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CHARACTERIZATION OF TREES WITH EQUAL 2-DOMINATION NUMBER AND DOMINATION NUMBER PLUS TWO

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Abstract

Let $G = (V(G), E(G))$ be a simple graph, and let k be a positive integer. A subset D of $V(G)$ is a k-dominating set if every vertex of $V(G) - D$ is dominated at least k times by D. The k-domination number $\gamma_k(G)$ is the minimum cardinality of a k-dominating set of G. In [5] Volkmann showed that for every nontrivial tree T, $\gamma_2(T) \geq \gamma_1(T) + 1$ and characterized extremal trees attaining this bound. In this paper we characterize all trees T with $\gamma_2(T) = \gamma_1(T) + 2$.

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1. Introduction

In a graph $G = (V(G), E(G)) = (V, E)$ of order $n(G)$, or simply n when the graph G is clear from the context, the *neighborhood* $N_G(v) = N(v)$ of a vertex $v \in V(G)$ consists of the vertices adjacent with v, and $N_G[v] = N[v] = N(v) \cup \{v\}$ is the *closed neighborhood*. If S is a subset of vertices, then the subgraph induced by S in G is denoted $G[S]$. The *degree* of a vertex v, denoted by $\deg_G(v)$, is the size of its open neighborhood. A vertex of degree one is called a *leaf*, and its neighbor is called a *support vertex*. We also denote the set of leaves of a graph G by $L(G)$ and the set of support vertices by $S(G)$. A tree T is a *double star* if it contains exactly two vertices that are not leaves. A double star with respectively p and q leaves attached at each support vertex is denoted by $S_{p,q}$. The *subdivision graph* of a graph G is that graph obtained from G by replacing each edge uv of G by a vertex w and edges uw and vw. If a tree T is a subdivision graph of a nontrivial tree T', then we say that T is a *subdivided tree*, and the $n(T') - 1$ new vertices resulting from the subdivision of the edges of T' are called *subdivision vertices*. Note that a subdivided tree has order at least three and at least one subdivision vertex. The *corona graph* $G \circ K_1$ of a graph G is the graph constructed from a copy of G, where for each vertex $v \in V(G)$, a new vertex v' and a pendant edge vv' are added. Let P_n denote the path graph of order n.

Let k be a positive integer. A subset $D \subseteq V(G)$ is a *k-dominating set* of the graph G, if $|N_G(v) \cap D| \geq k$ for every $v \in V(G) - D$. The *k-domination number* $\gamma_k(G)$ is the minimum cardinality among the k-dominating sets of G. Note that the 1-domination number $\gamma_1(G)$ is the usual *domination number* $\gamma(G)$. A set $S \subseteq V(G)$ is *independent* if no edge of G has its two endvertices in S.

We make a couple of straightforward observations.

Observation 1. *For every graph G and positive integer k, every vertex with degree at most $k-1$ belongs to every $\gamma_k(G)$-set.*

Observation 2. *For any tree T of order at least three, there exists a $\gamma(T)$-set that contains no leaves of T.*

The following results will be useful for the next.

Theorem 3 (Fink and Jacobson [2] 1985)**.** *If T is a tree of order n, then $\gamma_2(T) \geq (n+1)/2$, with equality if and only if $T = P_1$ or T is the subdivided graph of another tree.*

Theorem 4 (Volkmann [5] 2007). *For every nontrivial tree T, $\gamma_2(T) \geq \gamma(T) + 1$ with equality if and only if T is a subdivided star, the corona of a star, or a subdivided double star.*

Let $\mathcal{T}$ be the family of extremal trees achieving equality in Theorem 4, that is, $\mathcal{T}$ is the family of nontrivial trees T, where T is a subdivided star, the corona of a star, or a subdivided double star. For a subdivided tree in $\mathcal{T}$, we let $B(T)$ denote the set of subdivided vertices. Note that the corona of a star can also be described as a subdivided star with an added leaf adjacent to its center vertex. Thus, if T'' is the subdivision graph of a star T', then for the corona T of a star T', we let $B(T) = B(T'')$. Note that the paths P_2 and P_4 are coronas of stars, and for the path P_2, $B(T) = \emptyset$, and for the path P_4, $B(T)$ consists of exactly one support vertex. For any tree in $\mathcal{T}$, we let $A(T) = V(T) - B(T)$. (Note that if T is a subdivision of a tree T', then $A(T) = V(T')$ and if T is a corona, that is, a subdivision of a star T' with a leaf neighbor u added to its center, then $A(T) = V(T') \cup \{u\}$).

