there cannot exist a color-dominating vertex of color 3, a contradiction. Hence, $c(y_j) = c(z_j)$ is satisfied for every integer $j \in \{1, \ldots, t\}$. Since $\{y_j, z_l\} \in E(H_2)$ for $j, l \in \{1, \ldots, t\}$, $j \neq l$, we may assume w.l.o.g. that $c(y_j) = c(z_j) = j$ for $j = 1, \ldots, t$. This implies $c(N(x_1)) = \{1, \ldots, t\}$ and $c(x_1) \geq t + 1 \geq 4$. Thus, c cannot be a proper vertex coloring by 3 colors, a contradiction. ■

Remark 6. By case analysis we can prove that every graph of order at most 6 is b-continuous and the graph $H_1(1, 3)$ shown in Figure 1 is the only non-b-continuous graph of order 7. This bipartite graph has b-chromatic number 4 but does not allow a b-coloring by 3 colors.

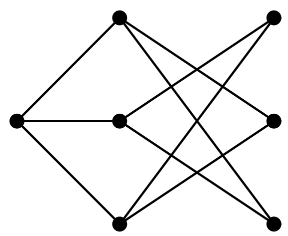


Figure 1. The smallest non-b-continuous graph.

3. UPPER BOUNDS ON $\chi_b(G)$

In the next four theorems we establish new bounds on the b-chromatic number. The proof of these results follows at the end of the section.

Theorem 7. *For every non-complete graph G, $\chi_b(G) \leq \lceil \frac{n(G)+\omega(G)}{2} \rceil - 1$.*

Corollary 8. *Every triangle-free graph G of order $n \geq 3$ satisfies $\chi_b(G) \leq \lceil \frac{n}{2} \rceil$. This bound is sharp for the cycle C_5 and every complete bipartite graph $K_{r,r}$ where a perfect matching is removed.*

The graph $H_1(s,t)$ is a sharpness example for Theorem 7 as well since $\chi_b(H_1(s,t)) = s+t = \Delta(H_1(s,t))+1 = \lceil \frac{n(H_1(s,t))+\omega(H_1(s,t))}{2} \rceil - 1$. Moreover, we also can find sharpness examples G for which $\lceil \frac{n(G)+\omega(G)}{2} \rceil - 1 < \Delta(G)+1$.

Observation 9. *For every integer $l \geq 2$ there exists a graph G of order $n = 3(l-1)$ such that $\omega(G) = l$ and $\chi_b(G) = \lceil \frac{n+l}{2} \rceil - 1$.*

Proof. Let $G = (V, E)$ be a graph with vertex set $V = X \cup Y \cup Z$ and edge set $E = A \cup B \cup C$ such that

$$\begin{aligned} X &= \{x_1, \ldots, x_{l-1}\},\ Y = \{y_1, \ldots, y_{l-1}\},\ Z = \{z_1, \ldots, z_{l-1}\}, \\ A &= \big\{\{x_i, x_j\} \mid 1 \leq i < j \leq l-1\big\}, \\ B &= \big\{\{x_i, z_j\} \mid 1 \leq i, j \leq l-1\big\}, \text{ and} \\ C &= \big\{\{x_i, y_j\}, \{y_i, z_j\} \mid 1 \leq i, j \leq l-1,\ i \neq j\big\}. \end{aligned}$$

G has $n = 3(l-1)$ vertices. Since $X \cup \{z_1\}$ is a clique with l vertices, and Y and Z are independent sets it is easy to see that $\omega(G) = \chi(G) = l$.

We consider the coloring c_b of G that has the color classes $\{x_1\}, \ldots, \{x_{l-1}\}$, $\{y_1, z_1\}, \ldots, \{y_{l-1}, z_{l-1}\}$. Obviously, c_b is a proper coloring of G by $2(l-1)$ colors. Moreover, we can easily check that x_i and z_i are color-dominating vertices for every $i \in \{1, \ldots, l-1\}$. Hence, every color class contains a color-dominating vertex, i.e., c_b is a b-coloring of G. It follows from this and from Theorem 7 that $\chi_b(G) = 2(l-1) = \lceil \frac{n+l}{2} \rceil - 1$. ■

It is known (cf. [7] and [10]) that for every graph G,

$$\chi(G) \leq \frac{n(G)-\alpha(G)+\omega(G)+1}{2} = \frac{n(G)-\alpha(G)+\Delta(G)+1}{2} - \frac{\Delta(G)-\omega(G)}{2l} \quad \text{for } l = 1, \text{ and}$$

$$\chi(G) \leq \left\lceil \frac{\omega(G)}{2} + \frac{n(G)+\Delta(G)}{4} \right\rceil = \left\lceil \frac{n(G)+\omega(G)}{2} - \frac{n(G)-\Delta(G)}{2l} \right\rceil \quad \text{for } l = 2.$$

Since these bounds look similar to the bounds on $\chi_b(G)$ given in Proposition 3 and Theorem 7, one might ask whether these bounds on $\chi(G)$ are even general upper bounds on $\chi_b(G)$ for some $l \geq 1$.

Proposition 10. *There exists no integer $l \geq 1$ such that*

(a) $\chi_b(G) \leq \lceil \frac{n(G)-\alpha(G)+\Delta(G)+1}{2} - \frac{\Delta(G)-\omega(G)}{2l} \rceil$ *for every graph G, or*

(b) $\chi_b(G) \leq \lceil \frac{n(G)+\omega(G)}{2} - \frac{n(G)-\Delta(G)}{2l} \rceil$ *for every graph G.*

Proof. Let $l, s, t \geq 1$ and $H := H_1(s,t)$. By Lemma 5 it follows that:

– if $t \geq l+2$, then $\lceil \frac{n(H)-\alpha(H)+\Delta(H)+1}{2} - \frac{\Delta(H)-\omega(H)}{2l} \rceil \leq s+t-1 < \chi_b(H)$,

– if $t \geq 3l-1$, then $\lceil \frac{n(H)+\omega(H)}{2} - \frac{n(H)-\Delta(H)}{2l} \rceil \leq s+t-1 < \chi_b(H)$. ■

Let $\theta(G)$ denote the clique cover number of G, i.e., $\theta(G) = \chi(\overline{G})$. Kouider and Zaker [9] proved that every graph G with clique cover number $\theta(G) = t$ satisfies $\chi_b(G) \leq \frac{t^2\omega(G)}{2t-1}$. This can slightly be improved to:

Theorem 11. *For every graph G with clique cover number $\theta(G) \leq t$, $\chi_b(G) \leq \lfloor \frac{t\omega(G)+(t-1)n(G)}{2t-1} \rfloor$.*

Note that the pigeonhole principle yields $n(G) \leq t\omega(G)$ for every graph G with $\theta(G) \leq t$ (since $n(G)$ vertices are distributed among $\theta(G) \leq t$ cliques in a minimum clique cover). Hence, $\frac{t\omega(G)+(t-1)n(G)}{2t-1} \leq \frac{t^2\omega(G)}{2t-1}$. So the upper bound from Theorem 11 is never larger than the bound of Kouider and Zaker, and even improves the bound in case of $n(G) < t\omega(G)$.

Let $\nu(\overline{G})$ denote the matching number of the complement $\overline{G}$, i.e., the cardinality of a maximum matching in $\overline{G}$.

Theorem 12. *For every graph G, $\chi_b(G) \leq n(G) - \lceil \frac{2\nu(\overline{G})}{3} \rceil$.*

Corollary 13. *If $\overline{G}$ has a perfect matching, then $\chi_b(G) \leq \lfloor \frac{2n(G)}{3} \rfloor$.*

At last we mention a bound that seems not so useful in the first place but has the advantage that it can be determined in polynomial time and it yields good results in particular for graphs whose minimum degree is close to its maximum degree.

Theorem 14. *For every graph G, $\chi_b(G) \leq \lfloor \frac{2n(G)-\Delta(G)-\delta(G)-3}{3n(G)-2\Delta(G)-\delta(G)-4} n(G) \rfloor$.*

Corollary 15. *If G is a $(n(G)-1-r)$-regular graph (i.e., $\overline{G}$ is r-regular), then $\chi_b(G) \leq \lfloor \frac{2r-1}{3r-1} n(G) \rfloor$.*

For $r = 1$ we obtain $\chi_b(G) \leq \lfloor \frac{n(G)}{2} \rfloor$. It will turn out in Corollary 28 that this is the exact value on $\chi_b(G)$. Moreover, for $r = 2$ we deduce that $\chi_b(G) \leq \lfloor \frac{3n(G)}{5} \rfloor$. As we will see in Theorem 31, this bound is attained in case that $\overline{G}$ is connected. For $r = 3$, there also exist sharpness examples for the bound $\chi_b(G) \leq \lfloor \frac{5n(G)}{8} \rfloor$. Figure 2 shows the complement of a 12-regular graph G of order 16 and a b-coloring of G by $10 = \frac{5n(G)}{8}$ colors. Incidently, this graph G is also a sharpness example for the upper bounds from Theorems 11 and 12 since $\omega(G) = 7$, $\theta(G) = 3$, and $\nu(\overline{G}) = 8$.

