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COMPLETE MINORS, INDEPENDENT SETS, AND CHORDAL GRAPHS

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Abstract

The Hadwiger number $h(G)$ of a graph G is the maximum size of a complete minor of G. Hadwiger's Conjecture states that $h(G) \geq \chi(G)$. Since $\chi(G)\alpha(G) \geq |V(G)|$, Hadwiger's Conjecture implies that

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$\alpha(G)h(G) \geq |V(G)|$. We show that $(2\alpha(G) - \lceil \log_\tau(\tau\alpha(G)/2)\rceil)h(G) \geq |V(G)|$ where $\tau \approx 6.83$. For graphs with $\alpha(G) \geq 14$, this improves on a recent result of Kawarabayashi and Song who showed $(2\alpha(G) - 2)$ $h(G) \geq |V(G)|$ when $\alpha(G) \geq 3$.

Keywords: clique minor, independence number, Hadwiger conjecture, chordal graphs.

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1. Introduction

Hadwiger's Conjecture [10] from 1943 states the following (see [18] for a survey):

Conjecture. For every k-chromatic graph G, K_k is a minor of G.

Hadwiger's Conjecture for $k = 4$ was proved by Dirac [6], the case $k = 5$ was shown equivalent to the Four Color Theorem [1, 2, 16] by Wagner [19] and the case $k = 6$ was shown equivalent to the Four Color Theorem by Robertson *et al.* [17]. Hadwiger's Conjecture for $k \geq 7$ remains open. Let $h(G)$ denote the Hadwiger number, the size of the largest complete minor of G. Since $\alpha(G)\chi(G) \geq |V(G)|$, Hadwiger's Conjecture implies the following conjecture, which was observed in [8, 14] and [21].

Conjecture. For every graph G, $\alpha(G)h(G) \geq |V(G)|$.

This conjecture seems weaker than Hadwiger's Conjecture, however Plummer, Stiebitz, and Toft [15] showed that for graphs with $\alpha(G) = 2$, the two conjectures are equivalent. In 1981, Duchet and Meyniel [8] showed that $(2\alpha(G) - 1)h(G) \geq |V(G)|$. No general improvement on this theorem has been made for the case $\alpha(G) = 2$. Seymour asked for any improvement on this result for $\alpha(G) = 2$, conjecturing that there exists an $\epsilon > 0$ such that if $\alpha(G) = 2$, then G has a complete minor of size $(1/3 + \epsilon)n$. Recently, Kawarabayashi, Plummer, and Toft [11] showed that $(4\alpha(G) - 3)$ $h(G) \geq 2\,|V(G)|$ when $\alpha(G) \geq 3$ and Kawarabayashi and Song [12] improved this to $(2\alpha(G) - 2)h(G) \geq |V(G)|$ when $\alpha(G) \geq 3$. Wood [20] proved $(2\alpha(G) - 1)(2h(G) - 5) \geq 2\,|V(G)| - 5$ for all graphs G. Our main result is to improve the bound for graphs with $\alpha(G) \geq 14$.

Theorem 1. *Let G be an n-vertex graph. Then $K_{\lceil n/r \rceil}$ is a minor of G, where*

$$r = 2\alpha(G) - \lceil \log_\tau(\tau\alpha(G)/2) \rceil \text{ and } \tau = \frac{2\sqrt{2}}{\sqrt{2}-1} \approx 6.83.$$

Using a more careful analysis, we are able to improve the result for $\alpha(G) = 5$.

Theorem 2. *Let G be an n-vertex graph with $\alpha(G) = 5$. Then $K_{5n/38}$ is a minor of G.*

The proof of Theorem 2 appears in the appendix which is posted online [3].

A graph G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G. For two vertex sets $T, S \subseteq V(G)$, we say T *touches* S if $T \cap S \neq \emptyset$ or there is an edge $xy \in E(G)$ with $x \in T$ and $y \in S$. For $T \subseteq V(G)$, we define $\alpha(T) = \alpha(G[T])$ and $N(T) = \{x \in V(G) : \exists y \in T, xy \in E(G)\} = \cup_{v \in T} N(v)$. If H is a subgraph of G and $T \subseteq V(G)$, then we define $H \cap T = G[V(H) \cap T]$. H is a *spanning subgraph* of G if H is a subgraph of G and $V(H) = V(G)$. A graph G is *chordal* if G has no induced cycle of length at least 4. A vertex is *simplicial* if its neighborhood is a clique. A *simplicial elimination ordering* is an order $v_n, \ldots, v_1$ in which vertices can be deleted so that each vertex v_i is a simplicial vertex of the graph induced by $\{v_1, \ldots, v_i\}$. A *partial simplicial elimination ordering* is an ordered vertex set $U = \{v_1, \ldots, v_k\} \subseteq V(G)$, such that for each $v_i v_j \notin E(G)$ with $i < j$ and $v_i, v_j \in U$ and each component C of $G - \{v_1, \ldots, v_j\}$, at most one of v_i or v_j touches C. Dirac [7] proved that a graph is chordal if and only if it has a simplicial elimination ordering, and Berge [4] observed that by greedily coloring the vertices of a simplicial elimination ordering one obtains an $\omega(G)$-coloring of G, proving that chordal graphs are perfect.

Let $f : V(G) \to \mathbb{Q}^+$ be a weight function on $V(G)$. For $A \subseteq V(G)$, define $f(A) = \sum_{v \in A} f(v)$. Then the *weighted independence number of G relative to f* is

$$\alpha_f(G) = \max \{f(A) : A \text{ is an independent set in } G\}.$$

We shall need the following result.

Theorem 3. *Let H be a perfect graph and f a weight function on $V(H)$. Then*

$$\omega(H) \geq \left\lceil \frac{f(V(H))}{\alpha_f(H)} \right\rceil.$$

The goal of our algorithm is to find a minor H of G such that H is a chordal graph, and then to devise a weight function on the vertices of H to which we apply Theorem 3. Most of the time, the weight of a vertex v in H is the number of vertices of G which are contracted to v. The algorithm builds the minor H by using two operations: extension and breaking. The key property is that at each step, the algorithm uses the operations to increase the number of vertices in a partial simplicial elimination ordering. Once all vertices are included in the partial simplicial elimination ordering, we have a simplicial elimination ordering, so that the algorithm has produced a chordal graph.

In Section 2.2, we provide an algorithm which yields an alternate proof of Kawarabayashi and Song's [12] result.

Theorem 4. *Let G be an n-vertex graph. Then $K_{\lceil r \rceil}$ is a minor of G, where*

$$r = \begin{cases} n & \alpha(G) = 1, \\ \frac{n}{3} & \alpha(G) = 2, \\ \frac{n}{2\alpha(G)-2} & \alpha(G) \geq 3. \end{cases}$$

We say that G has an *odd complete minor of order at least* ℓ if there are ℓ disjoint trees in G such that every two of them are joined by an edge, and in addition, all the vertices of the trees are two-colored in such a way that edges within trees are bichromatic and edges between trees are monochromatic. Gerards and Seymour conjectured that if a graph has no odd complete minor of order ℓ, then it is $(\ell - 1)$-colorable. This is substantially stronger than Hadwiger's Conjecture. The algorithm used by Duchet and Meyniel to prove $(2\alpha(G) - 1)h(G) \geq |V(G)|$ produces odd complete minors. The algorithm in our alternate proof of Kawarabayashi and Song's [12] result in Section 2.2 can be shown to produce an odd complete minor, so every graph with $\alpha(G) \geq 3$ has an odd complete minor of size at least $n/(2\alpha(G) - 2)$. With a little more work, we can show that our algorithm in Section 2.3 not only produces a complete minor but actually produces an odd complete minor. Therefore every graph G has an odd complete minor of size at least $n/(2\alpha(G) - \log_\tau(\tau\alpha(G)/2))$.