Thus, by Theorem 4, if T is a tree and T is not in $\mathcal{T}$, then $\gamma_2(T) \geq \gamma(T)+2$. Our aim in this paper is to characterize all trees T with $\gamma_2(T) = \gamma(T)+2$. We close this section by the following observation.

Observation 5. *If $T \in \mathcal{T}$, then $A(T)$ is a $\gamma_2(T)$-set. Moreover, if $T \in \mathcal{T}$ and $T \neq P_4$, then $A(T)$ is the unique $\gamma_2(T)$-set.*

2. The Families $\mathcal{G}$ and $\mathcal{F}$

Let $\mathcal{T}_1$ denote the subdivided stars, $\mathcal{T}_2$ the coronas of stars, and $\mathcal{T}_3$ the subdivided double stars of $\mathcal{T}$. Thus, $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$. Recall that $L(T)$ denotes the set of leaves of T and $S(T)$ the set of support vertices. Let $X = X(T)$ consist of the leaf adjacent to the vertex of maximum degree if $T \in \mathcal{T}_2$ and $T \neq P_2$, and $X = \emptyset$ otherwise. We also let $H = H(T)$ consist of the center vertex if $T \in \mathcal{T}_3$ and $H = \emptyset$ otherwise.

Observation 6. *If T is a tree in $\mathcal{T}$ of order at least three, then every vertex of $B(T)$ is either a support vertex or the center vertex if $T \in \mathcal{T}_3$.*

We define the following families of trees $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ and $\mathcal{G}_4$, and let $\mathcal{G} = \bigcup_{i=1}^{4} \mathcal{G}_i$, where

- $\mathcal{G}_1$ is the family of trees obtained by a path $P_2 = uv$ and a tree $T' \in \mathcal{T}$ different to the path P_4, by adding an edge uw, where $w \in B(T') - H(T')$.
- $\mathcal{G}_2$ is the family of trees obtained by a tree $T \in \mathcal{T}$ different to the path P_2, by adding a new vertex attached to any support vertex of T.
- $\mathcal{G}_3$ is the family of trees obtained by a path P_3 and a tree $T' \in \mathcal{T}_2 \cup \mathcal{T}_3$ different to P_2 and P_4, by adding an edge xy, where x is any leaf of P_3 and $y \in L(T') - X$.
- $\mathcal{G}_4$ is the family of trees that are a subdivision graph of a caterpillar having three or four support vertices and the remaining vertices of the caterpillar are leaves.

A tree T is in $\mathcal{F}$ if it can be constructed using one of the following operations.

- **Operation** $\mathcal{F}_0$: Let T_1 and T_2 be in $\mathcal{T}$, each of order at least three. Form T from $T_1 \cup T_2$ by adding an edge xy, where $x \in B(T_1) - H(T_1)$ and $y \in B(T_2) - H(T_2)$.
- **Operation** $\mathcal{F}_1$: Let $T_1 \in \mathcal{T}_1$ and $T_2 \in \mathcal{T}_1$. Form T from $T_1 \cup T_2$ by adding an edge xy, where $x \in V(T_1)$, $y \in A(T_2)$.
- **Operation** $\mathcal{F}_2$: Let $T_1 \in \mathcal{T}_3$ and $T_2 \in \mathcal{T}_1$. Form T from $T_1 \cup T_2$ by adding an edge xy, where $x \in H(T_1)$ and $y \in A(T_2)$.
- **Operation** $\mathcal{F}_3$: Let $T_1 \in \mathcal{T}$ and $T_2 \in \mathcal{T}_2 \cup \mathcal{T}_3$ with $T_2 \neq P_2$. Form T from $T_1 \cup T_2$ by adding an edge xy, where $x \in B(T_1) - H(T_1)$ and $y \in A(T_2) - L(T_2)$.
- **Operation** $\mathcal{F}_4$: Let T_1 and T_2 be in $\mathcal{T}$, each of order at least four. Form T from $T_1 \cup T_2$ by adding an edge xy, where either $x \in A(T_1) - L(T_1)$ and $y \in A(T_2) - L(T_2)$, or $x \in L(T_1) - X$, $y \in A(T_2) - L(T_2)$ and at least T_1 or T_2 is in $\mathcal{T}_1$.
- **Operation** $\mathcal{F}_5$: Let $T_1 \in \mathcal{T}_2$ and $T_2 \in \mathcal{T}$ but not both a path P_2. Form T from $T_1 \cup T_2$ by adding a path xzy, where x is a vertex of maximum degree in T_1, $y \in A(T_2) - X(T_2)$ and z is a new vertex.
- **Operation** $\mathcal{F}_6$: Let $T_1 \in \mathcal{T}_1$ and $T_2 \in \mathcal{T}_3$. Form T from $T_1 \cup T_2$ by adding a path $xvwzy$, where v, w, z are new vertices, $x \in A(T_1), y \in A(T_2)$, and at least one of x and y is not in $L(T_1) \cup L(T_2)$ or $x \in L(T_1), y \in L(T_2)$ and $T_1 = P_3$.
- **Operation** $\mathcal{F}_7$: Let $T_1 \in \mathcal{T}_1$ and $T_2 \in \mathcal{T}_1$. Form T from $T_1 \cup T_2$ by adding a path $xvwzy$, where $x \in A(T_1)$, $y \in A(T_2)$ and v, w, z are new vertices.