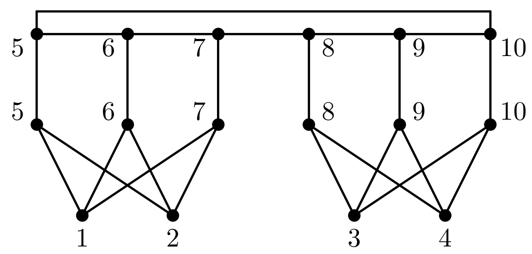


Figure 2. Complement of a $(n(G)-4)$-regular graph G with $\chi_b(G) = \frac{5n(G)}{8}$.

In the literature, the problem of deciding which d-regular graphs G satisfy $\chi_b(G) = d+1$ is often discussed. Corollary 15 yields:

Corollary 16. *If G is a d-regular graph with $\frac{2n(G)}{3} - 1 \leq d \leq n(G) - 2$, then $\chi_b(G) < d+1$.*

***Proof of Theorems* 7, 11, 12, *and* 14.** We set $n := n(G)$, $\omega := \omega(G)$, $\delta := \delta(G)$, and $\Delta := \Delta(G)$.

Let c be a b-coloring of G by k colors and let $V_1, \ldots, V_k$ be the corresponding color classes such that $|V_1| \leq |V_2| \leq \cdots \leq |V_k|$. For $i = 1, \ldots, k$ a color-dominating vertex $v_i \in V_i$ is chosen. By a we denote the number of color classes of cardinality 1. Obviously, $n \geq a + 2(k-a)$, i.e., $a \geq 2k - n$.

Assume that $a > 0$.
Let $A := \{v_1, \ldots, v_a\}$, $B := \{v_{a+1}, \ldots, v_k\}$, and $C := V(G) \setminus (A \cup B)$.

Since all vertices in $A \cup B$ are color-dominating and the vertices in A are the only vertices with colors $1, \ldots, a$ it follows that the vertices in A are pairwise adjacent and every vertex in B is adjacent to all vertices in A. Hence:

Fact 1. $\{v_i, v_j\} \in E(G)$ for $i \in \{1, \ldots, a\}$, $j \in \{1, \ldots, k\} \setminus \{i\}$.

(a) If $\omega < n$, then it is easy to see that $a < k$. So the vertex v_{a+1} exists, and according to Fact 1 the vertices $v_1, \ldots, v_{a+1}$ induce a clique of order $a+1$. This implies $\omega \geq a+1 \geq 2k-n+1$ and therefore $k \leq \lfloor \frac{n+\omega-1}{2} \rfloor = \lceil \frac{n+\omega}{2} \rceil - 1$.

(b) Let $t \geq \theta(G)$ and let $\{Q_1, \ldots, Q_t\}$ be a clique cover of G with t cliques. The pigeonhole principle implies that there is an integer $i \in \{1, \ldots, t\}$ such that Q_i contains at least $s := \lceil \frac{|B|}{t} \rceil = \lceil \frac{k-a}{t} \rceil$ vertices from the set B. W.l.o.g. assume that $v_{a+1}, \ldots, v_{a+s} \in Q_i$. Since Q_i is a clique, $v_{a+1}, \ldots, v_{a+s}$ are pairwise adjacent. Together with Fact 1 this implies that the vertices $v_1, \ldots, v_{a+s}$ induce a clique of order $a+s$. Hence, $\omega \geq a+s = a+\lceil \frac{k-a}{t} \rceil = \lceil \frac{k+a(t-1)}{t} \rceil$ and by $a \geq 2k-n$ we deduce that $\omega \geq \lceil \frac{k+(2k-n)(t-1)}{t} \rceil$. This yields $k \leq \lfloor \frac{t\omega+(t-1)n}{2t-1} \rfloor$.

(c) Set $q := \nu(\overline{G})$ and let M be a maximum matching of $\overline{G}$ with q edges. By Fact 1 we know that every edge from M either has both end vertices in $V(G) \setminus A$ or one end vertex in A and the other in C. Let p be the number of edges $\{u, v\} \in M$ such that $u \in A$ and $v \in C$. Thus, $p \leq |C| = n-k$. For the $q-p$ remaining matching edges with both end vertices in $V(G)\setminus A$ we deduce that $q - p \leq \lfloor \frac{n-a-p}{2} \rfloor$. By use of the inequalities $p \leq n-k$ and $a \geq 2k-n$, it follows $2q \leq n+p-a \leq 3n-3k$ and finally $k \leq \lfloor n - \frac{2q}{3} \rfloor = n - \lceil \frac{2q}{3} \rceil$.

(d) Fact 1 yields $N_{\overline{G}}(u) \subseteq C$ for every vertex $u \in A$. Hence, there exist at least $a \cdot \delta(\overline{G})$ edges $\{u, v\} \in E(\overline{G})$ such that $u \in A$ and $v \in C$. Every vertex $v \in C$ belongs to a color class of cardinality larger than 1. So there exists a vertex $w \in B$ such that $\{v, w\} \in E(\overline{G})$. Therefore, v can be adjacent to at most $\Delta(\overline{G}) - 1$ vertices in A. Since $|C| = n-k$, the last two sentences imply $a \cdot \delta(\overline{G}) \leq (\Delta(\overline{G}) - 1)(n-k)$. This inequality along with $a \geq 2k-n$ yields $k \leq \lfloor \frac{\Delta(\overline{G})+\delta(\overline{G})-1}{\Delta(\overline{G})+2\delta(\overline{G})-1} n \rfloor = \lfloor \frac{2n-\Delta-\delta-3}{3n-2\Delta-\delta-4} n \rfloor$.

Assume that $a = 0$.
By $a \geq 2k-n$ we immediately obtain $k \leq \lfloor \frac{n}{2} \rfloor$. We show now that this bound is never larger than the bounds obtained for the case $a > 0$.

(a) Since $\omega \geq 1$ we deduce that $\lfloor \frac{n}{2} \rfloor \leq \lfloor \frac{n+\omega-1}{2} \rfloor = \lceil \frac{n+\omega}{2} \rceil - 1$.

(b) By the inequality $n \leq t\omega$ mentioned above we obtain

$$\lfloor \tfrac{n}{2} \rfloor = \lfloor \tfrac{n(2t-1)}{2(2t-1)} \rfloor = \lfloor \tfrac{n+2(t-1)n}{2(2t-1)} \rfloor \leq \lfloor \tfrac{t\omega+2(t-1)n}{2(2t-1)} \rfloor \leq \lfloor \tfrac{2t\omega+2(t-1)n}{2(2t-1)} \rfloor = \lfloor \tfrac{t\omega+(t-1)n}{2t-1} \rfloor.$$

(c) Since $2q \leq n$ we have $\lfloor \frac{n}{2} \rfloor \leq \lfloor \frac{2n}{3} \rfloor = \lfloor n - \frac{n}{3} \rfloor \leq \lfloor n - \frac{2q}{3} \rfloor = n - \lceil \frac{2q}{3} \rceil$.

(d) Clearly, $a = 0$ is only possible if G is not complete. So, $\delta \leq n-2$ and

$$\lfloor \tfrac{n}{2} \rfloor = \lfloor \tfrac{3n-2\Delta-\delta-4}{2(3n-2\Delta-\delta-4)} n \rfloor = \lfloor \tfrac{2(2n-\Delta-\delta-3)+2+\delta-n}{2(3n-2\Delta-\delta-4)} n \rfloor \leq \lfloor \tfrac{2n-\Delta-\delta-3}{3n-2\Delta-\delta-4} n \rfloor. \quad \blacksquare$$

4. Results for Special Graphs

4.1. Disconnected graphs

In case that the complement $\overline{G}$ is disconnected we know the following:

If $\overline{G}_1, \ldots, \overline{G}_r$ are the components of $\overline{G}$, then G is a join of the graphs $G_i = \overline{\overline{G}}_i$ $(i = 1, \ldots, r)$, i.e., $G = G_1 + \cdots + G_r$. It is well-known that a join

$G = G_1 + \cdots + G_r$ has chromatic number $\chi(G) = \sum_{i=1}^{r} \chi(G_i)$. An analogous result was shown for the b-chromatic number:

Proposition 17 (Hoàng and Kouider, [4])**.** *Let G be a graph with disconnected complement $\overline{G}$ and let $\overline{G}_1, \ldots, \overline{G}_r$ be the components of $\overline{G}$. Then $\chi_b(G) = \sum_{i=1}^{r} \chi_b(G_i)$.*

Now we investigate the case that G is disconnected.

Let $G_1, \ldots, G_r$ be the components of G. In [4] it was already mentioned that $\chi_b(G) \geq \max_{1 \leq i \leq r} \chi_b(G_i)$. Moreover, from Theorem 1 we can deduce:

Theorem 18. *Let G be a disconnected graph with components $G_1, \ldots, G_r$. Then $\chi_b(G) \leq \sum_{i=1}^{r}(n(G_i) - \chi_b(\overline{G}_i)) + 1$.*

Proof. Since $\overline{G}$ is the join of the graphs $\overline{G}_1, \ldots, \overline{G}_r$ Proposition 17 yields $\chi_b(\overline{G}) = \sum_{i=1}^{r} \chi_b(\overline{G}_i)$. So by Theorem 1 we obtain $\chi_b(G) \leq n(G) + 1 - \chi_b(\overline{G}) = \sum_{i=1}^{r} n(G_i) + 1 - \sum_{i=1}^{r} \chi_b(\overline{G}_i) = \sum_{i=1}^{r}(n(G_i) - \chi_b(\overline{G}_i)) + 1$. ■

Lemma 19. *If c is a b-coloring of a disconnected graph G with components $G_1, \ldots, G_r$, then for each $i \in \{1, \ldots, r\}$ the component G_i contains at most $\chi_b(G_i)$ color-dominating vertices of pairwise different colors.*

Proof. Let c be a b-coloring of G by k colors. Suppose that there exists an integer $h \in \{1, \ldots, r\}$ such that the component G_h contains k_h color-dominating vertices $v_1, \ldots, v_{k_h}$ of pairwise different colors where $k_h > \chi_b(G_h) \geq 1$. Let c_h be the coloring c restricted to the subgraph G_h and w.l.o.g. assume that $c_h(v_i) = i$ for $i = 1, \ldots, k_h$.