The rest of the paper is organized as follows: in Section 2 we define the operations and define the algorithms, in Section 3 we prove some lemmas and theorems about the operations used during the algorithms, and in Section 4 we analyze the algorithm. In the appendix posted online [3] we specialize the algorithm to $\alpha(G) = 5$, and by changing the weight function we find a complete minor of size at least $5n/38$, which is slightly larger than the $n/8 = 5n/40$ produced by the general algorithm.

After completing our work, we learned that Fox [9] proved using claw-free structural theorems of Chudnovsky and Fradkin [5] that every graph G has a complete minor of size at least $\frac{|V(G)|}{(2-c)\alpha(G)}$ where $c \approx 0.017$. Our result is better when $\alpha(G) \leq 230$ and has the advantage of producing an odd complete minor.

2. Definition of the Algorithms

The algorithm first builds a family of disjoint vertex sets spanning connected graphs which partition $V(G)$ and a spanning subgraph of G. We start with the empty family and at each step apply an operation which either adds a new set to the family, adds vertices to an existing set in the family, or updates the spanning subgraph. To identify the spanning subgraph, we color the edges of G: initially all edges are blue and during the algorithm we color some edges red. We denote the spanning subgraph induced on the blue edges by G_b. When we color some edges red, we make sure that each element in the family spans a connected graph in G_b. Once we have obtained a partition of $V(G)$, we define a graph H by starting with G_b and contracting each set of the partition to a single vertex. We need the spanning subgraph G_b because starting from G and contracting each set in the partition might not yield a chordal graph. Throughout this paper, a subscript of G is implied on α and N.

2.1. Operations used in the algorithm

There are two operations that are carried out by the algorithm: extending and breaking. We are given a labeled (ordered) family of disjoint vertex sets $\mathcal{F}$ and a red/blue coloring of the edges of G. Let $U = V(G) - \cup_{T\in\mathcal{F}}T$, and let G_b be the spanning subgraph of blue edges. We define the following operations:

Extending T into X by k: Let $T \in \mathcal{F}$, let $X \subseteq U$ such that $G_b[X \cup T]$ is connected, there are no red edges between T and X, and let $k \in \mathbb{Z}^+$ such that $k \leq \alpha(X - N(T))$. The operation extends T into X by k by adding at most $2k$ vertices from X into T so that the new $G_b[T]$ is still connected and we increase $\alpha(T)$ by at least k. When extending T into X, the order of the sets in $\mathcal{F}$ is unchanged. In the extension we always follow the algorithm described in the proof of Lemma 12, which shows that such an extension is possible. Extending T into X by k is always *acceptable*.

Breaking X by k: Let k be a positive integer, and let $X \subseteq U$ such that X does not touch $U - X$ in G (i.e., X is a union of some components of $G[U]$).

Step (a). For any $T \in \mathcal{F}$ and any component D of $G[X]$ with $\alpha(D - N(T)) = \alpha(D)$, we color all edges between T and D red.

Step (b). If there exists a component D of $G[X]$ with independence number at least k, let I be an independent set in D with $|I| \geq k$ and let v be any vertex in I. Add $T = \{v\}$ to $\mathcal{F}$ and then extend T into $D - T$ by $k - 1$. Lemma 13 shows that $T, D - T, k$ satisfy the conditions in the extension. We then set $X := X - T$ and continue Step (b) until every component in $G[X]$ has independence number strictly less than k. The new sets produced are added last in the ordering of $\mathcal{F}$.

Definition. We say that breaking X by k is *acceptable* if both of the following conditions hold before we start breaking (before Step (a)):

- For all $T \in \mathcal{F}$ and every component D of $G[X]$ either the edges between T and D are already red, or $\alpha(D - N(T)) = \alpha(D)$ (the edges will become red in Step (a)), or $\alpha(D - N(T)) < k$.
- For every component D of $G[X]$, $\alpha(D) < 2k$.

In other words, an acceptable breaking means each set T in the original family and each component D of U will either have the edges between T and D colored red or touch with blue edges every set born during Step (b) in D, and the new sets will touch each other as well.

Definition. We say that $\mathcal{F}$ is *formed by acceptable operations* in G if $\mathcal{F}$ is formed by starting with the empty family and then performing any sequence of acceptable operations. When we extend T into X by k we say that the *amount* of the extension is k. For $T \in \mathcal{F}$, define $\operatorname{ext}(T)$ to be one plus the total amount of extensions of T, which includes the extensions in the breaking when T was born and all other extensions of T.

In Theorem 5 we show that we obtain a chordal graph when we start with the graph G_b and contract each set of the partition.

Theorem 5. *Let $\mathcal{F}$ be a partition of $V(G)$ formed by acceptable operations in G, and let G_b be the spanning subgraph of blue edges. Let H be the graph obtained by starting from G_b and contracting each set of $\mathcal{F}$ to a single vertex. Then H is a chordal graph.*

Lemma 6. *Consider a family $\mathcal{F}$ formed by a sequence of operations in G. Then for every $T \in \mathcal{F}$, $|T| \leq 2\,\mathrm{ext}(T) - 1$. Also, $\mathrm{ext}(T) \leq \alpha(T)$.*

An acceptable breaking of X by k requires that for each component D of $G[X]$ and each $T \in \mathcal{F}$ we have $\alpha(D - N(T))$ in the correct range. The following lemma shows that we can control $\alpha(D-N(T))$ by using the extension operation.

Lemma 7. *Let T' be the set formed by extending T into X by k. Then $\alpha(X - N(T)) - k \geq \alpha(X - T' - N(T'))$. That is, extending T into X by k using the procedure in Lemma 12 reduces $\alpha(X - N(T))$ by at least k.*

2.2. The $2\alpha(G) - 2$ algorithm

Let $n = |V(G)|$. We are going to build a partition $\mathcal{F}$ of $V(G)$ using only a sequence of breaking operations. At any stage of the algorithm, let $U = V(G) - \cup_{T\in\mathcal{F}} T$.

Case $\alpha(G) = 1$. Note that this conclusion is obvious but we give a detailed argument to make the reader more familiar with the definitions. The algorithm is to break $V(G)$ by 1. This is an acceptable operation because before the breaking $\mathcal{F}$ is empty and every component of G has independence number 1. Breaking $V(G)$ by 1 does not color any edges red because the family before the breaking is empty, and so the breaking results in a family of singleton sets $\mathcal{F} = \{\{v\} : v \in V(G)\}$ with $G_b = G$. Theorem 5 shows G is chordal, and using the weight function $f(v) = 1$ we have that the total weight is n and $\alpha_f(G) = \alpha(G) = 1$. Thus Theorem 3 shows that $\omega(G) \geq n$.

Case $\alpha(G) = 2$. We first break $V(G)$ by 2. This is acceptable because before the breaking $\mathcal{F}$ is empty and every component of G has independence number at most 2. No edges are colored red, and so this breaking results in a family $\mathcal{F}$ of disjoint induced P_3s (P_3 is the unique connected graph on three vertices with independence number 2). This family $\mathcal{F}$ is maximal because the remaining vertices (the set U) induce a disjoint union of cliques. We next break U by 1. This is acceptable because each $T \in \mathcal{F}$ dominates U so $\alpha(U - N(T)) = 0$, and each component of $G[U]$ is a clique. Also, no edge is colored red because each P_3 in $\mathcal{F}$ dominates U. Thus the two breaking operations produce a partition of $V(G)$ into a maximal family of induced P_3s and singleton sets of the remaining vertices, with all edges colored blue ($G_b = G$). We now contract each P_3 to form the graph H, and use the weight function $f(v) = 3$ for a vertex v obtained by contracting a P_3, and $f(v) = 1$ otherwise. Thus $f(v)$ records the number of vertices in the set in $\mathcal{F}$ that is contracted down to v, and the total weight $f(V(H)) = n$. Theorem 5 shows that H is a chordal graph. To compute $\alpha_f(H)$, take any independent set I in H. This independent set corresponds to a pairwise non-touching subfamily $\mathcal{I}$ of $\mathcal{F}$. Since no edges are colored red, $\mathcal{I}$ is pairwise non-touching in G. Then either $\mathcal{I}$ contains one P_3 and nothing else (in which case $f(I) = 3$) or at most two single vertices (in which case $f(I) = 2$). Thus $\alpha_f(H) \leq 3$ so Theorem 3 shows that $\omega(H) \geq \lceil n/3 \rceil$, that is we have a complete minor of G of size at least $\lceil n/3 \rceil$.