- **Operation** $\mathcal{F}_8$: Let $T_1 \in \mathcal{T}_3$ and $T_2 \in \mathcal{T}_3$. Form T from $T_1 \cup T_2$ by adding a path $xvwzy$, where v, w, z are new vertices, $x \in A(T_1) - L(T_1), y \in A(T_2) - L(T_2)$.

3. Trees T with $\gamma_2(T) = \gamma(T) + 2$

Theorem 7. *A tree T satisfies $\gamma_2(T) = \gamma(T) + 2$ if and only if $T \in \mathcal{G} \cup \mathcal{F}$.*

Proof. Let T be a tree with $\gamma_2(T) = \gamma(T)+2$ and S any $\gamma_2(T)$-set. For any vertex $x \in V - S$, let $S_x = N(x) \cap S$. Clearly $|S_x| \geq 2$. Since T is a tree, for every pair of vertices x, y in $V - S$, $|S_x \cap S_y| \leq 1$. Let x, y be two adjacent vertices of $V - S$ and let T_x, T_y the subtrees of T obtained by removing the edge xy. Note that each of T_x and T_y has order at least three since $|S_x| \geq 2$ and $|S_y| \geq 2$. Then $S \cap V(T_x)$ and $S \cap V(T_y)$ are two 2-dominating sets of T_x and T_y, respectively. Hence $\gamma_2(T_x) + \gamma_2(T_y) \leq |S \cap V(T_x)| + |S \cap V(T_y)| = \gamma_2(T)$. On the other hand if D_x (respectively, D_y) is any $\gamma(T_x)$-set (respectively, $\gamma(T_y)$-set), then $D_x \cup D_y$ is a dominating set of T and so $\gamma(T) \leq \gamma(T_x) + \gamma(T_y)$. Also by Theorem 4, $\gamma_2(T_x) \geq \gamma(T_x) + 1$ and $\gamma_2(T_y) \geq \gamma(T_y)+1$. Therefore we obtain $\gamma(T)+2 = \gamma_2(T) \geq \gamma_2(T_x)+\gamma_2(T_y) \geq \gamma(T_x)+1+\gamma(T_y)+1 \geq \gamma(T)+2$, implying equality throughout the inequality chain, in particular $\gamma_2(T_x) = \gamma(T_x)+1$ and $\gamma_2(T_y) = \gamma(T_y)+1$. It follows that each of T_x and T_y belongs to $\mathcal{T} - \{P_2\}$, where $x \in B(T_x)$ and $y \in B(T_y)$. If $y \in H(T_y)$, then $S(T_x) \cup S(T_y) \cup H(T_x)$ (possibly $H(T_x) = \emptyset$) is a dominating set of T of size less than $\gamma_2(T) - 2$, a contradiction. Hence $y \notin H(T_y)$ and likewise $x \notin H(T_x)$. Therefore $T \in \mathcal{F}$ since it can be constructed using Operation $\mathcal{F}_0$. From now on we may assume that $V - S$ is independent.