Since there are color-dominating vertices in G_h, the component G_h must contain vertices of all k colors. Hence, $|c_h(V(G_h))| = k \geq k_h > \chi_b(G_h)$ and we deduce that c_h is not a b-coloring of G_h. So there exists a color class without a color-dominating vertex. We recolor all vertices from this color class by suitable other colors to obtain a coloring of G_h by $k - 1$ colors. We repeat this recoloring of color classes in order to decrease the number of colors as long as it is possible. If it is not possible anymore, then the obtained coloring, say c'_h, is a b-coloring of G_h. Hence, the number of remaining colors k' satisfies $k' \leq \chi_b(G_h)$.

During the whole recoloring procedure the vertex v_i for $i \in \{1, \ldots, k_h\}$ has had neighbors in all other existing color classes. So the color class of color i was not recolored (but may be enlarged). This implies that $\{1, \ldots, k_h\} \subseteq c'_h(V(G_h))$ and therefore $k' \geq k_h$.

Altogether we obtain $k_h \le k' \le \chi_b(G_h)$, a contradiction to $k_h > \chi_b(G_h)$. ■

The last lemma immediately implies:

Theorem 20. *Let G be a disconnected graph with components $G_1, \dots, G_r$. Then $\chi_b(G) \le \sum_{i=1}^{r} \chi_b(G_i)$.*

A simple sharpness example for this bound is the graph $G = G_1 \cup G_2$ in Figure 3. Here, $\chi_b(G_1) = \chi_b(G_2) = 2$ and $\chi_b(G) = 4$ (a b-coloring by 4 colors is given in the picture).

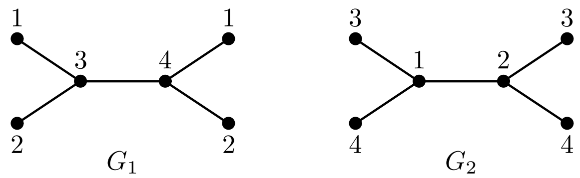


Figure 3. Disconnected graph $G = G_1 \cup G_2$ with $\chi_b(G) = \chi_b(G_1) + \chi_b(G_2)$.

Note that Theorem 20 outperforms Theorem 18 in the majority of cases. Only if $\chi_b(G_i) + \chi_b(\overline{G}_i)$ is close to $n(G_i) + 1$ (compare Theorem 1) for most of the integers $i \in \{1, \dots, r\}$ may Theorem 18 yield the better upper bound.

Now we investigate the b-continuity property for disconnected graphs. One can easily check that not all disconnected graphs are b-continuous. So we can only achieve partial results.

Proposition 21. *Let G be a disconnected graph with components $G_1, \dots, G_r$. Then G has a b-coloring by k colors for $\max_{1 \le i \le r} \chi_b(G_i) \le k \le \chi_b(G)$.*

Proof. Obviously, there exists a b-coloring of G by $\chi_b(G)$ colors. Now let c be a b-coloring of G by k colors where $\max_{1 \le i \le r} \chi_b(G_i) < k \le \chi_b(G)$.

For $h = 1, \dots, r$ and $i = 1, \dots, k$ let c_h denote the coloring c restricted to the subgraph G_h, let V_i^h be the set of vertices of color i in $V(G_h)$, and let v_i be a color-dominating vertex of color i.

We now construct from c a b-coloring c' of G by $k-1$ colors.
For $h = 1, \dots, r$ we do the following: In case of $V_k^h = \emptyset$ we set $c'_h := c_h$, else:

If V_k^h has no color-dominating vertex, we recolor all vertices in V_k^h by suitable colors different from k. This yields a new coloring c'_h of G_h without color k.

If V_k^h has a color-dominating vertex, then G_h contains vertices of all $k > \chi_b(G_h)$ colors. Thus, c_h is not a b-coloring of G_h. Hence, there exists a color subclass V_j^h, $j < k$, that has no color-dominating vertex. We swap the two sets V_j^h and V_k^h. After this, V_k^h does not contain a color-dominating vertex anymore and we construct a new coloring c'_h of G_h without color k.

The union of all colorings $c'_1, \ldots, c'_r$ of the subgraphs $G_1, \ldots, G_r$ yields a new coloring c' of G by $k-1$ colors. Moreover, we notice that for each $i \in \{1, \ldots, k-1\}$ vertex v_i still has its original color and has a neighbor of color j (not necessarily the same vertex as before) for every $j \in \{1, \ldots, k-1\} \setminus \{i\}$. Thus, $v_1, \ldots, v_{k-1}$ are color-dominating vertices of colors $1, \ldots, k-1$ implying that c' is a b-coloring of G by $k-1$ colors.

So by induction, G has a b-coloring by k colors for $\max_{1 \le i \le r} \chi_b(G_i) \le k \le \chi_b(G)$. ■

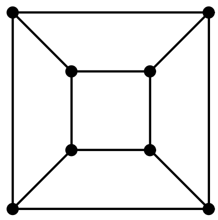


Figure 4. Cube graph Q_3.

It is known that the cube graph Q_3 (see Figure 4) is not b-continuous. By considering the unions $Q_3 \cup Q_3$, $Q_3 \cup P_3$, and $Q_3 \cup P_2$ we notice that the former two are b-continuous while the latter is not. So we deduce that the union of two non-b-continuous graphs can be b-continuous, and the union of a b-continuous and a non-b-continuous graph can be b-continuous or not. However, the next observation implies that the union of two b-continuous graphs is always b-continuous.

Observation 22. *Let G be a disconnected graph with components $G_1, \ldots, G_r$. If there is an integer $h \in \{1, \ldots, r\}$ such that $\chi_b(G_h) \ge \max_{1 \le i \le r} \chi_b(G_i) - 1$ and G_h is b-continuous, then G is b-continuous.*

Proof. Let G_h be the component with the mentioned properties. This and $\chi(G) = \max_{1 \le i \le r} \chi(G_i) \ge \chi(G_h)$ imply that G_h has a b-coloring by k colors for every k satisfying $\chi(G) \le k \le \max_{1 \le i \le r} \chi_b(G_i) - 1$. The b-coloring of the subgraph G_h can easily be extended to a b-coloring of G by k colors by coloring the component G_i for $i \in \{1, \ldots, r\} \setminus \{h\}$ with $\chi(G_i)$ colors.

From this and Proposition 21 we conclude that G has a b-coloring by k colors for each k satisfying $\chi(G) \leq k \leq \chi_b(G)$. Thus, G is b-continuous. ∎

At last we present a theorem that allows us to determine the complete b-spectrum of a disconnected graph G only by considering the single components $G_1, \ldots, G_r$. This result is useful in particular for the case that r is large and the components $G_1, \ldots, G_r$ have simple structures.

Let $C_k(G)$ be the set of all colorings of the graph G by k colors. For a coloring $c \in C_k(G)$ let n_c be the number of color classes that contain a color-dominating vertex. We define $d_k(G) := \max_{c \in C_k(G)} n_c$ for $\chi(G) \leq k \leq \Delta(G) + 1$ and $d_k(G) := 0$ for $k > \Delta(G) + 1$. Note that $d_k(G) \geq 0$ for all $k \geq \chi(G)$.

Theorem 23. *Let G be a disconnected graph with components $G_1, \ldots, G_r$. Then G has a b-coloring by k colors if and only if $\chi(G) \leq k \leq \sum_{i=1}^{r} d_k(G_i)$.*

Proof. ($\Rightarrow$) Assume that G has a b-coloring c by k colors.
For $h = 1, \ldots, r$ let a_h denote the number of color classes that have a color-dominating vertex in G_h and let c_h be the coloring c restricted to the subgraph G_h. It is obvious that $\chi(G) \leq k \leq \sum_{h=1}^{r} a_h$.

If $a_h = 0$, then $a_h \leq d_k(G_h)$ is trivial. If $a_h > 0$, then all k colors occur in G_h and therefore c_h is a coloring of G_h by k colors. Hence, $a_h = n_{c_h} \leq \max_{c \in C_k(G_h)} n_c = d_k(G_h)$.

So, $a_h \leq d_k(G_h)$ for $h = 1, \ldots, r$ which yields $\sum_{h=1}^{r} a_h \leq \sum_{h=1}^{r} d_k(G_h)$. Altogether we obtain $\chi(G) \leq k \leq \sum_{h=1}^{r} a_h \leq \sum_{h=1}^{r} d_k(G_h)$.

($\Leftarrow$) Assume that $\chi(G) \leq k \leq \sum_{i=1}^{r} d_k(G_i)$.
We order the components of G in the way that $d_k(G_1) \geq \cdots \geq d_k(G_r)$. Moreover, let $r' := \max\{i \mid 1 \leq i \leq r,\, d_k(G_i) > 0\}$.