Case $\alpha(G) \geq 3$. Initially, $U = V(G)$ and $\mathcal{F} = \emptyset$.

Step 1. Let C be any component of $G[U]$. If $\alpha(C)$ is 1 or 2, then we break C like in the above two cases. If $\alpha(C) \geq 3$, then we break C by $\alpha(C) - 1$.

Step 2. We now update $U := U - \cup_{T \in \mathcal{F}} T$ and continue Step 1 with a new C until $U = \emptyset$.

First, all the breakings are acceptable. Consider a component C we are about to break in Step 1. Now consider any set T that has already been produced, say T was born when C' was broke. If C is not contained in C' then there are no edges between T and C so $\alpha(C - N(T)) = \alpha(C)$. If $C \subseteq C'$, then $\alpha(T) = \alpha(C') - 1$ so that $\alpha(C' - N(T)) \leq 1$ which implies that $\alpha(C - N(T)) \leq 1$. Thus breaking C by $\alpha(C) - 1$ is acceptable. Because $\alpha(C - N(T)) \leq 1$, the only possibility for edges to be colored red in Step 1 is when we choose a component C with $\alpha(C) = 1$. Thus for each $T \in \mathcal{F}$, we have $G[T] = G_b[T]$.

Now consider the graph H formed by starting with G_b and contracting each set of $\mathcal{F}$. Consider the weight function f on $V(H)$ where we assign to each vertex of H the size of the set of $\mathcal{F}$ which it came from. Thus the total weight of f on H is n. By Theorem 5 we know that H is a chordal graph.

Next, we show that $\alpha_f(H) \leq 2\alpha(G) - 2$. Consider any independent set I in H. This corresponds to a pairwise non-touching (in G_b) subfamily $\mathcal{I}$ of $\mathcal{F}$. By Lemma 6, $|T| \leq 2\,\mathrm{ext}(T) - 1$ so that we can bound the total weight of I as follows:

$$f(I) = \sum_{T\in\mathcal{I}} |T| \leq 2\sum_{T\in\mathcal{I}} \mathrm{ext}(T) - |I|\,.$$

If $|I| = 1$ then the largest breaking we ever do is by $\alpha(G) - 1$ which produce sets with $\mathrm{ext}(T) \leq \alpha(G) - 1$ which have size at most $2\alpha(G) - 3$. So assume $|\mathcal{I}| \geq 2$.

Claim 8. For any pairwise non-touching family $\mathcal{I}$ in G_b, $\sum_{T\in\mathcal{I}} \mathrm{ext}(T) \leq \alpha(G)$.

Proof. Define $\mu(\mathcal{I})$ to be the total number of red edges between sets of $\mathcal{I}$. Assume we have a counterexample to Lemma 8 where $\mu(\mathcal{I})$ is minimized. In other words, a pairwise non-touching family $\mathcal{I}$ in G_b where $\sum_{T\in\mathcal{I}} \mathrm{ext}(T) > \alpha(G)$ and $\mu(\mathcal{I})$ is minimized. If $\mu(\mathcal{I}) = 0$, then $\mathcal{I}$ is a pairwise non-touching family in G so that $\sum_{T\in\mathcal{I}} \alpha(T) \leq \alpha(G)$. By Lemma 6, $\mathrm{ext}(T) \leq \alpha(T)$ so $\sum_{T\in\mathcal{I}} \mathrm{ext}(T) \leq \alpha(G)$.

Assume now that $\mu(\mathcal{I}) \geq 1$ and take some $T, R \in \mathcal{I}$ where there is a red edge between T and R. We will produce a subfamily $\mathcal{I}'$ spanning fewer red edges. For edges to be colored red one of T or R must be a single vertex. Assume $|R| = 1$, and let C be the component containing R (with $\alpha(C) = 1$) chosen in Step 1 which caused the edges between T and R to be colored red. Thus $\alpha(C - N(T)) = 1$ so there exists a vertex $v \in V(C) - N(T)$. Let $\mathcal{I}' = \mathcal{I} - R + \{v\}$. Note that since $v \in V(C)$ and $\alpha(C) = 1$, $\{v\} \in \mathcal{F}$. We now show that v does not touch any other set in $\mathcal{I}'$. Say that there exists an $S \in \mathcal{I} \cap \mathcal{I}'$ where S touches v in G. Since $v \in V(C) - N(T)$, we must have $T \neq S$. First assume $\mathrm{ext}(S) = 1$, so that $S = \{s\}$ for some vertex s. Then since s touches v we must have $s \in V(C)$. Note that when singletons are born, their component must be a clique. But then s touches R using a blue edge, contradicting that $S \in \mathcal{I}$. So we can assume $\mathrm{ext}(S) \geq 2$.

First assume T is indexed lower than S, and let C' be the component chosen in Step 1 when T was born. Then $\alpha(T) \geq \mathrm{ext}(T)$ and $\mathrm{ext}(T)$ is one plus the number of extensions during the breaking so $\mathrm{ext}(T) = \alpha(C') - 1$. Since S touches v and T is connected to v by a path of length 2 using a vertex of C we have that S is contained inside C'. Since $\alpha(S) \geq \mathrm{ext}(S) \geq 2$ we must have T touching S using blue edges, contradicting that both are in $\mathcal{I}$. Now assume that S is indexed lower than T, and let C' be the component chosen in Step 1 when S was born. Then $\alpha(S) \geq \text{ext}(S) = \alpha(C') - 1$ and since S touches v and T is connected to v by a path of length 2 using a vertex of C we have that T is contained inside C'. Since $\alpha(T) \geq \text{ext}(T) \geq 2$ we have that S touches T using blue edges, contradicting that both are in $\mathcal{I}$. Thus v does not touch any other set in $\mathcal{I}'$, so $\mathcal{I}'$ is pairwise non-touching in G_b and we have reduced the number of red edges. Also, $\sum_{T \in \mathcal{I}'} \text{ext}(T) = \sum_{T \in \mathcal{I}} \text{ext}(T) > \alpha(G)$ contradicting that $\mathcal{I}$ was a minimum counterexample. ■

Using Claim 8, we can immediately complete the proof since then $f(I) \leq 2\alpha(G) - |I| \leq 2\alpha(G) - 2$. To summarize, we can find a complete minor of G of size $\lceil r \rceil$, where r is defined as in Theorem 4. □

2.3. The $2\alpha(G) - \log_\tau(\tau\alpha(G)/2))$ algorithm

Given a graph G, we use the operations of breaking and extending to produce a partition $\mathcal{F}$ of $V(G)$ and a spanning subgraph G_b of blue edges. When we start the algorithm, $\mathcal{F}$ will be the empty family and $G_b = G$. The improvement from $2\alpha(G) - 2$ to $2\alpha(G) - \log_\tau(\tau\alpha(G)/2)$ comes from breaking each component C by $\lceil(\alpha(C) + 1)/2\rceil$ so we produce sets of size approximately $\alpha(C)$, and then we extend the sets of $\mathcal{F}$ before future breakings only if it would prevent the breaking from being acceptable.