Assume that $|S_u| \geq 4$ for some vertex $u \in V - S$. Then $\{u\} \cup S - S_u$ is a dominating set of T with cardinality at most $\gamma_2(T) - 3$, a contradiction. Thus every vertex of $V - S$ has degree two or three.

Now let x be a vertex of $V - S$ of degree three. Let $y \in S_x$ such that the subtrees obtained by removing the edge xy are both nontrivial. If such a vertex y does not exist, then $T = K_{1,3}$ that belongs to $\mathcal{G}_2$. Hence suppose that y exists. Then $S \cap V(T_x)$ is a 2-dominating set of T_x and likewise $S \cap V(T_y)$ for T_y. Thus $\gamma_2(T_x) + \gamma_2(T_y) \leq |S \cap V(T_x)| + |S \cap V(T_y)| = \gamma_2(T)$. Moreover if D_x (respectively, D_y) is any $\gamma(T_x)$-set (respectively, $\gamma(T_y)$-set), then $D_x \cup D_y$ is a dominating set of T and so $\gamma(T) \leq \gamma(T_x) + \gamma(T_y)$. Using Theorem 4 we obtain $\gamma(T) + 2 = \gamma_2(T) \geq \gamma_2(T_x) + \gamma_2(T_y) \geq \gamma(T_x) + 1 + \gamma(T_y) + 1 \geq \gamma(T) + 2$, implying equality throughout the inequality chain, in particular $\gamma_2(T_x) = \gamma(T_x) + 1$ and $\gamma_2(T_y) = \gamma(T_y) + 1$. It follows that each of T_x and T_y belongs to $\mathcal{T}$, where $x \in B(T_x)$ and $y \in A(T_y)$. Note that since $x \in B(T_x)$, T_x has order at least three. If T_x and T_y are in $\mathcal{T}_1$, then T can be constructed using Operation $\mathcal{F}_1$. Thus assume that at least one of T_x and T_y is in $\mathcal{T}_2 \cup \mathcal{T}_3$, say $T_y \in \mathcal{T}_2 \cup \mathcal{T}_3$. Since $x \in B(T_x)$, by Observation 6, x is either a support vertex or the center vertex if $T_x \in \mathcal{T}_3$.

First assume that x is a support vertex. Suppose that $y \in L(T_y)$ and let w be the unique neighbor of y in T_y. Since $T_y \in \mathcal{T}_2 \cup \mathcal{T}_3$ either $w \in B(T_y)$ or $w \in A(T_y)$ if $y \in X$. In addition let z be the second neighbor of w if $T_y \in \mathcal{T}_3$. Now if $T_y = P_2$, then $T_x \neq P_4$ for otherwise T is a corona of a path P_3 and so by Theorem 4, $\gamma_2(T) = \gamma(T) + 1$, a contradiction. It follows that T belongs to $\mathcal{G}_1$. Suppose now that $T_y \neq P_2$. Then for all possibilities of T_x to be in $\mathcal{T}$, and $T_y \in \mathcal{T}_2 \cup \mathcal{T}_3$ with $T_y \neq P_2$, the set $S(T_x) \cup S(T_y) \cup H(T_x) \cup \{z\} - \{w\}$ (possibly $H(T_x) = \emptyset$ if $T_x \notin \mathcal{T}_3$) is a dominating set of T of size $\gamma_2(T) - 3$, a contradiction. Thus $y \in A(T_y) - L(T_y)$ and so T can be constructed using Operation $\mathcal{F}_3$.

