

## CONNECTED GLOBAL OFFENSIVE $k$ -ALLIANCES IN GRAPHS

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### Abstract

We consider finite graphs  $G$  with vertex set  $V(G)$ . For a subset  $S \subseteq V(G)$ , we define by  $G[S]$  the subgraph induced by  $S$ . By  $n(G) = |V(G)|$  and  $\delta(G)$  we denote the order and the minimum degree of  $G$ , respectively. Let  $k$  be a positive integer. A subset  $S \subseteq V(G)$  is a *connected global offensive  $k$ -alliance* of the connected graph  $G$ , if  $G[S]$  is connected and  $|N(v) \cap S| \geq |N(v) - S| + k$  for every vertex  $v \in V(G) - S$ , where  $N(v)$  is the neighborhood of  $v$ . The *connected global offensive  $k$ -alliance number*  $\gamma_o^{k,c}(G)$  is the minimum cardinality of a connected global offensive  $k$ -alliance in  $G$ .

In this paper we characterize connected graphs  $G$  with  $\gamma_o^{k,c}(G) = n(G)$ . In the case that  $\delta(G) \geq k \geq 2$ , we also characterize the family of connected graphs  $G$  with  $\gamma_o^{k,c}(G) = n(G) - 1$ . Furthermore, we present different tight bounds of  $\gamma_o^{k,c}(G)$ .

**Keywords:** alliances in graphs, connected global offensive  $k$ -alliance, global offensive  $k$ -alliance, domination.

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### 1. TERMINOLOGY AND INTRODUCTION

We consider finite, undirected and simple graphs  $G$  with vertex set  $V(G)$ . The number of vertices  $|V(G)|$  of a graph  $G$  is called the *order* and is denoted by  $n = n(G)$ . The *neighborhood*  $N(v) = N_G(v)$  of a vertex  $v$  consists of the

vertices adjacent to  $v$  and  $d(v) = d_G(v) = |N(v)|$  is the *degree* of  $v$ . By  $\delta = \delta(G)$  and  $\Delta = \Delta(G)$ , we denote the *minimum degree* and the *maximum degree* of the graph  $G$ , respectively. For a subset  $S \subseteq V(G)$ , we define by  $G[S]$  the subgraph induced by  $S$ .

The complete graph of order  $n$  is denoted by  $K_n$ , and  $K_{s,t}$  is the complete bipartite graph with the two parts of cardinality  $s$  and  $t$ .

Two vertices that are not adjacent in a graph  $G$  are said to be *independent*. A set  $I$  of vertices is *independent* if every two vertices of  $I$  are independent. The *independence number*  $\alpha(G)$  of a graph  $G$  is the maximum cardinality among the independent sets of vertices of  $G$ .

A *vertex-cut* in a connected graph  $G$  is a set  $S$  of vertices of  $G$  such that  $G - S$  is disconnected. The *connectivity*  $\kappa(G)$  of a graph  $G$  is the minimum cardinality of a vertex-cut of  $G$  if  $G$  is not complete, and  $\kappa(G) = n - 1$  if  $G$  is isomorphic to the complete graph  $K_n$ .

Kristiansen, Hedetniemi and Hedetniemi [9] introduced several types of alliances in graphs, including defensive and offensive alliances. As a generalization of the offensive alliance, Shafique and Dutton [11, 12] defined the global offensive  $k$ -alliance for a positive integer  $k$  as follows. A subset  $S \subseteq V(G)$  is a *global offensive  $k$ -alliance* of the graph  $G$  if  $|N(v) \cap S| \geq |N(v) - S| + k$  for every vertex  $v \in V(G) - S$ . The *global offensive  $k$ -alliance number*  $\gamma_o^k(G)$  is the minimum cardinality of a global offensive  $k$ -alliance in  $G$ . A global offensive  $k$ -alliance set of the minimum cardinality of a graph  $G$  is called a  $\gamma_o^k(G)$ -set. Results on global offensive  $k$ -alliances were given, for example, by Bermudo, Rodríguez-Velázquez, Sigarreta and Yero [1], Chellali [2], Chellali, Haynes, Randerath and Volkmann [3] and Fernau, Rodríguez and Sigarreta [4].

In this paper, we are interested in connected global offensive  $k$ -alliances. Analogously to the definition above, a subset  $S \subseteq V(G)$  is a *connected global offensive  $k$ -alliance* of the connected graph  $G$ , if  $G[S]$  is connected and  $|N(v) \cap S| \geq |N(v) - S| + k$  for every vertex  $v \in V(G) - S$ . The *connected global offensive  $k$ -alliance number*  $\gamma_o^{k,c}(G)$  is the minimum cardinality of a connected global offensive  $k$ -alliance in  $G$ . A connected global offensive  $k$ -alliance set of the minimum cardinality of a connected graph  $G$  is called a  $\gamma_o^{k,c}(G)$ -set.