Given a graph G, set $G_b = G$ so all edges are colored blue and set $\mathcal{F} = \emptyset$. We pick C to be any component of $G[V(G) - \cup_{T \in \mathcal{F}} T]$. If $\alpha(C) = 1$, we break C by 1 which constitutes Step C. So assume $\alpha(C) \geq 2$, and run the following substeps inside C, which constitutes Step C.

Substep 1. For each $T \in \mathcal{F}$ with $\alpha(C - N(T)) = \alpha(C)$, color all edges between T and C red. Then let $b = \left\lceil \frac{\alpha(C)+1}{2} \right\rceil$, and let $A = V(C)$. Partition $\mathcal{F}$ into three classes.

- $\mathcal{H}_0 = \{T \in \mathcal{F} :$ all edges between T and C are colored red$\}$,
- $\mathcal{H}_1 = \left\{T \in \mathcal{F} - \mathcal{H}_0 : \alpha(C - N(T)) < \sqrt{2}(b-1)\right\}$,
- $\mathcal{H}_2 = \mathcal{F} - \mathcal{H}_0 - \mathcal{H}_1$.

Substep 2. For any $T \in \mathcal{H}_1$ and any component D of $G[A]$ with $b \leq \alpha(D - N(T)) < \alpha(D)$, we extend T into $V(D)$ by $\alpha(D - N(T)) - b + 1$. We then update $A := A - T$ and continue Substep 2 until no pair T, D satisfies

$b \leq \alpha(D - N(T)) < \alpha(D)$. Note, for the first T selected during Substep 2 we will have $D = C$ so that T satisfies $b \leq \alpha(C - N(T)) < \alpha(C)$ and thus is extended.

If there exists some $T \in \mathcal{H}_1$ which was not extended during Substep 2 and some component D of $G[A]$ such that $b \leq \alpha(D - N(T)) = \alpha(D)$ then we do not continue to Substep 3, instead we finish Step C. Otherwise, continue to Substep 3.

Substep 3. For any $T \in \mathcal{H}_2$ and any component D of $G[A]$ with $b \leq \alpha(D - N(T)) < \alpha(D)$, we extend T into $V(D)$ by $\alpha(D - N(T)) - b + 1$. We then update $A := A - T$ and continue Step 3 until no pair T, D satisfies $b \leq \alpha(D - N(T)) < \alpha(D)$.

Substep 4. Break A by b.

If $\mathcal{F}$ is not yet a partition of $V(G)$, pick a new component C.

In Sections 3 and 4, we prove that using this algorithm we can find a complete minor of G of size $\lceil n/r \rceil$, where r is defined in Theorem 1.

3. Analysis of the Operations

3.1. Proofs of Theorems 3 and 5

If $V(G) = \{v_1, \ldots, v_n\}$ and $H_1, \ldots, H_n$ are pairwise disjoint graphs, then the *composition* $G[H_1, \ldots, H_n]$ is the graph formed by the vertex disjoint union of $H_1, \ldots, H_n$ plus the edges xy where $x \in V(H_i), y \in V(H_j)$ and $v_i v_j \in E(G)$. In 1972, Lovász [13] proved that a composition of perfect graphs is perfect.

***Proof of Theorem* 3.** First, we modify f by multiplying each weight by their common denominator so that $f : V(H) \to \mathbb{Z}^+$. Multiplying every weight by a constant does not change $f(V(H))/\alpha_f(H)$. For $v \in V(H)$, define $H_v = f(v)K_1$ to be an independent set of size $f(v)$. Then define $H' = H[H_{v_1}, \ldots, H_{v_n}]$ as a composition of H. Then $f(V(H)) = |V(H')|$, $\omega(H) = \omega(H')$, and $\alpha_f(H) = \alpha(H')$. (If I' is a maximal independent set in H', then either $H_v \subseteq I'$ or $H_v \cap I' = \emptyset$ because H_v is an independent set and every vertex in H_v has the same neighborhood.) Since H' is a perfect graph, we have

$$\omega(H) = \omega(H') = \chi(H') \geq \left\lceil \frac{|V(H')|}{\alpha(H')} \right\rceil = \left\lceil \frac{f(V(H))}{\alpha_f(H)} \right\rceil . \qquad \blacksquare$$

We say that $\mathcal{F} = \{T_1, \dots, T_k\}$ is a *partial simplicial elimination ordering* in G if for every non-touching pair T_i, T_j with $i < j$ and for every component C of $G - T_1 - \dots - T_j$, at most one of T_i or T_j touches C. This corresponds exactly to a partial simplicial elimination ordering in the graph obtained by contracting each set of $\mathcal{F}$. We first prove that using acceptable operations we get a partial simplicial elimination ordering in the blue subgraph.

Theorem 9. *Let $\mathcal{F}_0$ be a partial simplicial elimination ordering in G, and let $\mathcal{F}$ be any family formed by starting with $\mathcal{F}_0$ and performing any sequence of acceptable operations. Let G_b be the spanning subgraph of blue edges after the operations. Then $\mathcal{F}$ is a partial simplicial elimination ordering in G_b.*

Proof. Let G_b be the spanning subgraph of G of the blue edges at the end of all operations. We need to prove that after every acceptable operation we have a partial simplicial elimination ordering.

Lemma 10. *Let $\mathcal{F}$ be a partial simplicial elimination ordering in G_b. Let U be the set of vertices $V(G) - \cup_{T \in \mathcal{F}} T$, and let $X \subseteq U$ be a union of some components of $G[U]$. Let k be an integer such that breaking X by k is an acceptable operation. Then the family obtained by breaking X by k is a partial simplicial elimination ordering in G_b.*

Proof. Let $\mathcal{F} = \{T_1, \dots, T_m\}$ be the original family, and let $R_1, \dots, R_\ell$ be the sets produced when we broke X by k. We consider a non-touching pair in G_b, and show that the pair satisfies the condition for a partial simplicial elimination ordering. We only need to consider pairs which contain at least one R_j.

Let T_i, R_j be a non-touching pair in G_b. Let D be the component of $G[X]$ containing R_j (then D is also a component of $G[U]$). We first show that all edges between T_i and D are colored red.

First assume T_i does not touch R_j in G. Then $k = \alpha(R_j) \leq \alpha(D - N(T_i))$ so by the condition in the definition of acceptable breaking we have $\alpha(D - N(T_i)) = \alpha(D)$ or all edges between T_i and D are colored red, thus after the breaking all edges between T_i and D are red.

Now assume the edges between T_i and R_j are red. Then we either had all edges between T_i and D red before the breaking or we colored all edges between T_i and D red during the breaking. Thus we have all the edges between T_i and D colored in red.

Let C be any component of $G_b - T_1 - \cdots - T_m - R_1 - \cdots - R_j$. We want to show that at least one of T_i or R_j does not touch C using blue edges. C is either contained inside D or disjoint from D, because D is a component of $G[U] = G - T_1 - \cdots - T_m$ and C is a connected subgraph of $G[U]$. If C is disjoint from D then R_j does not touch C in G. If C is contained inside D, then T_i does not touch C using blue edges because all edges between T_i and C are red.