Suppose now that x is not a support vertex. Thus $x \in H(T_x)$ and hence $T_x \in \mathcal{T}_3$. We shall show that $T_y \in \mathcal{T}_1$. Assume that T_y is in $\mathcal{T}_2 \cup \mathcal{T}_3$ and suppose that y is not a leaf. Then since $y \in A(T_y)$, y is either a neighbor of $H(T_y)$ if $T_y \in \mathcal{T}_3$ or y is the neighbor of $X(T_y)$ if $T_y \in \mathcal{T}_2$ (in the later case y is a support vertex). Anyway it can be seen that $S(T_x) \cup S(T_y) \cup Q$ is a dominating set of T of size $\gamma_2(T) - 3$, where $Q = \{y\}$ if $T_y \in \mathcal{T}_3$ and $Q = \emptyset$ otherwise. Hence y is a leaf in T_y. Let u be the unique neighbor of y in T_y. Clearly if $T_y = P_2$, then $S(T_x) \cup \{y\}$ is a dominating set of T of size less than $\gamma_2(T) - 2$, a contradiction. Thus $T_y \neq P_2$ and so u is a support vertex in T_y. But then $S(T_x) \cup S(T_y) \cup \{y\} \cup H(T_y) - \{u\}$ (possibly $H(T_y) = \emptyset$ if $T_y \notin \mathcal{T}_3$) is a dominating set of T of size less than $\gamma_2(T) - 2$, a contradiction too. Consequently $T_y \in \mathcal{T}_1$ and so T is constructed using Operation $\mathcal{F}_2$. From now on we may suppose that every vertex in $V - S$ has degree two.

Suppose now that T contains a support vertex w with at least two leaves. If $w \in V - S$, then by the previous assumption $\deg_T(w) = 2$ and so $T = P_3$ but then $\gamma_2(T) = \gamma(T) + 1$, a contradiction. Thus $w \in S$. Let w' be any leaf neighbor of w and consider the tree $T' = T - \{w'\}$. Clearly $\gamma(T') = \gamma(T)$ and $\gamma_2(T') \leq \gamma_2(T) - 1$. Therefore $\gamma(T') + 1 \leq \gamma_2(T') \leq \gamma_2(T) - 1 = (\gamma(T) + 2) - 1 = \gamma(T') + 1$, implying that $\gamma_2(T') = \gamma(T') + 1$. By Theorem 4 $T' \in \mathcal{T}$ and $T' \neq P_2$. Hence $T \in \mathcal{G}_2$. We may assume for the next that every support vertex is adjacent to exactly one leaf.

We now suppose that the subgraph $G[S]$ contains an edge uv for which the removing provides two nontrivial subtrees. Let T_u and T_v the resulting subtrees, where $u \in V(T_u)$ and $v \in V(T_v)$. By a similar argument to that used above we have $\gamma(T) + 2 = \gamma_2(T) \geq \gamma_2(T_u) + \gamma_2(T_v) \geq \gamma(T_u) + 1 + \gamma(T_v) + 1 \geq \gamma(T) + 2$ and so $\gamma_2(T_u) = \gamma(T_u) + 1$, $\gamma_2(T_v) = \gamma(T_v) + 1$. Hence each of T_u and T_v is in $\mathcal{T}$, where $u \in A(T_u)$ and $v \in A(T_v)$. Also each T_u and T_v has order at least three for otherwise S is not minimal since either $S - \{u\}$ or $S - \{v\}$ is 2-dominating set of T. We also note that if $T_u \in \mathcal{T}_2$ and $u \in X(T_u)$, then $S - \{u\}$ is 2-dominating set of T, a contradiction. Thus if $T_u \in \mathcal{T}_2$, then $u \notin X(T_u)$ and similarly if $T_v \in \mathcal{T}_2$, then $v \notin X(T_v)$. Now if u and v are both not leaves, then $|V(T_u)| \geq 4$ and $|V(T_v)| \geq 4$, and therefore T is constructed using Operation $\mathcal{F}_4$. Assume now that u and v are both leaves in T_u and T_v, respectively. If T_u and T_v belong to $\mathcal{T}_1$, then T is constructed by using Operation $\mathcal{F}_1$. Thus at least one of T_u and T_v is in $\mathcal{T}_2 \cup \mathcal{T}_3$, say $T_v \in \mathcal{T}_2 \cup \mathcal{T}_3$. If $T_u = P_3$, then $T_v \neq P_4$ for otherwise $T = P_7 \in \mathcal{T}$. Consequently $T \in \mathcal{G}_3$. Thus we assume that each of T_u and T_v has order at least four and recall that $u \notin X(T_u)$ and $v \notin X(T_v)$. Let u' be the support vertex of T_u adjacent to u and let v' the support of T_v adjacent to v. If $T_u \in \mathcal{T}_2$, then $S(T_u) \cup S(T_v) \cup \{v\} \cup H(T_v) - (\{u', v'\}$ is a dominating set of T of size less than $\gamma_2(T) - 2$, a contradiction. Thus $T_u \notin \mathcal{T}_2$ and likewise $T_v \notin \mathcal{T}_2$. Hence, without loss of generality, either $T_u \in \mathcal{T}_1$ and $T_v \in \mathcal{T}_3$ or $T_u, T_v \in \mathcal{T}_3$. Since for both cases $T_v \in \mathcal{T}_3$, let v'' be the second neighbor of v' in T_v. If $T_u \in \mathcal{T}_1$ and $T_v \in \mathcal{T}_3$, then $S(T_u) \cup S(T_v) \cup \{u, v''\} - \{u', v'\}$ is a dominating set of T of size $\gamma_2(T) - 3$. If $T_u, T_v \in \mathcal{T}_3$, then $S(T_u) \cup S(T_v) \cup H(T_u) \cup \{u, v''\} - \{u', v'\}$ is a dominating set of T of size $\gamma_2(T) - 3$. Both cases yield to a contradiction. Finally assume, without loss of generality, that u is a leaf in T_u and v is not a leaf in T_v. By examining case by case, it can be seen that at least one of T_u or T_v must be in $\mathcal{T}_1$. For the remaining cases T admits a dominating set of T of size $\gamma_2(T) - 3$. Thus T can be constructed by Operation $\mathcal{F}_4$.