We construct a b-coloring c of G by k colors as follows:
At first we choose a coloring $c_1 \in C_k(G_1)$ such that the color classes of the colors $1, \ldots, d_k(G_1)$ have a color-dominating vertex. By the definition of $d_k(G_1)$ and since $0 < d_k(G_1) \leq k$, the coloring c_1 always exists. After this we choose a coloring $c_2 \in C_k(G_2)$ such that the color classes of the colors $1 + d_k(G_1), \ldots, d_k(G_1) + d_k(G_2)$ have a color-dominating vertex, and we continue analogously with $c_3, \ldots, c_{r'}$ (note that we have to calculate modulo k if we exceed color k).

In summary, we color the graph G_h for $h = 2, \ldots, r'$ by k colors in such a way that the color classes of the colors $1 + (\sum_{i=1}^{h-1} d_k(G_i)) \bmod k, \ldots, 1 + (\sum_{i=1}^{h} d_k(G_i) - 1) \bmod k$ contain a color-dominating vertex. This coloring is denoted by c_h. We notice that c_h must satisfy $n_{c_h} = d_k(G_h)$ and by the definition of $d_k(G_h)$ and because of $d_k(G_h) > 0$ the coloring c_h exists. If $r' < r$, then for $h = r'+1, \dots, r$ we choose a coloring $c_h \in C_{\chi(G_h)}(G_h)$ (recall that $\chi(G_h) \leq \chi(G) \leq k$).

The union of all colorings $c_1, \dots, c_r$ of the subgraphs $G_1, \dots, G_r$ yields a coloring c of G by k colors. Moreover, since $k \leq \sum_{i=1}^{r} d_k(G_i) = \sum_{i=1}^{r'} d_k(G_i)$ the selection of the colorings $c_1, \dots, c_{r'}$ guarantees that every color class contains at least one color-dominating vertex. Therefore, c is a proper b-coloring of G by k colors. ■

4.2. Graphs with independence number 2

It is well-known that every graph G with independence number 2 satisfies $\chi(G) = n(G) - \nu(\overline{G}) \geq \lceil \frac{n(G)}{2} \rceil$. So by $\chi_b(G) \geq \chi(G)$ and Theorem 12 we immediately obtain:

Proposition 24. *If G is a graph with independence number $\alpha(G) = 2$, then $n(G) - \nu(\overline{G}) \leq \chi_b(G) \leq n(G) - \lceil \frac{2\nu(\overline{G})}{3} \rceil$.*

Note that the upper bound may also be written in terms of $\chi(G)$ as $\chi_b(G) \leq \lfloor \frac{n(G)+2\chi(G)}{3} \rfloor$. Moreover, since $\chi(G) \leq \frac{\Delta(G)+\omega(G)+2}{2}$ is satisfied for every graph G with $\alpha(G) = 2$ ([7]) we can deduce another upper bound on $\chi_b(G)$ with respect to the maximum degree $\Delta(G)$, namely $\chi_b(G) \leq \frac{n(G)+\Delta(G)+\omega(G)+2}{3}$.

By use of Ramsey numbers we can improve Theorem 7 for graphs with independence number 2. The *Ramsey number $R(s,t)$* is the smallest integer N such that every graph of order at least N contains a clique with s or an independent set with t vertices (cf. [11]).

Theorem 25. *If G is a graph with independence number $\alpha(G) = 2$ and clique number $\omega(G) \leq n(G)-4$, then $\chi_b(G) \leq \lfloor \frac{n(G)+\omega(G)+1-\sqrt{n(G)-\omega(G)+3}}{2} \rfloor$.*

Proof. Let $n := n(G)$ and $\omega := \omega(G) \leq n-4$. We consider a b-coloring of G by k colors. Let $V_1, \dots, V_k$ be the corresponding color classes such that $|V_1| \leq |V_2| \leq \ldots \leq |V_k|$. Choose a color-dominating vertex $v_i \in V_i$ for all $i = 1, \dots, k$. Again, $a = 2k - n$ shall denote the number of color classes of cardinality 1 (note that $a < k$ since G is not complete). Since the vertices $v_1, \dots, v_a$ induce a clique and vertex v_i is adjacent to all vertices in $\{v_1, \dots, v_a\}$ for $i \in \{a+1, \dots, k\}$, it follows that $a < \omega$ and the vertices

$v_{a+1}, \ldots, v_k$ must not induce a clique of order larger than $\omega - a$. Hence, $k - a = n - k \leq R(\omega - a + 1, 3) - 1$.

If $\omega - a + 1 = 2$, then $k - a = n - k \leq 2$ implying $n \leq k + 2 \leq a + 4 = \omega + 3$, a contradiction to $\omega \leq n - 4$. Hence, $\omega - a + 1 \geq 3$.

It is well-known that the Ramsey number $R(s, 3)$ satisfies $R(s, 3) < \binom{s+1}{2}$ for $s \geq 3$. So we deduce that $n - k \leq \binom{\omega - a + 2}{2} - 2 = \binom{\omega - 2k + n + 2}{2} - 2$. This inequality yields $k^2 - k(n + \omega + 1) \geq \frac{2n + 4 - (n + \omega + 2)(n + \omega + 1)}{4}$ and by completing the square we obtain $\left[k - \frac{1}{2}(n + \omega + 1)\right]^2 \geq \frac{2n + 4 - (n + \omega + 2)(n + \omega + 1)}{4} + \frac{(n + \omega + 1)^2}{4} = \frac{n - \omega + 3}{4}$. Hence, $\left|k - \frac{1}{2}(n + \omega + 1)\right| \geq \frac{\sqrt{n - \omega + 3}}{2}$ and since $k - \frac{1}{2}(n + \omega + 1) < 0$ (compare Theorem 7) we conclude that $k \leq \frac{1}{2}(n + \omega + 1 - \sqrt{n - \omega + 3})$. ■

Recall that the graph G whose complement is shown in Figure 2 has 16 vertices and b-chromatic number 10. Moreover, it satisfies $\alpha(G) = 2$ and $\omega(G) = 7$. Hence, it is a sharpness example for the upper bound from Theorem 25 since $\lfloor \frac{n(G) + \omega(G) + 1 - \sqrt{n(G) - \omega(G) + 3}}{2} \rfloor = 12 - \lceil \sqrt{3} \rceil = 10$.

Theorem 26. *If G is a graph with independence number $\alpha(G) = 2$, then G is b-continuous.*

Proof. If $\chi_b(G) = \chi(G)$, then G is obviously b-continuous. Now assume that $\chi_b(G) > \chi(G)$. Let c be a b-coloring of G by $k > \chi(G)$ colors and let $V_1, \ldots, V_k$ be the corresponding color classes such that $|V_i| = 1$ for $i = 1, \ldots, a$ and $|V_i| = 2$ for $i = a + 1, \ldots, k$. Moreover, for $i = 1, \ldots, k$ we choose a color-dominating vertex $v_i \in V_i$ and for $i = a + 1, \ldots, k$ we denote the vertex in $V_i \setminus \{v_i\}$ by w_i. Additionally, we set $A := \{v_1, \ldots, v_a\}$ and D shall denote the set of vertices from $V(G) \setminus A$ which are adjacent to all vertices in A.

Clearly, A induces a clique in G and $M := \bigcup_{i=a+1}^{k} V_i$ is a matching in $\overline{G}$. We now construct from c a b-coloring c' of G by $k - 1$ colors.

Since M is not a maximum matching of $\overline{G}$ ($|M| = n(G) - k < n(G) - \chi(G) = \nu(\overline{G})$), there exists a M-augmenting path in $\overline{G}$. We choose such a path $P = (p_1, p_2, \ldots, p_{2l})$ of minimum length $2l - 1$. By the definition of a M-augmenting path, $\{p_{2j-1}, p_{2j}\} \notin M$ for $j = 1, \ldots, l$ and $\{p_{2j}, p_{2j+1}\} \in M$ for $j = 1, \ldots, l-1$. We augment M with this path P obtaining the matching $M' := (M \setminus \bigcup_{j=1}^{l-1} \{p_{2j}, p_{2j+1}\}) \cup \bigcup_{j=1}^{l} \{p_{2j-1}, p_{2j}\}$. Now we consider the new coloring c' with color classes $V'_1, \ldots, V'_{k-1}$ where the $a - 2$ vertices from $V(G) \setminus V(M')$ yield the color classes $V'_1, \ldots, V'_{a-2}$ of cardinality 1 and the

$k-a+1$ matching edges from M' induce the color classes $V'_{a-1},\dots,V'_{k-1}$ of cardinality 2. Obviously, c' is a proper vertex coloring of G by $k-1$ colors. We have to prove that c' is also a b-coloring.

Because of $\alpha(G)=2$ it suffices to verify that every color class V'_f for $f \in \{1,\dots,k-1\}$ contains a vertex that is adjacent to all vertices in $A' := \bigcup_{i=1}^{a-2} V'_i$. Since every vertex that is unmatched by M' was also unmatched by M it follows that $A' \subset A$. This implies that every vertex $v \in A \cup D$ is adjacent to all vertices $w \in A'$ $(v \neq w)$.