A subset  $D \subseteq V(G)$  is a  *$k$ -dominating set* of the graph  $G$  if  $|N_G(v) \cap D| \geq k$  for every  $v \in V(G) - D$ . The  *$k$ -domination number*  $\gamma^k(G)$  is the minimum cardinality among the  $k$ -dominating sets of  $G$ . Note that the 1-domination number  $\gamma^1(G)$  is the usual *domination number*  $\gamma(G)$ . A subset  $D \subseteq V(G)$  is

a connected  $k$ -dominating set of a connected graph  $G$ , if  $D$  is a  $k$ -dominating set of  $G$  and the induced subgraph  $G[D]$  is connected. The *connected  $k$ -domination number*  $\gamma_o^{k,c}(G)$  is the minimum cardinality among the connected  $k$ -dominating sets of  $G$ .

In [5, 6], Fink and Jacobson introduced the concept of  $k$ -domination. For a comprehensive treatment of domination in graphs, see the monographs by Haynes, Hedetniemi and Slater [7, 8].

In this paper we characterize the connected graphs  $G$  with  $\gamma_o^{k,c}(G) = n(G)$ . If  $G$  is a connected graph with  $\delta(G) \geq k \geq 3$ , then we show that  $\gamma_o^{k,c}(G) = n(G) - 1$  if and only if  $G$  is isomorphic to the complete graph  $K_{k+1}$  or  $K_{k+2}$ . In addition, we derive different sharp bounds on  $\gamma_o^{k,c}(G)$ , as for example,  $\gamma_o^{k,c}(G) \leq 2\gamma_o^k(G) - k + 1$ .

## 2. MAIN RESULTS

**Observation 1.** *If  $k \geq 1$  is an integer, then  $\gamma_o^{k,c}(G) \geq \gamma_o^{k,c}(G)$  for any connected graph  $G$ .*

**Proof.** If  $S$  is a  $\gamma_o^{k,c}(G)$ -set, then  $G[S]$  is connected and every vertex of  $V(G) - S$  has at least  $k$  neighbors in  $S$ . Thus  $S$  is a connected  $k$ -dominating set of  $G$  and so  $\gamma_o^{k,c}(G) \leq |S| = \gamma_o^{k,c}(G)$ . ■

In view of Observation 1, each lower bound of  $\gamma_o^{k,c}(G)$  is also a lower bound of  $\gamma_o^{k,c}(G)$ . Now we characterize all connected graphs  $G$  with the property that  $\gamma_o^{k,c}(G) = n(G)$ .

**Observation 2.** *Let  $k \geq 2$  be an integer, and let  $G$  be a connected graph of order  $n \geq 2$ . Then  $\gamma_o^{k,c}(G) = n$  if and only if all vertices of  $G$  are either cut-vertices or vertices of degree less than  $k$ .*

**Proof.** If each vertex of  $G$  is either a cut-vertex or has degree less than  $k$ , then the definition of the connected global offensive  $k$ -alliance number leads to  $\gamma_o^{k,c}(G) = n$  immediately.

Conversely, assume that  $\gamma_o^{k,c}(G) = n$ . Suppose to the contrary that  $G$  contains a non-cut-vertex  $u$  with  $d_G(u) \geq k$ . This implies that  $G - u$  is a connected graph. Since  $d_G(u) \geq k$ , we deduce that  $V(G - u)$  is a connected global offensive  $k$ -alliance of  $G$ . Therefore we obtain the contradiction  $\gamma_o^{k,c}(G) \leq n - 1$ , and the proof is complete. ■

**Corollary 3.** *Let  $k \geq 2$  be an integer. If  $T$  is a tree, then  $\gamma_o^{k,c}(T) = n(T)$ .*

**Corollary 4.** *If  $k \geq 2$  is an integer, and  $G$  is a connected graph with  $\delta(G) \geq k$ , then  $\gamma_o^{k,c}(G) \leq n(G) - 1$ .*

Next we derive a characterization of all connected graphs  $G$  with  $\gamma_o^{k,c}(G) = n(G) - 1$  when  $\delta(G) \geq k \geq 2$ .