Assume now that $G[S]$ contains at least one edge but each one is pendant in T. Let $u \in S$ be a support and $v \in S$ its unique leaf. Let w be a vertex of $V - S$ adjacent to u for which the removing provides two nontrivial subtrees. If such a vertex does not exist, then T is a corona of a star and by Theorem 4, $\gamma_2(T) = \gamma(T) + 1$, a contradiction. Hence w exists and let r be the second neighbor of w in S. Consider the nontrivial subtrees T_r and T_u obtained by removing w (remember that w has degree two in T). Then $\gamma(T) + 2 = \gamma_2(T) \geq \gamma_2(T_u) + \gamma_2(T_r) \geq \gamma(T_u) + 1 + \gamma(T_r) + 1 \geq \gamma(T) + 2$ and so $\gamma_2(T_u) = \gamma(T_u) + 1$ and $\gamma_2(T_r) = \gamma(T_r) + 1$. It follows that T_u and T_r belong to $\mathcal{T}$, where $u \in A(T_u)$ and $r \in A(T_r)$. Moreover, since $u, v \in A(T_u)$ and u is a support vertex either $T_u = P_2$ or $T_u \in \mathcal{T}_2$ and u is the center vertex of T_u. Also T_u and T_r can not both be a path P_2 for otherwise $T = P_5$ and $\gamma_2(T) = \gamma(T)+1$, a contradiction. On the other hand if $T_r \in \mathcal{T}_2$ and $T_r \neq P_2$, then $r \notin X(T_r)$ for otherwise S would also contain the support vertex of r in T_r, say r', but in this case removing the edge rr' from $G[S]$ provides two nontrivial subtrees and such a case has been already considered. Thus $r \in A(T_r) - X(T_r)$ and therefore T can be constructed by Operation $\mathcal{F}_5$.

Now we can assume that S is independent. Since $V - S$ is an independent set in which every vertex has degree two, T is the subdivision graph of a tree T_0. Assume that S contains a vertex x of degree $k \geq 2$ such that $T - N[x]$ provides k nontrivial subtrees $T_1, T_2, \ldots, T_k$. Then $S \cap V(T_i)$ is a 2-dominating set of T_i for every i and clearly $\gamma(T) \leq 1 + \sum_{i=1}^{k} \gamma(T_i)$. Hence

$$\gamma(T)+2 = \gamma_2(T) \geq 1+\sum_{i=1}^{k}\gamma_2(T_i) \geq 1+\sum_{i=1}^{k}(\gamma(T_i)+1) \geq \gamma(T)+k \geq \gamma(T)+2,$$