Now consider a non-touching pair R_i, R_j in G_b with $i < j$. Assume R_i and R_j are contained in the same component D of $G[X]$. Then since $2k > \alpha(D)$ we must have R_i touching R_j in G so touching in G_b (we only color edges red which have exactly one endpoint in an existing set). Thus we must have R_i and R_j in different components of $G[X]$. So let C be any component of $G_b - T_1 - \cdots - T_m - R_1 - \cdots - R_j$, so that $V(C) \subseteq U$. Then $C[V(C) \cap X]$ is contained inside some component of $G_b[X]$ so $C[V(C) \cap X]$ cannot touch both R_i and R_j in G_b. Since there are no edges between X and $U - X$ in G, if R_i has no edges to $C[V(C) \cap X]$ then R_i has no edges to C and similarly if R_j has no edges to $C[V(C) \cap X]$. Thus at most one of R_i or R_j touches C using blue edges. □

Lemma 11. *Let $\mathcal{F}$ be a partial simplicial elimination ordering in G_b. Let $X \subseteq V(G) - \cup_{T \in \mathcal{F}} T$ where $G_b[X]$ is connected, and let T_i be an element of $\mathcal{F}$. Then the family obtained by extending T_i into X is still a partial simplicial elimination ordering in G_b.*

Proof. Let $\mathcal{F} = \{T_1, \ldots, T_m\}$ before the extension, and let T' be the set T_i plus the vertices added during the extension. Now consider a $T_j \in \mathcal{F}$ where T_j does not touch T' in G_b, let $\ell = \max\{i, j\}$ and consider any component C of $G - T_1 - \cdots - T_\ell - T'$. Let D be the component of $G - T_1 - \cdots - T_\ell$ which contains C. Because $\mathcal{F}$ is a partial simplicial elimination ordering, at least one of T_i or T_j does not touch D using blue edges. Since $G[X]$ is connected, X is either contained inside $V(D)$ or disjoint from $V(D)$. If X is not contained inside $V(D)$, then $D = C$ and at least one of T_i or T_j does not touch C using blue edges. Extension does not change this, so one of T' or T_j does not touch C using blue edges. If X is contained inside $V(D)$, then T_i touches D using blue edges (T_i touches the new vertices in T' and we only extend using blue edges) so T_j does not touch D using blue edges so does not touch C using blue edges. □

Clearly, Lemma 10 and Lemma 11 imply Theorem 9. ■

***Proof of Theorem* 5.** Assume that $\mathcal{F}$ is formed by acceptable operations. Initially, we have that $\mathcal{F}_0 = \emptyset$ which is trivially a partial simplicial elimination ordering. Then by Theorem 9, $\mathcal{F}$ is a partial simplicial elimination ordering in G_b. Let H be the graph obtained from G_b by contracting each $T_i \in \mathcal{F}$ into a single vertex $v_i \in V(H)$.

We show that H is chordal by giving a simplicial elimination ordering of H. We order the vertices of H according to the ordering of the sets of $\mathcal{F}$. For each $v_i \in V(H)$, define $B_i = N(v_i) \cap \{v_1, \ldots, v_i\}$. Assume we had $v_j, v_k \in B_i$ with $j < k < i$ where $v_j v_k \notin E(H)$. Let D be the component of $G_b - T_1 - \cdots - T_k$ which contains T_i. Then T_j does not touch T_k in G_b, so by the condition on partial simplicial elimination ordering one of T_j or T_k does not touch D in G_b. This contradicts that $v_j, v_k \in B_i$, so B_i spans a clique. This happens for each i, yielding that H is a chordal graph. ■

3.2. Some properties of the operations

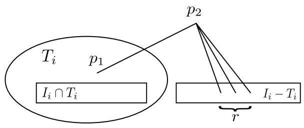
In this subsection, let G be any graph and G_b any spanning subgraph of G. Let $T, X \subseteq V(G)$ and k any integer with $T \cap X = \emptyset$, $G_b[T]$ connected, $G_b[X \cup T]$ connected, no red edges between T and X, and $k \leq \alpha(X - N(T))$. (These are the conditions when we extend T into X by k during the algorithm.)

Lemma 12. *It is possible to extend T into X by k such that $G_b[T]$ remains connected and $\alpha(T)$ increases by at least k and $|T|$ increases by at most $2k$.*

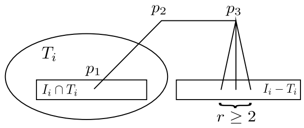
Proof. Let $T_0 = T$ so T_0 is the initial T. We use the following algorithm to produce $T_0 \subseteq T_1 \subseteq \cdots \subseteq T_k$, where $\alpha(T_i) \geq \alpha(T_0) + i$ and $|T_i| \leq |T_0| + 2i$. (Note that we do not define T_i for every $i < k$.) Initially, let I_0 be any maximal independent set in $G[X - N(T_0)]$ with $|I_0| \geq k$.

Assume we have defined T_i and I_i with $i < k$. We now show how to define T_{i+r} and I_{i+r} for some $1 \leq r \leq k - i$.

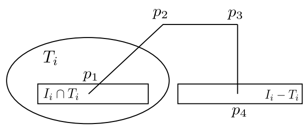
Step 1. Choose P to be a shortest path in $G_b[X \cup T_0]$ between T_i and $I_i - T_i$. The length of P is at most three because I_i is a maximal independent set in $G_b[X - N(T_0)]$. The algorithm maintains that there are no edges between T_i and $I_i - T_i$ when $i < k$, so the length of P is at least two.



(a) Case 1



(b) Case 2



(c) Case 3

Figure 1. Extensions of T_i.

Step 2.

Case 1. Consider when $P = (p_1, p_2, p_3)$ with $p_1 \in T_i$ and $p_3 \in I_i - T_i$. Then we add p_2 and

$$r = \min\{k - i, |N(p_2) \cap (I_i - T_i)|\} \geq 1$$

vertices from $N(p_2) \cap (I_i - T_i)$ to T_i to form T_{i+r}. Let $I_{i+r} = I_i$. Thus $\alpha(T_{i+r}) = \alpha(T_i) + r$ and $|T_{i+r}| = |T_i| + 1 + r$.

Case 2. Consider when $i \leq k - 2$ and $P = (p_1, p_2, p_3, p_4)$ with $p_1 \in T_i$ and $|N(p_3) \cap (I_i - T_i)| \geq 2$. Here, we add p_2, p_3, and

$$r = \min\{k - i, |N(p_3) \cap (I_i - T_i)|\} \geq 2$$

vertices of $N(p_3) \cap (I_i - T_i)$ to T_i to form T_{i+r}. Let $I_{i+r} \subset G[X - T_i - N(T_i)]$ be a maximal independent set containing I_i. Then $\alpha(T_{i+r}) \geq \alpha(T_i) + r$ and $|T_{i+r}| = |T_i| + 2 + r$. Since $r \geq 2$, the increase in the number of vertices is at most twice the increase of i.

Case 3. Consider when $P = (p_1, p_2, p_3, p_4)$ with $p_1 \in T_i$ and $N(p_3) \cap I_i = \{p_4\}$. We set $I_{i+1} = I_i - \{p_4\} + \{p_3\}$ and then extend I_{i+1} to be a maximal independent set in $G[X - T_i - N(T_i)]$. Then I_{i+1} is still a maximal independent set of size at least k, and we can now add p_2 and p_3 to T_i to get T_{i+1}. This increases the number of vertices by two and the independence number by one.

Case 4. Consider when $i = k - 1$ and $P = (p_1, p_2, p_3, p_4)$ with $p_1 \in T_i$ and $|N(p_3) \cap I_i| \geq 2$. Here, we add p_2 and p_3 to T_i to get T_k. Let $I_k = I_i - \{p_4\} + \{p_3\}$. Thus $\alpha(T_k) \geq \alpha(T_0) + k$ and $|T_k| \leq |T_i| + 2$.