So if V'_f contains a vertex from $A\cup D$, then it has a color-dominating vertex. Now assume that V'_f contains no vertex from $A\cup D$. Since $\{v_1,\dots,v_k\} \subseteq A\cup D$, this is only possible if there exist two integers $i,j \in \{a+1,\dots,k\}$ such that $V'_f = \{w_i,w_j\}$. We reconsider the M-augmenting path P in $\overline{G}$. Obviously, $p_1, p_{2l} \in A$. W.l.o.g. we assume that $p_1 = v_1$, $p_{2l} = v_2$, and w_i and w_j occur in P in this order. Since $v_i, v_j \in D$, there are two integers $g,h \in \{a+1,\dots,k\} \setminus \{i,j\}$ such that $P = (v_1, w_g, v_g, \dots, v_i, w_i, w_j, v_j, \dots, v_h, w_h, v_2)$. If there exists a vertex $v_3 \in A \setminus \{v_1,v_2\} = A'$ that is not adjacent to w_i, then $P' = (v_1, w_g, v_g, \dots, v_i, w_i, v_3)$ is a M-augmenting path of length smaller than $2l-1$. This contradicts the choice of P. Hence, w_i is adjacent to all vertices in A' and therefore w_i is a color-dominating vertex of V'_f.

Since every color class has a color-dominating vertex, c' is a b-coloring by $k-1$ colors. So by induction we deduce that G is b-continuous. ■

4.3. Graphs with large minimum degree

It is obvious that $\chi_b(G) = \chi(G) = n(G)$ is satisfied for a graph G with minimum degree $\delta(G) = n(G)-1$.

Theorem 27. *Let G be a graph of order n and minimum degree $\delta(G) = n-2$. Moreover, let ζ be the number of vertices of degree $n-2$. Then $\chi_b(G) = \chi(G) = n - \frac{\zeta}{2}$.*

Proof. Let $s := \frac{\zeta}{2}$. Obviously, the ζ vertices of degree $n-2$ induce a matching M of cardinality s in $\overline{G}$. Let $M = \{e_1,\dots,e_s\}$ such that $e_i = \{u_i,v_i\} \in E(\overline{G})$ for $i \in \{1,\dots,s\}$. We notice that $V(G)\setminus\{v_1,\dots,v_s\}$ induces a clique Q of order $n-s$ in G. Hence, $\chi_b(G) \geq \chi(G) \geq \omega(G) \geq n-s$.

Suppose that there is a b-coloring c of G by $n-s+a$ colors for $a \geq 1$. W.l.o.g. let $c(V(Q)) = \{1,\dots,n-s\}$ such that $c(u_h) = h$ for $h \in \{1,\dots,s\}$. Since the clique Q can contain at most $n-s$ color-dominating vertices, there exists an integer $i \in \{1,\dots,s\}$ such that v_i is a color-dominating vertex with color $c(v_i) > n - s$. Moreover, since $\{u_i, v_i\} \notin E(G)$, there exists an integer $j \in \{1, \dots, s\} \setminus \{i\}$ such that $v_j \in N(v_i)$ and $c(v_j) = i$. Because of $\{u_i, v_j\} \in E(G)$ and $c(u_i) = c(v_j)$, c is not a proper coloring of G, a contradiction. Thus, $\chi_b(G) \le n - s$ and altogether, $\chi_b(G) = \chi(G) = n - s = n - \frac{\zeta}{2}$. ■

Corollary 28. *If G is a $(n(G)-2)$-regular graph, then $\chi_b(G) = \chi(G) = \frac{n(G)}{2}$.*

Since $\chi_b(G) = \chi(G)$ for every graph G with minimum degree $\delta(G) \ge n(G) - 2$, we conclude:

Proposition 29. *If G is a graph with minimum degree $\delta(G) \ge n(G) - 2$, then G is b-continuous.*

For a connected graph G with maximum degree $\Delta(G) = 2$ a *segmentation* $\mathcal{S}(G)$ of G shall denote a set of disjoint paths that cover all vertices of G.

Let $P^1, P^2 \in \mathcal{S}(G)$.

If $V(P^1) \cup V(P^2)$ induces a path of order $|V(P^1)| + |V(P^2)|$ in G, then we say that P^1 and P^2 are *consecutive*.

If P^1 and P^2 are non-consecutive, then there exists a so-called *separating set* $\{Q^1, \dots, Q^l\} \subseteq \mathcal{S}(G)$ of $l \ge 1$ paths such that by setting $Q^0 := P^1$ and $Q^{l+1} := P^2$ the subset $V(Q^i) \cup V(Q^{i+1})$ induces a path of order $|V(Q^i)| + |V(Q^{i+1})|$ in G for $i \in \{0, \dots, l\}$. In case that every separating set for P^1 and P^2 contains at least two paths of order 2 we say that P^1 and P^2 are *separated by at least* two paths of order 2.

Note that if G is a path, then the separating set is unique and if G is a cycle, then there exist exactly two distinct separating sets.

Lemma 30. *Let G be a graph of order $n \ge 4$, minimum degree $\delta(G) = n-3$, and with connected complement $\overline{G}$. Moreover, let c be a vertex coloring of G by k colors where $V_1, \dots, V_k$ are the corresponding color classes.*

Then c is a b-coloring of G if and only if $\{\overline{G}[V_1], \dots, \overline{G}[V_k]\}$ is a segmentation of $\overline{G}$ into paths of order 1 and 2 such that any two distinct paths of order 1 are separated by at least two paths of order 2.

Proof. Since $\overline{G}$ is connected and $\Delta(\overline{G}) = n - 1 - \delta(G) = 2$, $\overline{G}$ is isomorphic to a cycle C_n or a path P_n of order $n \ge 4$. Moreover, because of $\alpha(G) = \omega(\overline{G}) = 2$, it is obvious that c is a proper vertex coloring of G if and only if $|V_i| \in \{1, 2\}$ and $\overline{G}[V_i] \simeq P_{|V_i|}$ for $i \in \{1, \dots, k\}$. This implies that c is a proper vertex coloring of G if and only if $\mathcal{S}(\overline{G}) := \{\overline{G}[V_1], \ldots, \overline{G}[V_k]\}$ is a segmentation of $\overline{G}$ into paths of order 1 and 2.

In the following let $V_h = \{u_h, v_h\}$, $V_i = \{u_i\}$ and $V_j = \{u_j\}$ denote three distinct color classes of cardinality 2 and 1, respectively ($h, i, j \in \{1, \ldots, k\}$).

($\Rightarrow$) Assume that c is a b-coloring of G.
– Suppose that $\overline{G}[V_i]$ and $\overline{G}[V_j]$ are consecutive. Then $\{u_i, u_j\} \in E(\overline{G})$ and u_i has no neighbor in color class V_j. Hence, V_i has no color-dominating vertex, a contradiction.
– Suppose that $\{\overline{G}[V_h]\}$ is a separating set for $\overline{G}[V_i]$ and $\overline{G}[V_j]$ and w.l.o.g. let $\overline{G}[V_h \cup V_i \cup V_j] = (u_i, u_h, v_h, u_j)$. Since $\{u_i, u_h\} \in E(\overline{G})$ and $\{v_h, u_j\} \in E(\overline{G})$, the vertex u_h has no neighbor in color class V_i and v_h has no neighbor in color class V_j. So there is no color-dominating vertex in V_h, a contradiction.

It follows from this that any two distinct paths of order 1 are separated by at least two paths of order 2.

($\Leftarrow$) Assume that $\mathcal{S}(\overline{G})$ is a segmentation of $\overline{G}$ into paths of order 1 and 2 such that any two distinct paths of order 1 are separated by at least two paths of order 2.
– Consider the color class V_i of cardinality 1.
Since $\overline{G}[V_i]$ and $\overline{G}[V_j]$ are non-consecutive, it follows $\{u_i, u_j\} \in E(G)$. Moreover, because of $\alpha(G) = 2$, u_i has at least one neighbor in each color class of cardinality 2. So, u_i is a color-dominating vertex of the color class V_i.
– Consider the color class V_h of cardinality 2.
Since $\Delta(\overline{G}) = 2$ and $\{u_h, v_h\} \in E(\overline{G})$, it follows that $|N_{\overline{G}}(u_h) \setminus \{v_h\}| \leq 1$ and $|N_{\overline{G}}(v_h) \setminus \{u_h\}| \leq 1$. If $|N_{\overline{G}}(u_h) \setminus \{v_h\}| = 0$ or ($|N_{\overline{G}}(u_h) \setminus \{v_h\}| = 1$ and $w_u \in N_{\overline{G}}(u_h) \setminus \{v_h\}$ belongs to a color class of cardinality 2), then u_h has a neighbor in each other color class, i.e., u_h is a color-dominating vertex. The same can be shown for v_h. So it remains to consider the case where $|N_{\overline{G}}(u_h) \setminus \{v_h\}| = |N_{\overline{G}}(v_h) \setminus \{u_h\}| = 1$ and $w_u \in N_{\overline{G}}(u_h) \setminus \{v_h\}$, $w_v \in N_{\overline{G}}(v_h) \setminus \{u_h\}$ belong to color classes of cardinality 1. Because of $\omega(\overline{G}) = 2$, we know $w_u \neq w_v$. So, w.l.o.g. let $w_u = u_i$ and $w_v = u_j$. Then $\{\overline{G}[V_h]\}$ is a separating set for $\overline{G}[V_i]$ and $\overline{G}[V_j]$, a contradiction to the properties of the segmentation. Hence, we can deduce that V_h contains a color-dominating vertex.