implying equality throughout the inequality chain, in particular $k = 2$, that is $\deg_T(x) = 2$, $\gamma_2(T_i) = \gamma(T_i)+1$ for every $i = 1, 2$. Hence each of T_1 and T_2 belongs to $\mathcal{T}$. Let $N(x) = \{x', x''\}$ and assume, without loss of generality, that $S_{x'} = \{y', x\}$ and $S_{x''} = \{y'', x\}$, where $y' \in V(T_1)$ and $y'' \in V(T_2)$. Clearly $y' \in A(T_1)$ and $y'' \in A(T_2)$. Since S is independent, $T_1 \notin \mathcal{T}_2$ and $T_2 \notin \mathcal{T}_2$. Assume that y' and y'' are both leaves. If $T_1, T_2 \in \mathcal{T}_3$, then let y_1 be the neighbor of y' and $z_1 \neq y'$ be the neighbor of y_1 in T_1, and define similarly y_2 and z_2 in T_2. Then $S(T_1) \cup S(T_2) \cup \{z_1, x', x'', z_2\} - \{y_1, y_2\}$ is a dominating set of T of size less than $\gamma_2(T) - 2$, a contradiction. Thus, without loss of generality, $T_1 \in \mathcal{T}_1$ and $T_2 \in \mathcal{T}_1 \cup \mathcal{T}_3$. If T_1 has order three, then T is obtained by using Operation $\mathcal{F}_6$ (when $T_2 \in \mathcal{T}_3$) or Operation $\mathcal{F}_7$ (when $T_2 \in \mathcal{T}_1$). Hence suppose that T_1 has order at least five. Now if $T_2 \in \mathcal{T}_3$, then let us use the notation of y_1, z_1, y_2, z_2 as have been defined above. Then $S(T_1) \cup S(T_2) \cup \{y', x'', z_2\} - \{y_1, y_2\}$ is a dominating set of T of size less than $\gamma_2(T) - 2$, a contradiction. Thus $T_1 \in \mathcal{T}_1$ and $T_2 \in \mathcal{T}_1$, and therefore T can be constructed by Operation $\mathcal{F}_7$. For the next we will assume that at least one of x and y is not in $L(T_1) \cup L(T_2)$. If T_1 and T_2 are in $\mathcal{T}_1$, then T is constructed using Operation $\mathcal{F}_7$. Hence either ($T_1 \in \mathcal{T}_1$ and $T_2 \in \mathcal{T}_3$) or ($T_1 \in \mathcal{T}_3$ and $T_2 \in \mathcal{T}_3$). In the first case T is constructed using Operation $\mathcal{F}_6$. In the later case it can be seen that $y' \in A(T_1) - L(T_1)$ and $y'' \in A(T_2) - L(T_2)$ for otherwise T admits a dominating set of size less than $\gamma_2(T) - 2$, a contradiction. Thus T is obtained by using Operation $\mathcal{F}_8$.

Finally assume that for every vertex $x \in S$ of degree at least two the forest $T - N[x]$ contains a component of size one. Hence every vertex of S is either a leaf or at distance two from some leaf. Using this fact and since T is the subdivision graph of a tree T_0, it follows that every vertex of T_0 is either a support vertex or a leaf, that is $V(T_0) = S(T_0) \cup L(T_0)$. Let n_0 be the order of T_0. Then $|V(T)| = n = 2n_0 - 1$ and by Theorem 3, $\gamma_2(T) = \frac{n+1}{2} = n_0$, implying that $\gamma(T) = n_0 - 2$. Suppose that a support vertex x in T_0 is adjacent to at least three other support vertices, say u, v and w. Let u', v', w' be the subdivision vertices resulting by subdividing edges xu, xv and xw. Clearly $u', v', w' \in B(T)$ and $B(T)$ is a dominating set of T of size $n_0 - 1$ but then $\{x\} \cup B(T) - \{u', v', w'\}$ is a dominating set of T with cardinality $n_0 - 3$, a contradiction. Hence every support vertex of T_0 is adjacent to at most two other support vertices, more precisely T_0 is a caterpillar whose support vertices induce a path. If T_0 has one or two support vertices, then $T \in \mathcal{T}_1$ or $T \in \mathcal{T}_3$, respectively, and by Theorem 4, $\gamma_2(T) = \gamma(T) + 1$, a contradiction. Hence $|S(T_0)| \geq 3$. Suppose that $|S(T_0)| \geq 5$ and let $u_1, u_2, \ldots, u_5$ be five consecutive support vertices. Let v_i be the subdivision vertex resulting by subdividing the edge $u_i u_{i+1}$, where $1 \leq i \leq 4$. Then $\{u_2, u_4\} \cup B(T) - \{v_1, v_2, v_3, v_4\}$ is a dominating set of T of size $n_0 - 3$, a contradiction. It follows that T_0 is a caterpillar with three or four support vertices. Hence $T \in \mathcal{G}_4$.