**Theorem 5.** *Let  $k \geq 2$  be an integer, and let  $G$  be a connected graph of order  $n$  and minimum degree  $\delta$ .*

- (i) *If  $\delta \geq 2$ , then  $\gamma_o^{2,c}(G) = n - 1$  if and only if  $G$  is a cycle or  $G$  is isomorphic to the complete graph  $K_4$ .*
- (ii) *If  $\delta \geq k \geq 3$ , then  $\gamma_o^{k,c}(G) = n - 1$  if and only if  $G$  is isomorphic to the complete graph  $K_{k+1}$  or  $K_{k+2}$ .*

**Proof.** Obviously, if  $G$  is a cycle or  $G$  is isomorphic to  $K_4$ , then  $\gamma_o^{2,c}(G) = n - 1$ , and if  $G$  is isomorphic to the complete graphs  $K_{k+1}$  or  $K_{k+2}$ , then  $\gamma_o^{k,c}(G) = n - 1$ .

Conversely, assume that  $\gamma_o^{k,c}(G) = n - 1$ , and let  $P = u_1 u_2 \dots u_t$  be the longest path in  $G$ . The condition  $\delta \geq k \geq 2$  implies that  $u_1 \neq u_t$  and  $G - \{u_1, u_t\}$  is a connected subgraph of  $G$ . If  $u_1$  and  $u_t$  are not adjacent in  $G$ , then we arrive at the contradiction that  $V(G) - \{u_1, u_t\}$  is a connected global offensive  $k$ -alliance of  $G$ . In the remaining case that  $u_1$  and  $u_t$  are adjacent in  $G$ , we observe that  $C = u_1 u_2 \dots u_t u_1$  is a Hamiltonian cycle of  $G$ , because  $P$  is the longest path in  $G$ . This yields  $t = n$ .

(i) Assume that  $k = 2$ . Suppose that the Hamiltonian cycle  $C = u_1 u_2 \dots u_n u_1$  has a chord. If, without loss of generality,  $u_1 u_s$  with  $3 \leq s \leq n - 1$  is a chord of  $C$ , then we obtain the contradiction that  $V(G) - \{u_2, u_n\}$  is a connected global offensive 2-alliance of  $G$  or  $u_2$  and  $u_n$  are adjacent. Therefore assume in the following that  $u_2$  and  $u_n$  are adjacent. If  $n = 4$ , then  $G = K_4$ . If  $n \geq 5$ , then we distinguish the cases  $s = 3$  and  $s \geq 4$ .

Assume first that  $s = 3$ . Then we obtain the contradiction that  $V(G) - \{u_2, u_4\}$  is a connected global offensive 2-alliance of  $G$  or  $u_2$  and  $u_4$  are adjacent. If  $u_2$  and  $u_4$  are adjacent, then we have the contradiction that  $V(G) - \{u_3, u_n\}$  is a connected global offensive 2-alliance of  $G$  or  $u_3$  and  $u_n$  are adjacent. However, if  $u_3$  and  $u_n$  are adjacent, then  $d_G(u_2), d_G(u_n) \geq 4$ , and thus we arrive at the contradiction that  $V(G) - \{u_2, u_n\}$  is a connected global offensive 2-alliance of  $G$ .

Assume now that  $s \geq 4$ . Then we obtain the contradiction that  $V(G) - \{u_1, u_3\}$  is a connected global offensive 2-alliance of  $G$  or  $u_1$  and  $u_3$  are

adjacent. If  $u_1$  and  $u_3$  are adjacent, then we have the contradiction that  $V(G) - \{u_3, u_n\}$  is a connected global offensive 2-alliance of  $G$  or  $u_3$  and  $u_n$  are adjacent. However, if  $u_3$  and  $u_n$  are adjacent, then  $d_G(u_1), d_G(u_n) \geq 4$ , and thus we arrive at the contradiction that  $V(G) - \{u_1, u_n\}$  is a connected global offensive 2-alliance of  $G$ .

(ii) Assume that  $k \geq 3$ . In the following all indices are taken modulo  $n$ . If the vertices  $u_i$  and  $u_{i+2}$  are not adjacent for any index  $i$  with  $1 \leq i \leq n$ , then the hypothesis  $\delta \geq k \geq 3$  leads to the contradiction that  $V(G) - \{u_i, u_{i+2}\}$  is a connected global offensive  $k$ -alliance of  $G$ . Hence assume that  $u_i$  and  $u_{i+2}$  are adjacent for each index  $i \in \{1, 2, \dots, n\}$ . Now let  $s$  be an arbitrary integer with  $3 \leq s \leq n - 3$ . If  $u_i$  and  $u_{i+s}$  are not adjacent, then  $V(G) - \{u_i, u_{i+s}\}$  is a connected global offensive  $k$ -alliance of  $G$ , since there exists the edge  $u_{i-1}u_{i+1}$  in  $G$ . Therefore it remains the case that  $G$  is a complete graph. If  $G$  is isomorphic to  $K_{k+1}$  or  $K_{k+2}$ , then  $\gamma_o^{k,c}(G) = n - 1$ . However, if  $G$  is isomorphic to  $K_q$  for any integer  $q \geq k + 3$ , then  $V(G) - \{u_1, u_2\}$  is a connected global offensive  $k$ -alliance of  $G$ . This contradiction completes the proof of Theorem 5. ■