We conclude that every color class of cardinality 1 and 2, respectively, has a color-dominating vertex. Thus, c is a b-coloring of G. ∎

Theorem 31. *Let G be a graph of order $n \geq 3$ and minimum degree $\delta(G) = n-3$ such that $\overline{G}$ is connected. Then*

$$\chi_b(G) = \begin{cases} \lfloor \frac{3n}{5} \rfloor, & \text{if } \overline{G} \simeq C_n \ \vee \ \big(\overline{G} \simeq P_n \ \wedge \ 2 \mid (n \bmod 5)\big), \\ \lceil \frac{3n}{5} \rceil, & \text{if } \overline{G} \simeq P_n \ \wedge \ 2 \nmid (n \bmod 5). \end{cases}$$

Proof. Since $\overline{G}$ is connected and $\Delta(\overline{G}) = 2$, $\overline{G} \simeq C_n$ or $\overline{G} \simeq P_n$.
Let $n = 3$. If $\overline{G} \simeq C_3$, then G is an empty graph and therefore $\chi_b(G) = 1 = \lfloor \frac{3n}{5} \rfloor$. If $\overline{G} \simeq P_3$, then $G = K_1 \cup K_2$ yielding $\chi_b(G) = 2 = \lceil \frac{3n}{5} \rceil$.

Now let $n \geq 4$ and c be a b-coloring of G by k colors where $V_1, \ldots, V_k$ are the corresponding color classes.

By Lemma 30 we know that $\{\overline{G}[V_1], \ldots, \overline{G}[V_k]\}$ is a segmentation of $\overline{G}$ into paths of order 1 and 2 such that any two distinct paths of order 1 are separated by at least two paths of order 2.

Let p and q denote the number of color classes of cardinality 2 and 1, respectively. Then we obtain $q = n-2p$ and $k = p+q = p+(n-2p) = n-p$.

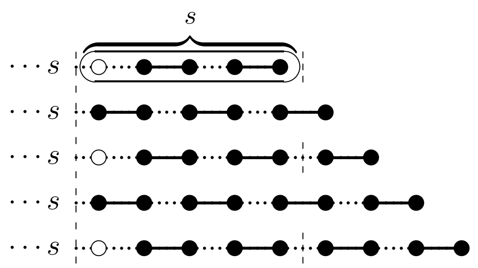
Moreover, since any two distinct paths of order 1 are separated by at least two paths of order 2, we deduce that $p \geq 2q$ if $\overline{G} \simeq C_n$ and $p \geq 2(q-1)$ if $\overline{G} \simeq P_n$. In case of $\overline{G} \simeq P_n$ and $2 \mid (n \bmod 5)$ we also can verify $p \geq 2q$ as follows:

$2 \mid (n \bmod 5)$ implies $\exists Q \in \mathbb{Z}\ \exists R \in \{0, 2, 4\} : n = 5Q + R$. Hence, $n = 5Q + R = 2p + q$ which yields $5(q - Q) = R + 2(2q - p)$. Thus, $5 \mid \big(R+2(2q-p)\big)$ and since R is even we further deduce that $10 \mid \big(R+2(2q-p)\big)$. This is not possible for $p = 2(q-1)$ and $p = 2q-1$.

Obviously, the b-chromatic number $\chi_b(G)$ is the largest possible value for k. Since $k = n-p$, we obtain this maximum integer k by minimizing p. So it remains to determine $p_{\min}$ which shall denote the smallest integer p that satisfies the inequality mentioned above.
– If $\overline{G} \simeq C_n$ or $\big(\overline{G} \simeq P_n$ and $2 \mid (n \bmod 5)\big)$, then $p \geq 2q = 2(n-2p)$ and therefore $p \geq \frac{2n}{5}$. Thus, $p_{\min} \geq \lceil \frac{2n}{5} \rceil$. There exists a segmentation of $\overline{G}$ with exactly $\lceil \frac{2n}{5} \rceil$ paths of order 2 (see Figure 5).

So we deduce that $p_{\min} = \lceil \frac{2n}{5} \rceil$ and therefore $\chi_b(G) = n - p_{\min} = \lfloor \frac{3n}{5} \rfloor$.
– If $\overline{G} \simeq P_n$ and $2 \nmid (n \bmod 5)$, then $p \geq 2(q-1) = 2(n-2p-1)$ and therefore $p \geq \frac{2(n-1)}{5}$. Hence, $p_{\min} \geq \lceil \frac{2(n-1)}{5} \rceil$. Moreover, since $\lceil \frac{2(n-1)}{5} \rceil = \lfloor \frac{2n}{5} \rfloor$ for $(n \bmod 5) \in \{1, 3\}$ we deduce that $p_{\min} \geq \lfloor \frac{2n}{5} \rfloor$. We can find a segmentation of $\overline{G}$ with exactly $\lfloor \frac{2n}{5} \rfloor$ paths of order 2 (see Figure 6).

This yields $p_{\min} = \lfloor \frac{2n}{5} \rfloor$ and therefore $\chi_b(G) = n - p_{\min} = \lceil \frac{3n}{5} \rceil$. ■



$n \bmod 5 = 0,\ p = 2q \quad = \lceil \frac{2n}{5} \rceil$

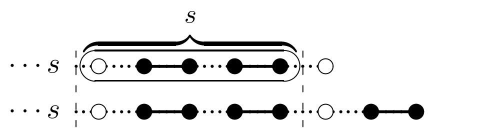
$n \bmod 5 = 1,\ p = 2q+3 = \lceil \frac{2n}{5} \rceil$

$n \bmod 5 = 2,\ p = 2q+1 = \lceil \frac{2n}{5} \rceil$

$n \bmod 5 = 3,\ p = 2q+4 = \lceil \frac{2n}{5} \rceil$

$n \bmod 5 = 4,\ p = 2q+2 = \lceil \frac{2n}{5} \rceil$

Figure 5. Segmentation of $\overline{G}$ with $\lceil \frac{2n}{5} \rceil$ paths of order 2.



$n \bmod 5 = 1,\ p = 2q-2 = \lfloor \frac{2n}{5} \rfloor$

$n \bmod 5 = 3,\ p = 2q-1 = \lfloor \frac{2n}{5} \rfloor$

Figure 6. Segmentation of $\overline{G}$ with $\lfloor \frac{2n}{5} \rfloor$ paths of order 2.

If the complement $\overline{G}$ is disconnected and $\overline{G}_1, \ldots, \overline{G}_r$ are the components of $\overline{G}$, then we already know from Subsection 4.1 that $\chi(G) = \sum_{i=1}^{r} \chi(G_i)$ and $\chi_b(G) = \sum_{i=1}^{r} \chi_b(G_i)$. This allows us to determine $\chi_b(G)$ for all graphs G with minimum degree $\delta(G) = n-3$.

Remark 32. Let G be a graph of order $n \geq 3$ and minimum degree $\delta(G) = n-3$ such that $\overline{G}$ is disconnected. Moreover let $\overline{G}_1, \ldots, \overline{G}_r$ be the components of $\overline{G}$ and $G_i = \overline{\overline{G}}_i$ $(i = 1, \ldots, r)$. As already mentioned above, $\chi_b(G) = \sum_{i=1}^{r} \chi_b(G_i)$ holds. So we only have to determine $\chi_b(G_i)$ for $i = 1, \ldots, r$.

Obviously, $\delta(G_i) \geq n(G_i) - 3$ and $\overline{G}_i$ is connected. If $\delta(G_i) = n(G_i) - 1$ or $\delta(G_i) = n(G_i) - 2$, then $G_i \simeq K_1$ or $G_i \simeq K_1 \cup K_1$, respectively. Hence we can deduce that $\chi_b(G_i) = 1$ in both cases. If $\delta(G_i) = n(G_i) - 3$, then we can apply Theorem 31 yielding $\chi_b(G_i) = \lfloor \frac{3n(G_i)}{5} \rfloor$ or $\chi_b(G_i) = \lceil \frac{3n(G_i)}{5} \rceil$ depending on $\overline{G}_i$.

If G is a $(n(G)-3)$-regular graph, then every component of $\overline{G}$ is a cycle. So we deduce:

Corollary 33. *If G is a $(n(G)-3)$-regular graph, then $\chi_b(G) = \sum_{i=1}^{r} \lfloor \frac{3n(G_i)}{5} \rfloor$ where $\overline{G}_1, \ldots, \overline{G}_r$ are the components of $\overline{G}$.*

Lemma 34. *Let G be a graph with minimum degree $\delta(G) = n(G) - 3$. Moreover, let $\overline{G}_1, \ldots, \overline{G}_s$ be the components of $\overline{G}$ which are not isomorphic to C_3 and d denotes the number of components of $\overline{G}$ which are isomorphic to C_3. Then $\chi_b(G) = \sum_{i=1}^{s} \chi_b(G_i) + d$ and $\chi(G) = \sum_{i=1}^{s} \chi(G_i) + d = \sum_{i=1}^{s} \lceil \frac{n(G_i)}{2} \rceil + d$.*

Proof. Let $G' := G_1 + \cdots + G_s$. Then G is the join of G' and d independent sets of cardinality 3 (the sets which induce the cycles of length 3 in $\overline{G}$). So by the properties of a join, $\chi_b(G) = \chi_b(G') + d = \sum_{i=1}^{s} \chi_b(G_i) + d$ and $\chi(G) = \chi(G') + d = \sum_{i=1}^{s} \chi(G_i) + d$. It remains to prove that $\chi(G_i) = \lceil \frac{n(G_i)}{2} \rceil$ for each $i \in \{1, \ldots, s\}$.