Conversely, if $T \in \mathcal{G} \cup \mathcal{F}$, then $T \notin \mathcal{T}$ and so by Theorem 4, $\gamma_2(T) \geq \gamma(T) + 2$. Equality can be checked by examining case by case the trees of $\mathcal{G} \cup \mathcal{F}$. ■

Observe that any tree $T \in \mathcal{T} \cup \mathcal{G} \cup \mathcal{F}$ has diameter at most 12, indeed the tree of larger diameter is obtained by using Operation $\mathcal{F}_7$ or $\mathcal{F}_8$. Consequently Theorems 4 and 7 imply the following corollary.

Corollary 8. *If T is a tree of diameter at least 13, then $\gamma_2(T) \geq \gamma(T) + 3$.*

4. Trees T with $\gamma_{\hat{o}}(T) = \gamma(T) + 2$

Hedetniemi, Hedetniemi, and Kristiansen [4] introduced several types of alliances in graphs, including the global strong offensive alliances defined as follow: A set $S \subseteq V(G)$ is a *global strong offensive alliance* (abbreviated, gsoa) of G if $|N[v] \cap S| > |N[v] - S|$ for every vertex $v \in V(G) - S$. The *global strong offensive number* $\gamma_{\hat{o}}(G)$ is the minimum cardinality of a global strong offensive alliance of G.

Note if S is any global strong offensive alliance of G, then every vertex of $V(G) - S$ has at least two neighbors in S. Thus S is a 2-dominating set of G, and we obtain $\gamma_2(G) \leq \gamma_{\hat{o}}(G)$. Using this fact, it has been observed in [1] that for every nontrivial tree T, $\gamma_{\hat{o}}(T) \geq \gamma(T) + 1$ with equality if and only if $T \in \mathcal{T}$.

Next we present a characterization of trees T with $\gamma_{\hat{o}}(T) = \gamma(T) + 2$. For this purpose let $\mathcal{F}'$ be the subfamily of $\mathcal{F}$ consisting of all trees constructed by performing Operation $\mathcal{F}_0$.

Theorem 9. *A tree T satisfies $\gamma_{\hat{o}}(T) = \gamma(T) + 2$ if and only if $T \in \mathcal{G} \cup (\mathcal{F} - \mathcal{F}')$.*

Proof. Let T be a tree with $\gamma_{\hat{o}}(T) = \gamma(T) + 2$ and S any $\gamma_{\hat{o}}(T)$-set. Clearly $\gamma_2(T) = \gamma(T) + 2$ and so S is also a $\gamma_2(T)$-set. For a vertex $x \in V - S$, let $S_x = N(x) \cap S$. Then since T is a tree, $|S_x \cap S_y| \leq 1$ for every pair of vertices x, y in $V - S$. Assume now that u, v are two adjacent vertices in $V - S$. Then since S is a $\gamma_{\hat{o}}(T)$-set, $|S_u| \geq 3$ and $|S_v| \geq 3$, and so $S \cup \{u, v\} - (S_u \cup S_v)$ is a dominating set of T with cardinality at most $|S \cup \{u, v\} - (S_u \cup S_v)| \leq \gamma_{\hat{o}}(T) - 4$, a contradiction. Thus $V - S$ is independent. Since S is a $\gamma_2(T)$-set, all steps in the proof of the Theorem 7 remain valid here and therefore $T \in \mathcal{G} \cup (\mathcal{F} - \mathcal{F}')$.

Conversely, every tree $T \in \mathcal{G} \cup (\mathcal{F} - \mathcal{F}')$ admits a $\gamma_2(T)$-set that is also a global strong offensive alliance of T. Thus $\gamma(T) + 2 \leq \gamma_2(T) \leq \gamma_{\hat{o}}(T) \leq \gamma_2(T) = \gamma(T) + 2$. Therefore $\gamma_{\hat{o}}(T) = \gamma(T) + 2$. ■

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