**Proposition 6.** *Let  $G$  be a graph of order  $n$ , and let  $k, p$  be two integers such that  $k \geq 1$  and  $-1 \leq p \leq \alpha(G) - 2$ . If  $\delta(G) \geq k$  and  $\kappa(G) \geq \alpha(G) - p$ , then*

$$\gamma_o^{k,c}(G) \leq n(G) - \alpha(G) + p + 1.$$

**Proof.** Let  $I \subset V(G)$  be an independent set of cardinality  $\alpha(G) - p - 1$ . The hypothesis  $\kappa(G) \geq \alpha(G) - p$  implies that  $G[V(G) - I]$  is connected. Since  $I$  is an independent set, the condition  $\delta(G) \geq k$  shows that each vertex in  $I$  has at least  $k$  neighbors in  $V(G) - I$ . Thus  $V(G) - I$  is a connected global offensive  $k$ -alliance of  $G$  such that  $|V(G) - I| \leq n - (\alpha(G) - p - 1)$ , and the proof is complete. ■

If  $H$  is the complete bipartite graph  $K_{k,k}$ , then  $\delta(H) = \alpha(H) = \kappa(H) = k$  and  $\gamma_o^{k,c}(H) = k + 1 = n(H) - \alpha(H) + 1$ . This example demonstrates that Proposition 6 is the best possible, at least for  $p = 0$ .

**Theorem 7.** *Let  $G$  be a connected graph and  $k$  an integer with  $1 \leq k \leq \Delta(G)$ . Then*

$$\gamma_o^{k,c}(G) \leq 2\gamma_o^k(G) - k + 1.$$

**Proof.** Let  $S$  be a  $\gamma_o^k(G)$ -set. Since  $k \leq \Delta(G)$ , we observe that  $|S| = \gamma_o^k(G) \leq n(G) - 1$ . Now let  $x \in V(G) - S$  be an arbitrary vertex.

If  $G[S \cup \{x\}]$  is connected, then the inequality  $k \leq \gamma_o^k(G)$  implies that  $\gamma_o^{k,c}(G) \leq \gamma_o^k(G) + 1 \leq 2\gamma_o^k(G) - k + 1$ , and we are done.

Thus assume next that  $G[S \cup \{x\}]$  is not connected. We will add successively vertices from  $V(G) - (S \cup \{x\})$  to  $S \cup \{x\}$  in order to decrease the number of components, at least one in each step, until we obtain a set of vertices whose induced subgraph is connected. Note that if we partition  $S \cup \{x\}$  into two parts  $A$  and  $B$  such that there is no edge between  $A$  and  $B$ , and we take vertices  $a \in A$  and  $b \in B$  such that the distance between  $a$  and  $b$  is minimum in  $G$ , then the property of  $S$  of being dominating implies that  $d_G(a, b) \leq 3$ . It follows that in each step of increasing  $S \cup \{x\}$  we need to add at most 2 vertices from  $V(G) - (S \cup \{x\})$ . Let  $r_1$  and  $r_2$  be the number of steps where we include one vertex and two vertices from  $V(G) - (S \cup \{x\})$ , respectively, and define  $r = r_1 + r_2$ . Let  $S_0 \subset S \cup \{x\}$  be the set of vertices of the component of  $G[S \cup \{x\}]$  to which  $x$  belongs, and let  $S_i \subset S$  be the set of vertices connected to  $\bigcup_{j=0}^{i-1} S_j$  in step  $i \geq 1$ . Clearly,  $|S_0| \geq k + 1$  and  $|S_i| \geq 1$  for  $1 \leq i \leq r$ . Furthermore, since  $S$  is a global offensive  $k$ -alliance, in the steps where two vertices from  $V(G) - (S \cup \{x\})$  are added, we observe that  $|S_i| \geq k + 1$ . This leads to

$$\gamma_o^k(G) = |S| = |S_0 - \{x\}| + \sum_{i=1}^r |S_i| \geq k + r_2(k + 1) + r_1$$