Since $\Delta(\overline{G}_i) \leq 2$ and $\overline{G}_i \not\simeq C_3$ it follows that $\alpha(G_i) = \omega(\overline{G}_i) \leq 2$. If $\alpha(G_i) = 1$, then $G_i \simeq K_1$ and $n(G_i) = 1$. Hence, $\chi(G_i) = 1 = \lceil \frac{n(G_i)}{2} \rceil$.

If $\alpha(G_i) = 2$, then $\overline{G}_i \simeq C_{n(G_i)}$ or $\overline{G}_i \simeq P_{n(G_i)}$. Therefore, $\overline{G}_i$ has matching number $\nu(\overline{G}_i) = \lfloor \frac{n(G_i)}{2} \rfloor$. Moreover for a graph G_i with independence number 2 it is well-known that $\chi(G_i) = n(G_i) - \nu(\overline{G}_i)$. So we deduce that $\chi(G_i) = n(G_i) - \lfloor \frac{n(G_i)}{2} \rfloor = \lceil \frac{n(G_i)}{2} \rceil$. ■

Proposition 35. *If G is a graph with minimum degree $\delta(G) = n(G) - 3$, then G is b-continuous.*

Proof. Let $\overline{G}_1, \ldots, \overline{G}_s$ be the components of $\overline{G}$ which are not isomorphic to C_3 and d denotes the number of components of $\overline{G}$ which are isomorphic to C_3. Additionally, let $G' := G_1 + \cdots + G_s$. By Lemma 34 we can deduce that $\chi_b(G) = \chi_b(G') + d$ and $\chi(G) = \chi(G') + d$. Moreover, since $\Delta(\overline{G'}) \leq 2$ and $\overline{G'}$ has no component which is isomorphic to C_3 it follows that $\alpha(G') \leq 2$.

If $\alpha(G') = 1$, then G' is complete and therefore b-continuous. In Theorem 26 it was proved that every graph with independence number 2 is b-continuous. So if $\alpha(G') = 2$, then G' is b-continuous as well.

It follows from this that G' has a b-coloring c' by k' colors for $\chi(G') \leq k' \leq \chi_b(G')$. Let $c'(V(G')) = \{1, \ldots, k'\}$. We can extend c' to a coloring c of G by $k' + d$ colors by coloring the d independent sets of cardinality 3 by d pairwise different colors from $\{k' + 1, \ldots, k' + d\}$. Due to the properties of a join it is easy to check, that c is a b-coloring of G. Thus, G has a b-coloring c by $k := k' + d$ colors for $\chi(G') \leq k' \leq \chi_b(G')$, i.e., for $\chi(G) = \chi(G') + d \leq k \leq \chi_b(G') + d = \chi_b(G)$. Hence, G is b-continuous. ■

There exist graphs G with minimum degree $\delta(G) = n(G) - 5$ which are not b-continuous, e.g. the bipartite graphs $H_1(1, 3)$ (see Figure 1) and Q_3 (see

Figure 4). So we ask:

Problem 36. Is every graph G with minimum degree $\delta(G) = n(G) - 4$ b-continuous?

Concerning the determination of the b-chromatic number of graphs G with minimum degree $\delta(G) = n(G) - 4$ we believe that a result like Theorem 31 cannot be achieved. We even guess that this determination could be $\mathcal{NP}$-hard.

However, there are several bounds on $\chi_b(G)$ for such graphs that we obtain from Section 3. If $\delta(G) = n(G) - 4$, then $\Delta(\overline{G}) = 3$ and therefore $\theta(G) = \chi(\overline{G}) \leq 4$. Moreover, by Brooks' Theorem we even can deduce that $\theta(G) \leq 3$ if no component of $\overline{G}$ is a K_4. Along with $n(G) \leq \theta(G)\omega(G)$ and $\chi_b(G) \geq \chi(G)$, Theorems 11 and 14 yield among others:

Corollary 37. *Let G be a graph with $\delta(G) = n(G) - 4$ and $\theta(G) = t$. Then*

(a) $\lceil \frac{n(G)}{4} \rceil \leq \lceil \frac{n(G)}{t} \rceil \leq \chi_b(G) \leq \lfloor \frac{t\omega(G)+(t-1)n(G)}{2t-1} \rfloor \leq \lfloor \frac{4\omega(G)+3n(G)}{7} \rfloor$,

(b) $\chi_b(G) \leq \lfloor \frac{2n(G)}{3} \rfloor$ *and* $\lfloor \frac{3n(G)}{4} \rfloor$ *if* $\Delta(G) = \delta(G) + 1$ *and* $\delta(G) + 2$, *resp.*,

(c) $\chi_b(G) \leq \lfloor \frac{5n(G)}{8} \rfloor$ *if G is regular,*

(d) $\lceil \frac{n(G)}{2} \rceil \leq \chi_b(G) \leq \lfloor \frac{5n(G)}{8} \rfloor$ *if G is regular and* $\alpha(G) = 2$,

(e) $\lceil \frac{n(G)}{2} \rceil \leq \chi_b(G) \leq \lfloor \frac{2\omega(G)+n(G)}{3} \rfloor$ *if $\overline{G}$ is bipartite.*

The graphs mentioned in (d) are graphs whose complements are cubic and triangle-free. The gap between lower and upper bound is here at most $\frac{n(G)}{8}$ and both bounds are sharp since the cycle C_6 satisfies $\chi_b(C_6) = 3 = \frac{n(C_6)}{2}$ and a sharpness example for $\frac{5n(G)}{8}$ was already given in Figure 2.

Incidently, for complements of bipartite graphs listed in (e) a method for determining the complete b-spectrum is established in [9].

4.4. Graphs with large clique or independence number

By Theorem 7 and Inequality (1) it follows that:

Proposition 38. *If G is a graph with clique number $\omega(G) \geq n(G) - 2$, then $\chi_b(G) = \omega(G)$. Moreover, $\chi_b(G) = n(G) - 1$ if and only if $\omega(G) = n(G) - 1$.*

Moreover, for $\omega(G) = n(G) - 3$ we deduce that $n(G) - 3 \leq \chi_b(G) \leq n(G) - 2$. The following theorem gives a characterization in which cases the lower and in which cases the upper bound is attained.

Theorem 39. *Let G be a graph of order n and clique number $\omega(G) = n-3$. If $\overline{G}$ contains a (not necessarily induced) subgraph H which is*

(a) *a path $(w_1, v_1, u_1, u_2, v_2, w_2)$ of length 5 such that $d_G(u_1) = d_G(u_2) = n-3$, $d_G(w_1) = d_G(w_2) = n-2$, and $d_G(v_1), d_G(v_2) \leq n-3$, or*

(b) *a cycle $(w_1, v_1, u_1, u_2, v_2, w_1)$ of length 5 such that $d_G(u_1) = d_G(u_2) = d_G(w_1) = n-3$ and $d_G(v_1), d_G(v_2) \leq n-3$,*

and $V(G) \setminus V(H)$ induces a clique in G, then $\chi_b(G) = n-2$. Otherwise, $\chi_b(G) = n-3$.

Proof. It suffices to prove that $\chi_b(G) = n-2$ if and only if the subgraph H exists in $\overline{G}$.

($\Rightarrow$) Assume that there exists a b-coloring c of G by $n-2$ colors.
Let $V_1, \ldots, V_{n-2}$ be the corresponding color classes satisfying $|V_1| \geq \cdots \geq |V_{n-2}|$. Moreover, for $i = 1, \ldots, n-2$ choose a color-dominating vertex $v_i \in V_i$.

Case 1. $|V_1| = 3$ and $|V_i| = 1$ for $i \in \{2, \ldots, n-2\}$.
Then the vertices $v_1, \ldots, v_{n-2}$ induce a clique of order $n-2$, a contradiction to $\omega(G) = n-3$.

Case 2. $|V_1| = |V_2| = 2$ and $|V_i| = 1$ for $i \in \{3, \ldots, n-2\}$.
Denote by w_1 and w_2 the vertices in $V_1 \setminus \{v_1\}$ and $V_2 \setminus \{v_2\}$, respectively. The vertices $v_3, \ldots, v_{n-2}$ are pairwise adjacent and therefore form a clique of order $n-4$. Moreover, $\{v_i, v_1\} \in E(G)$ and $\{v_i, v_2\} \in E(G)$ are satisfied for $i \in \{3, \ldots, n-2\}$. So it follows immediately $d(v_i) \geq n-3$ for $i \in \{3, \ldots, n-2\}$. The premise $\omega(G) = n-3$ yields $\{v_1, v_2\} \notin E(G)$. Because v_1 and v_2 are color-dominating vertices we deduce that $\{v_1, w_2\}, \{v_2, w_1\} \in E(G)$, and therefore $d(v_1) = d(v_2) = n-3$. Moreover, we know that w_1 is not to all vertices $v_3, \ldots, v_{n-2}$ adjacent since otherwise the vertices $w_1, v_2, \ldots, v_{n-2}$ would induce a clique with $n-2 > \omega(G)$ vertices. Therefore, $d(w_1) \leq n-3$. In the same way we can prove that w_2 is not to all vertices $v_3, \ldots, v_{n-2}$ adjacent and therefore $d(w_2) \leq n-3$.