Consider one step which did not produce T_k. If this step is Case 1, then we added the entire set $N(p_2) \cap I_i$ to T_i. In Case 2, we added the entire set

$N(p_3) \cap I_i$ to T_i. In Case 3, we added the entire $N(p_2) \cap I_{i+1}$ to T_i. In Case 4, we always produce T_k. Note that we always maintain that there are no edges between T_i and $I_i - T_i$ if $i < k$. If we ever added all vertices of I_i to T_i we would have increased $\alpha(T_i)$ to $\alpha(T_0) + k$ because $I_i \subseteq X - N(T_0)$ and $|I_i| \geq k$. We continue the algorithm if $i < k$ so we will eventually produce a T_k. ■

Lemma 13. *The extension of $T = \{v\}$ into $D - T$ by $k - 1$ during Step* (b) *of a breaking operation satisfies all the conditions of the extension.*

Proof. Any edge colored red has at least one endpoint in a set of $\mathcal{F}$, so that $G_b[D] = G[D]$ where G_b is the spanning subgraph of blue edges at the time of extension. Since D is a component of $G[U]$, we have $G_b[D]$ connected. Also, since $T \subseteq V(D)$ we have no red edges between T and $D - T$ at the time of extension. Finally, since we chose $v \in I$ where I is an independent set of size at least k, we have $\alpha(D - N(v)) \geq |I - v| \geq k - 1$. ■

***Proof of Lemma* 6.** The extending operation does not produce new sets, so the only way to produce a new set is by breaking some set X by k. In Step (b) of the breaking, we initially have $|T| = 1$, and then we extend T by $k - 1$ which adds at most $2k - 2$ new vertices, so T has at most $2k - 1$ vertices. Since the independence number increased by at least $k - 1$, we have $\alpha(T) \geq k$. Extending T by k increases the number of its vertices by at most $2k$ and its independence number by at least k. Thus $|T| \leq 2\,\mathrm{ext}(T) - 1$ and $\alpha(T) \geq \mathrm{ext}(T)$. ■

***Proof of Lemma* 7.** Let $B = G[X - N(T)]$. Assume towards a contradiction that $\alpha(B) - k + 1 \leq \alpha(X - T' - N(T'))$, and let $I = I_k$ be the independent set used at the end of the proof of Lemma 12. Then $\alpha(B) \geq \alpha(B \cap T') + \alpha(B - T' - N(T'))$. Since $B - T' - N(T') = X - T' - N(T')$, we have $\alpha(B) \geq \alpha(B \cap T') + \alpha(B) - k + 1$, i.e., $\alpha(B \cap T') < k$.

But $|I \cap T'| = k$ because the algorithm added at least k vertices of $I_k = I$ to form $T_k = T'$. Since $I \subseteq V(B)$, we have $\alpha(B \cap T') \geq k$, a contradiction. ■

4. THE ANALYSIS OF THE $2\alpha(G) - \log_\tau(\tau\alpha(G)/2)$ ALGORITHM

Definition. Let $f : \{0\} \cup \mathbb{R}^{\geq 1} \to \mathbb{R}, \tau \in \mathbb{R}$ satisfy the following properties:

P1: $f(0) = 0$,

P2: $f(4\sqrt{2}) \leq 1$,
P3: If $1 \leq x \in \mathbb{R}$, then $f(\tau x) \leq 1 + f(x)$,
P4: If $1 \leq x, y \in \mathbb{R}$, then $f(2\sqrt{2}x + 2\sqrt{2}y) \leq f(x) + f(y)$,
P5: If $0 \leq x \leq y \in \mathbb{Z}$, then $f(y) \leq f(x) + y - x$,
P6: If $1 \leq x \leq y \in \mathbb{Z}$ and $1 \leq r \in \mathbb{R}$, then $f(ry) \leq f(rx) + y - x$,
P7: f is non-decreasing so by property P4, if $x_1, \dots, x_k \in \mathbb{R}$ with $x_i \geq 1$, then $f(\sum_i x_i) \leq \sum_i f(x_i)$,
P8: If $1 \leq x \in \mathbb{Z}$, then either $\lceil \sqrt{2}(x-1) \rceil \geq \frac{\tau}{\tau - \sqrt{2}} x$ or $f(2\sqrt{2}x) \leq f(\tau)$,
P9: If $2 \leq x \in \mathbb{Z}$, then $f(2\sqrt{2}x) \leq 1 + f(x - \lceil (x+1)/2 \rceil)$,
P10: If $2 \leq x \in \mathbb{Z}$, then $\sqrt{2}x \leq \tau \lceil (x-1)/2 \rceil$,
P11: If $2 \leq x \in \mathbb{Z}$, then $f(\sqrt{2}x) \leq 2 \lceil (x-1)/2 \rceil$.

We can pick $f(x) = \max \left\{ \lceil \log_\tau(\tau x/(4\sqrt{2})) \rceil, 0 \right\}$ for $x > 0$ and $f(0) = 0$, where $\tau = 2\sqrt{2}/(\sqrt{2} - 1) \approx 6.83$.

The goal of this section is to prove the following which implies our main result:

Theorem 14. *The algorithm in Section* 2.3 *produces a complete minor of size* $\lceil n/(2\alpha(G) - f(2\sqrt{2}\alpha(G))) \rceil$.

To prove Theorem 14 we use Theorems 3 and 5, so we need to prove that the algorithm uses acceptable operations and give an upper bound for the weight of an independent set.

Notation. Let $\mathcal{F}$ be the partition after the algorithm terminates, and let G_b be the spanning subgraph of blue edges after the algorithm terminates. Let $\mathcal{F}_C$ be the family before Step C begins, and define $A_C = V(C) - \cup_{R \in \mathcal{F}_C} R$. If $\alpha(C) > 1$, define $\mathcal{F}_C^i$ to be the family right before Substep i of Step C, define $\mathcal{F}_C^5$ to be the family after all substeps of Step C are completed, and let $A_C^i = V(C) - \cup_{R \in \mathcal{F}_C^i} R$. For $T \in \mathcal{F}$ and $1 \leq i \leq 5$, define

$$T^{C,i} = \begin{cases} S & S \in \mathcal{F}_C^i \text{ and } S \subseteq T \text{ if there exists such an S}, \\ \emptyset & \text{otherwise.} \end{cases}$$

In other words, $T^{C,i}$ is the set in $\mathcal{F}_C^i$ that gets extended to T during the rest of the algorithm.

Define $\mathcal{H}_0(C)$, $\mathcal{H}_1(C)$, and $\mathcal{H}_2(C)$ to be the partition chosen in Substep 1 of Step C.

Lemma 15. *Let $T^{C,1} \in \mathcal{H}_1(C) \cup \mathcal{H}_2(C)$ such that $T^{C,1}$ was extended during Substep 2 or Substep 3 of Step C. This extension satisfies all the conditions of an extension.*

Proof. Let A be the set of vertices not in a set of $\mathcal{F}$ right before the extension, D the component of $G[A]$ which $T^{C,1}$ is extended into and G'_b the spanning subgraph of blue edges at the time of the extension. Since $T^{C,1} \notin \mathcal{H}_0(C)$ there are no red edges between $T^{C,1}$ and C so no red edges between $T^{C,1}$ and D. Since each red edge at the time of extension has at least one endpoint in $\mathcal{F}_C$, we have $G[V(D)] = G'_b[V(D)]$. Since D is a component of $G[A]$, we have that $G'_b[V(D)]$ is connected. Since we are extending $T^{C,1}$ we must have $\alpha(D - N(T^{C,1})) < \alpha(D)$ implying there exist edges between $T^{C,1}$ and D. These edges must be blue so $G'_b[V(D) \cup T^{C,1}]$ is connected. Finally, we extend by $\alpha(D - N(T^{C,1})) - b + 1$ which is smaller than $\alpha(D - N(T^{C,1}))$ since $b \geq 1$. ■

Lemma 16. *$\mathcal{F}$ is formed by acceptable operations.*

Proof. Consider Step C. If $\alpha(C) = 1$, we break C by 1. This breaking is acceptable because for each $T \in \mathcal{F}_C$ we either have $\alpha(C - N(T)) = 0 < 1$ or $\alpha(C - N(T)) = \alpha(C)$. So assume $\alpha(C) \geq 2$.