and therefore  $r_1 \leq \gamma_o^k(G) - k - r_2(k + 1)$ . As a further consequence, we see that  $S \cup \{x\}$  together with all vertices from  $V(G) - (S \cup \{x\})$  added in steps 1 to  $r$  form a connected global offensive  $k$ -alliance of  $G$ . Altogether, we deduce that

$$\begin{aligned} \gamma_o^{k,c}(G) &\leq |S| + 1 + r_1 + 2r_2 \\ &\leq \gamma_o^k(G) + 1 + \gamma_o^k(G) - k - r_2(k + 1) + 2r_2 \\ &= 2\gamma_o^k(G) - k + 1 - r_2(k + 1) + 2r_2 \\ &\leq 2\gamma_o^k(G) - k + 1, \end{aligned}$$

and the proof is complete. ■

If  $H$  is the complete bipartite graph  $K_{k,p}$ , then  $\gamma_o^k(H) = k$  and  $\gamma_o^{k,c}(H) = k + 1$ . This example shows that the bound given in Theorem 7 is tight.

**Theorem 8.** *Let  $G$  be a connected graph and  $k \geq 1$  an integer. If  $\delta(G) \geq k + 1$ , then*

$$\gamma_o^{k+1,c}(G) \leq \frac{\gamma_o^{k,c}(G) + n(G)}{2}.$$

**Proof.** Let  $S$  be a  $\gamma_o^{k,c}(G)$ -set, and let  $A$  be the set of isolated vertices in the subgraph  $G - S$ . Then the subgraph  $G - (S \cup A)$  contains no isolated vertices. If  $D$  is a minimum dominating set of  $G - (S \cup A)$ , then the well-known inequality of Ore [10] implies

$$|D| \leq \frac{|V(G) - (S \cup A)|}{2} \leq \frac{|V(G) - S|}{2} = \frac{n(G) - \gamma_o^{k,c}(G)}{2}.$$

If  $S' = S \cup D$ , then  $G[S']$  is connected. In addition, for each vertex  $v \in V(G) - (S' \cup A)$ , we have

$$\begin{aligned} |N(v) \cap S'| &= |N(v) \cap S| + |N(v) \cap D| \\ &\geq |N(v) - S| + k + 1 \\ &= |N(v) - S'| + |N(v) \cap D| + k + 1 \\ &\geq |N(v) - S'| + k + 2. \end{aligned}$$

Since  $\delta(G) \geq k + 1$ , every vertex of  $A$  has at least  $k + 1$  neighbors in  $S$ , and therefore  $S'$  is a connected global offensive  $(k + 1)$ -alliance of  $G$  and thus

$$\begin{aligned} \gamma_o^{k+1,c}(G) &\leq |S'| = |S| + |D| = \gamma_o^{k,c}(G) + |D| \\ &\leq \gamma_o^{k,c}(G) + \frac{n(G) - \gamma_o^{k,c}(G)}{2} = \frac{n(G) + \gamma_o^{k,c}(G)}{2}. \end{aligned} \quad \blacksquare$$

The inequality  $|N(v) \cap S'| \geq |N(v) - S'| + k + 2$  for each vertex  $v \in V(G) - (S' \cup A)$  in the proof of Theorem 8 leads immediately to the next result.

**Theorem 9.** *Let  $G$  be a connected graph and  $k \geq 1$  an integer. If  $\delta(G) \geq k + 2$ , then*

$$\gamma_o^{k+2,c}(G) \leq \frac{\gamma_o^{k,c}(G) + n(G)}{2}.$$

If  $H = K_{k+3}$ , then  $\gamma_o^{k+2,c}(H) = \gamma_o^{k+1,c}(H) = k + 2$  and  $\gamma_o^{k,c}(H) = k + 1$  and thus

$$\gamma_o^{k+2,c}(H) = \gamma_o^{k+1,c}(H) = k + 2 = \frac{\gamma_o^{k,c}(H) + n(H)}{2}.$$

Let  $k \geq 2$  be an even integer, and let  $F = K_{k+6} - M$ , where  $M$  is a perfect matching of the complete graph  $K_{k+6}$ . Then  $\gamma_o^{k+2,c}(F) = \gamma_o^{k+1,c}(F) = k + 4$  and  $\gamma_o^{k,c}(F) = k + 2$ , and so

$$\gamma_o^{k+2,c}(F) = \gamma_o^{k+1,c}(F) = k + 4 = \frac{\gamma_o^{k,c}(F) + n(F)}{2}.$$

These two graphs  $H$  and  $F$  demonstrate that Theorem 8 as well as Theorem 9 are the best possible.

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