If there exists an integer $g \in \{3, \ldots, n-2\}$ such that $\{w_1, v_g\} \notin E(G)$ and $\{w_2, v_g\} \notin E(G)$, then $d(v_g) = n-3$ and $(v_g, w_1, v_1, v_2, w_2, v_g)$ is a cycle of length 5 in $\overline{G}$.

If g does not exist, then there exist integers $h_1, h_2 \in \{3, \ldots, n-2\}$, $h_1 \neq h_2$, such that $\{w_1, v_{h_1}\}, \{w_2, v_{h_2}\} \notin E(G)$. Hence, $d(v_{h_1}) = d(v_{h_2}) = n-2$ and $(v_{h_1}, w_1, v_1, v_2, w_2, v_{h_2})$ is a path of length 5 in $\overline{G}$.

In both cases the vertices not contained in the path or the cycle, respectively, belong to $\{v_3, \ldots, v_{n-2}\}$ and therefore induce a clique in G.

Altogether the subgraph H exists in $\overline{G}$.

($\Leftarrow$) Assume that the subgraph H exists in $\overline{G}$.

Case 1. H is a path $(w_1, v_1, u_1, u_2, v_2, w_2)$ in $\overline{G}$ such that $d_G(u_1) = d_G(u_2) = n-3$, $d_G(w_1) = d_G(w_2) = n-2$, and $d_G(v_1), d_G(v_2) \leq n-3$.

Denote by $x_1, \ldots, x_{n-6}$ the vertices from $V(G) \setminus V(H)$ that induce a clique.

We define a coloring c of G by $n-2$ colors as follows:
Set $c(u_1) = c(v_1) = 1$, $c(u_2) = c(v_2) = 2$, $c(w_1) = 3$, $c(w_2) = 4$, and $c(x_i) = i+4$ for $i \in \{1, \ldots, n-6\}$.

Since $\{u_1, v_1\}, \{u_2, v_2\} \notin E(G)$ we can easily see that this coloring is proper. Moreover, because of $d(u_1) = n-3$ it follows $N(u_1) = V(G) \setminus \{u_1, v_1, u_2\}$. So u_1 has a neighbor in every other color class and therefore it is a color-dominating vertex of color 1. Analogously, we can show that u_2 is a color-dominating vertex of color 2. Since $d(w_1) = n-2$ we know $N(w_1) = V(G) \setminus \{w_1, v_1\}$, so w_1 is a color-dominating vertex of color 3. In the same way we can prove that w_2 is a color-dominating vertex of color 4. For $i \in \{1, \ldots, n-6\}$ we can deduce that $(V(G) \setminus \{v_1, v_2, x_i\}) \subseteq N(x_i)$. This implies that x_i is a color-dominating vertex of color $i+4$ for $i \in \{1, \ldots, n-6\}$.

Altogether, c is a b-coloring of G.

Case 2. H is a cycle $(w_1, v_1, u_1, u_2, v_2, w_1)$ in $\overline{G}$ such that $d_G(u_1) = d_G(u_2) = d_G(w_1) = n-3$ and $d_G(v_1), d_G(v_2) \leq n-3$.

Denote by $x_1, \ldots, x_{n-5}$ the vertices from $V(G) \setminus V(H)$ that induce a clique.

We define a coloring c of G by $n-2$ colors as follows:
Set $c(u_1) = c(v_1) = 1$, $c(u_2) = c(v_2) = 2$, $c(w_1) = 3$, and $c(x_i) = i+3$ for $i \in \{1, \ldots, n-5\}$.

Then we can show analogously to Case 1 that c is a b-coloring of G. ■

Theorem 2 and Inequality (1) yield $\chi_b(G) = 2$ for every graph G with $\alpha(G) = n(G) - 1$. In case of $\alpha(G) = n(G) - 2$ we obtain $2 \leq \chi_b(G) \leq 3$. The next theorem classifies the graphs with b-chromatic number 2 and 3:

Theorem 40. *Let G be a graph of order n and independence number $\alpha(G) = n-2$. If $\omega(G) = 3$ or G contains an induced path of order 5, then $\chi_b(G) = 3$. Otherwise, $\chi_b(G) = 2$.*

Proof. It suffices to prove that there exists a b-coloring by 3 colors if and only if $\omega(G) = 3$ or G contains an induced path of order 5.

Let I be a maximum independent set of G.

($\Rightarrow$) Assume that there exists a b-coloring c of G by 3 colors. Let V_1, V_2, V_3 be the corresponding color classes and for $i = 1, 2, 3$ choose a color-dominating vertex $v_i \in V_i$.

Case 1. $\{v_1, v_2, v_3\}$ induces a clique of order 3. Then $3 \leq \omega(G) \leq \chi_b(G) \leq 3$ and therefore $\omega(G) = 3$.

Case 2. $\{v_1, v_2, v_3\}$ does not induce a clique of order 3. W.l.o.g. let $\{v_1, v_2\} \notin E(G)$.

Suppose that $\{v_1, v_2\} \subseteq I$. Since v_1, v_2 are color-dominating vertices, it follows that $V(G)\backslash I$ has to contain vertices of all three colors, a contradiction to $|V(G) \setminus I| = 2$.

Suppose that I contains exactly one vertex from $\{v_1, v_2\}$, say $v_1 \in I$ and $v_2 \notin I$. Since $\{v_1, v_2\} \notin E(G)$ and v_1 is a color-dominating vertex of V_1 it follows that there must exist two vertices $w_2 \in N(v_1) \cap V_2$, $w_2 \neq v_2$, and $w_3 \in N(v_1) \cap V_3$. This implies $v_2, w_2, w_3 \in V(G) \setminus I$, again a contradiction.

Hence we conclude that $V(G) \setminus I = \{v_1, v_2\}$. Since v_3 is a color-dominating vertex and $v_3 \in I$, $\{v_1, v_3\}, \{v_2, v_3\} \in E(G)$. Moreover, because v_1 and v_2 are color-dominating and $\{v_1, v_2\} \notin E(G)$, there exist two vertices $u_1, u_2 \in I$ such that $u_1 \in N(v_1) \cap V_2$ and $u_2 \in N(v_2) \cap V_1$. Hence, $(u_1, v_1, v_3, v_2, u_2)$ is a path of order 5 in G which is an induced path since $u_1, u_2, v_3 \in I$, $u_1, v_2 \in V_2$, $v_1, u_2 \in V_1$, and $\{v_1, v_2\} \notin E(G)$.

($\Leftarrow$) Assume that $\omega(G) = 3$ or G contains an induced path of order 5. If $\omega(G) = 3$, then $3 = \omega(G) \leq \chi_b(G) \leq 3$ and therefore $\chi_b(G) = 3$.

Else $\omega(G) \leq 2$ and G contains an induced path (u_1, v_1, w, v_2, u_2) of order 5. Since $|V(G) \setminus I| = 2$, we can easily verify that $\{u_1, u_2, w\} \subseteq I$, $\{v_1, v_2\} = V(G) \setminus I$. Let $N_1 := N(v_1) \setminus N(v_2)$ and $N_2 := N(v_2) \setminus N(v_1)$. Note, that $N_1, N_2, N(v_1) \cap N(v_2) \subseteq I$.

We define a coloring c of G by 3 colors as follows: Set $c(x) = 1$ for $x \in \{v_1\} \cup N_2$, $c(x) = 2$ for $x \in \{v_2\} \cup N_1$, and $c(x) = 3$ for $x \in I \setminus (N_1 \cup N_2)$.

Since (u_1, v_1, w, v_2, u_2) is an induced path it follows that $u_1 \in N_1$, $u_2 \in N_2$, and $w \in N(v_1) \cap N(v_2)$. Moreover, it is clear that $\{v_1\} \cup N_2$, $\{v_2\} \cup N_1$, and $I \setminus (N_1 \cup N_2)$ are independent sets. Thus, the coloring c is proper.

Because of $c(u_1) = 2$, $c(w) = 3$, and $\{u_1, w\} \subseteq N(v_1)$ we deduce that v_1 is a color-dominating vertex of color 1. Analogously, we can prove that v_2 is a color-dominating vertex of color 2. Moreover, since $c(w) = 3$ and w is adjacent to v_1 and v_2 which have color 1 and 2, respectively, w is a color-dominating vertex of color 3.

Altogether, c is a b-coloring of G. ■

Since $\chi_b(G) \leq \chi(G) + 1$ for every graph G with clique number at least $n(G) - 4$ or independence number at least $n(G) - 2$, we immediately obtain:

Proposition 41. *If G is a graph with clique number $\omega(G) \geq n(G) - 4$ or independence number $\alpha(G) \geq n(G) - 2$, then G is b-continuous.*

Observation 42. *For every integer $n \geq 7$ there exist a non-b-continuous graph G_1 of order n and clique number $\omega(G_1) = n - 5$ and a non-b-continuous graph G_2 of order n and independence number $\alpha(G_2) = n - 3$.*

Proof. Let $n \geq 7$ and consider the graphs $H_1 := H_1(n-6, 3)$ and $H_2 := H_2(n-6, 3)$. According to Lemma 5, H_1 satisfies $\chi(H_1) = \omega(H_1) = n - 5$ and $\chi_b(H_1) = n - 3$ but it has no b-coloring by $n - 4$ colors. Moreover, for H_2 we deduce that $\alpha(H_2) = n - 3$, $\chi(H_2) = 2$, and $\chi_b(H_2) = 4$ but this graph has no b-coloring by 3 colors. Hence, H_1 and H_2 are non-b-continuous. ■

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