The coloring in Substep 1 can be viewed as breaking $V(C)$ by $\alpha(C) + 1$. For $T \in \mathcal{F}$, in Step (a) of breaking $V(C)$ by $\alpha(C) + 1$ we colors all edges red between T and D, where D is a component of C (i.e., $D = C$) with $\alpha(D - N(T)) = \alpha(D)$. Since we are breaking by $\alpha(C) + 1$, Step (b) of the breaking does not produce any new sets. This breaking is acceptable since for each $T \in \mathcal{F}$, $\alpha(C - N(T)) \leq \alpha(C) < \alpha(C) + 1$.

Extensions are always acceptable, so if Step C does not continue to Substep 3 after Substep 2 then Step C uses acceptable operations. So assume that Step C continues to Substep 3 and then consider the breaking in Substep 4. Since $b = \lceil (\alpha(C) + 1)/2 \rceil$, for any component D of $G[A^4_C]$ we have $\alpha(D) \leq \alpha(C) \leq 2b$. So consider some $T^{C,4} \in \mathcal{F}^4_C$ and some component D of $G[A^4_C]$. If $T^{C,1} \in \mathcal{H}_0(C)$, then $T^{C,1} = T^{C,4}$ and $\alpha(D - N(T^{C,4})) \leq \alpha(C - N(T^{C,4})) < b$ or all edges between $T^{C,4}$ and D are colored red.

If $T^{C,1} \in \mathcal{H}_1(C)$, then we considered extending $T^{C,1}$ in Substep 2. If we extended $T^{C,1}$ then by Lemma 7 we must have $\alpha(D - N(T^{C,4})) < b$. So assume $T^{C,1} = T^{C,4}$ and that $b \leq \alpha(D - N(T^{C,1}))$, and let D' be the component of $G[A^3_C]$ which contains D. Then $b \leq \alpha(D' - N(T^{C,1}))$ and since we continued to Substep 3 we must have $\alpha(D' - N(T^{C,1})) < \alpha(D')$. In this case we should have continued Substep 2 with the pair $T^{C,1}, D'$. Thus we must have $\alpha(D - N(T^{C,1})) < b$.

If $T^{C,1} \in \mathcal{H}_2(C)$ then we considered extending $T^{C,1}$ in Substep 3. If we extended $T^{C,1}$ then by Lemma 7 we must have $\alpha(D - N(T^{C,4})) < b$. So assume $T^{C,1} = T^{C,4}$. If $b \leq \alpha(D - N(T^{C,4})) < \alpha(D)$, then we should have continued Substep 3 with the pair $T^{C,1}, D$. Thus either $\alpha(D - N(T^{C,4})) < b$ or $\alpha(D - N(T^{C,4})) = \alpha(D)$, showing that the breaking in Substep 4 is acceptable. ∎

We now need to bound the maximum weight of an independent set. Define the set of *independent subfamilies of* $\mathcal{F}$ in G_b by

$$\mathrm{IND}_{G_b}(\mathcal{F}) = \{\mathcal{I} \subseteq \mathcal{F} : \mathcal{I} \text{ is a pairwise non-touching subfamily in } G_b\}.$$

Independent subfamilies of $\mathcal{F}$ correspond to independent sets in H.

Using the weight function which measures a set with its size, the total weight is $|V(G)|$. Then the total weight of $\mathcal{I} \in \mathrm{IND}_{G_b}(\mathcal{F})$ is $\sum_{T\in\mathcal{I}} |T|$. Using Lemma 6, we know that the weight of $\mathcal{I}$ is at most $2\sum_{T\in\mathcal{I}} \mathrm{ext}(T) - |\mathcal{I}|$. We will give an upper bound of $2\alpha(G) - f(2\sqrt{2}\alpha(G))$ on the weight. To do this, we prove the following inequality

$$f(2\sqrt{2}\alpha(G)) \leq |\mathcal{I}| + 2\alpha(G) - 2\sum_{T\in\mathcal{I}} \mathrm{ext}(T).$$

Note that when we analyzed the Section 2.2 algorithm, we showed that either $|\mathcal{I}|$ is at least 2 or that $\alpha(G) - \sum_{T\in\mathcal{I}} \mathrm{ext}(T)$ is at least 1. That is, we showed that in order for the total amount of extensions of sets in $\mathcal{I}$ to be $\alpha(G)$ we need more than one set.

Consider Figure 2, where the vertices of the tree are the steps run by the algorithm. Each step of the algorithm corresponds to a component, and the tree is the containment tree of these components. Let $\mathcal{I} \in \mathrm{IND}_{G_b}(\mathcal{F})$, with $T, S \in \mathcal{I}$. Say that T is born in the step labeled C_1 in the figure, and is extended during the steps labeled C_2, C_3, and C_4. Assume that S is born in the step labeled C_1' and is extended in the step labeled C_2'. We would like to prove by induction on the steps of the algorithm that

$$|\{Q \in \mathcal{I} : Q \subseteq V(C)\}| + 2\alpha(C) - 2\sum\{\mathrm{ext}(Q) : Q \in \mathcal{I}, Q \subseteq V(C)\}$$

is large. We are unable to prove this directly because when the induction reaches Step C_1 we must include $\mathrm{ext}(T)$ into the sum for the first time

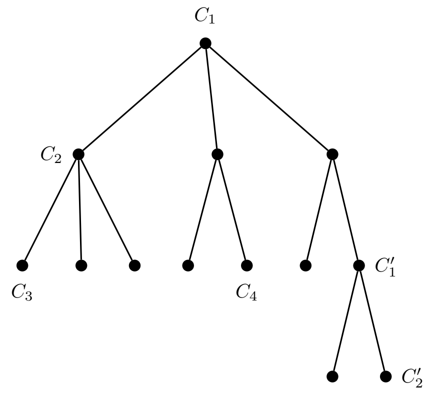


Figure 2. Containment tree of the steps run by the algorithm.

because Step C_1 is the first step where T is completely contained inside the component for the step. Instead, we would like our inductive bound for Step C_2 to include the amount of extensions of T carried out in steps C_2 and C_3 which is only part of $\mathrm{ext}(T)$, so that when we reach Step C_1 the inductive bounds for the smaller components contained inside C_1 already include most of the value $\mathrm{ext}(T)$. So we define a notion of the gap between $\alpha(C)$ and $\sum\{\mathrm{ext}(Q) : Q \in \mathcal{I}, Q \subseteq V(C)\}$ which allows us to include only the amount extensions of T into some subset of $V(C)$. Note that since $\mathcal{F}$ is a partial simplicial elimination ordering, we can have at most one set T which has part of its extensions inside C and part outside C. Define for any $T \in \mathcal{I}$

$$\begin{aligned} \mathrm{ext}(C,T) &= \text{the total amount of extensions of } T \text{ into } X \text{ where } X \subseteq V(C), \\ \mathrm{gap}(C,\mathcal{I},T) &= \alpha(C - N(T^{C,1})) - \mathrm{ext}(C,T) - \sum \big\{ \mathrm{ext}(Q) : Q \in \mathcal{I}, Q \neq T \\ &\qquad\qquad \text{and } Q \cap V(C) \neq \emptyset \big\}. \end{aligned}$$

Note that if $T \cap V(C) \neq \emptyset$ then for each Q in the sum we must have $Q \subseteq V(C)$ because $\mathcal{F}$ is a partial simplicial elimination ordering. In addition, define

$$\begin{aligned} \mathrm{ext}(C,\emptyset) &= 0, \\ \mathrm{gap}(C,\mathcal{I},\emptyset) &= \alpha(C) - \sum_{\substack{Q \in \mathcal{I} \\ Q \cap V(C) \neq \emptyset}} \mathrm{ext}(Q). \end{aligned}$$

In the next lemma, we show that $|\mathcal{I}| + 2\,\mathrm{gap}(C, \mathcal{I}, T)$ is large by induction on the number of steps carried out by the algorithm. For comparison with the Section 2.2 algorithm, Claim 8 proves $0 \le \mathrm{gap}(G, \mathcal{I}, \emptyset)$.

Lemma 17. *Consider any component C chosen as a Step during the algorithm. Let $\mathcal{I}_0 \in \mathrm{IND}_{G_b}(\mathcal{F})$, and let $\mathcal{I} = \{Q \in \mathcal{I}_0 : Q \cap V(C) \neq \emptyset\}$.*

(i) *If $\alpha(C) > 1$ and there exists $T \in \mathcal{I}$ with $T^{C,1} \neq \emptyset$ and $T^{C,1} \in \mathcal{H}_1(C)$ then*

$$f(\alpha(C - N(T^{C,1}))) \le |\mathcal{I}| + 2\,\mathrm{gap}(C, \mathcal{I}, T) - 1.$$

(ii) *If $\alpha(C) > 1$ and there exists $T \in \mathcal{I}$ with $T^{C,1} \neq \emptyset$ and $T^{C,1} \in \mathcal{H}_2(C)$ then*

$$f(\sqrt{2}\alpha(C - N(T^{C,1}))) \le |\mathcal{I}| + 2\,\mathrm{gap}(C, \mathcal{I}, T) - 1.$$

(iii) *Otherwise,*

$$f(2\sqrt{2}\alpha(C)) \le |\mathcal{I}| + 2\,\mathrm{gap}(C, \mathcal{I}, \emptyset).$$

Using Lemma 17, we can prove Theorem 14.

***Proof of Theorem* 14.** Let H be the graph formed from G_b by contracting each set of $\mathcal{F}$. Define g to be the weight function on $V(H)$ which assigns to each $v \in V(H)$ the size of the set of $\mathcal{F}$ which contracted to v. By Lemma 16 and Theorem 5 H is a perfect graph, so by Theorem 3 we just need to show that $\alpha_g(H) \le 2\alpha(G) - f(2\sqrt{2}\alpha(G))$.

Let $C_1, \ldots, C_k$ be the components of G. Let I be any independent set in H, which corresponds to a subfamily $\mathcal{I} \in \mathrm{IND}_{G_b}(\mathcal{F})$. Then define $\mathcal{I}_i = \{T \in \mathcal{I} : T \cap V(C_i) \neq \emptyset\}$. Since $C_1, \ldots, C_k$ are components of G we have $\mathcal{I}_i = \{T \in \mathcal{I} : T \subseteq V(C_i)\}$.

For each $T \in \mathcal{I}_i$, we have $T \subseteq V(C_i)$ which implies $T^{C_i,1} = \emptyset$. Thus we apply the bound in case (iii) of Lemma 17 for each component C_i to obtain

$$\sum_{1 \le i \le k} f(2\sqrt{2}\alpha(C_i)) \le \sum_{1 \le i \le k} |\mathcal{I}_i| + 2 \sum_{1 \le i \le k} \mathrm{gap}(C_i, \mathcal{I}_i, \emptyset). \tag{1}$$

Expanding the definition of $\mathrm{gap}(C_i, \mathcal{I}_i, \emptyset)$ in (1) gives,

$$\sum_{1 \le i \le k} f(2\sqrt{2}\alpha(C_i)) \le |\mathcal{I}| + 2 \sum_{1 \le i \le k} \alpha(C_i) - 2 \sum_{1 \le i \le k} \sum_{T \in \mathcal{I}_i} \mathrm{ext}(T). \tag{2}$$

Since $\sum_i \alpha(C_i) = \alpha(G)$ and each $T \in \mathcal{I}$ appears in exactly one $\mathcal{I}_i$, (2) simplifies to

$$\sum_{1\le i\le k} f(2\sqrt{2}\alpha(C_i)) \le |\mathcal{I}| + 2\alpha(G) - 2\sum_{T\in\mathcal{I}} \mathrm{ext}(T). \tag{3}$$

Using Lemma 6 to bound $|T|$ and rearranging (3), we have

$$g(I) = \sum_{T\in\mathcal{I}} |T| \le 2\sum_{T\in\mathcal{I}} \mathrm{ext}(T) - |\mathcal{I}| \le 2\alpha(G) - \sum_{1\le i\le k} f(2\sqrt{2}\alpha(C_i)).$$

By property P7,

$$g(I) \le 2\alpha(G) - f(2\sqrt{2}\sum_{1\le i\le k}\alpha(C_i)) = 2\alpha(G) - f(2\sqrt{2}\alpha(G)).$$

This says every independent set in H has weight at most $2\alpha(G)-f(2\sqrt{2}\alpha(G))$, we have $\alpha_g(H) \le 2\alpha(G) - f(2\sqrt{2}\alpha(G))$. ■

Before proving Lemma 17, we need some lemmas:

Lemma 18. *Let $T^{C,1} \in \mathcal{H}_1(C)$ and let R be a set born in Substep 4 of Step C. Then $T^{C,4}$ touches R with blue edges.*

Proof. Since $T^{C,1} \in \mathcal{H}_1(C)$, all edges between $T^{C,1}$ and C are colored blue at the start of Step C. We produced an R in Substep 4 so $\alpha(C) \ge 2$, so we consider extending $T^{C,1}$ in Substep 2. Since we continued to Substep 3 after Substep 2, we have for each component D of $G[A_C^3]$, $\alpha(D - N(T^{C,3})) < b$. Now consider the component D' of $G[A_C^4]$ which contains R. Then there exists a component D of $G[A_C^3]$ which contains D' so $\alpha(D' - N(T^{C,4})) \le \alpha(D - N(T^{C,4})) < b$ so no edges incident to $T^{C,4}$ are colored red during Substep 4. Also, since $b \le \alpha(R)$ by Lemma 6 we must have an edge of G between $T^{C,4}$ and R. This edge is colored blue at the end of the algorithm because no edges incident to $T^{C,4}$ are colored red during Substep 4 and all future edge colorings only color edges between an existing set in $\mathcal{F}$ and a vertex of A. ■

Lemma 19. *Let $T^{C,1} \in \mathcal{H}_2(C)$, and assume that $T^{C,1}$ was extended in Substep 3 of Step C. Let R be a set born during Substep 4 of Step C. Then $T^{C,4}$ touches R with blue edges.*

Proof. Since $T^{C,1} \in \mathcal{H}_2(C)$ and we extended $T^{C,1}$, let A be the subset of vertices of C not yet in a set at the time we extend $T^{C,1}$, and let D be the component of $G[A]$ where $b \leq \alpha(D - N(T^{C,1})) < \alpha(D)$. By Lemma 7, $\alpha(D - T^{C,4} - N(T^{C,4})) < b$. Also, consider any other component D' of $G[A]$ besides D. Since $\alpha(C)/2 < b < \alpha(D)$, we must have $\alpha(D') < b$ since $\alpha(D) + \alpha(D') \leq \alpha(C)$. Thus for all components D' of $G[A - T^{C,4}]$ we have $\alpha(D' - N(T^{C,4})) < b$. Now R is connected so R must be contained inside some component D'' of $G[A^4_C]$ which is contained inside some component D' of $G[A - T^{C,4}]$. If $T^{C,4}$ does not touch R in G, then $\alpha(D - N(T^{C,4})) \geq \alpha(R) \geq b$ which gives a contradiction. If $T^{C,4}$ touches R with red edges, then we must have colored the edges between $T^{C,4}$ and D'' red during Substep 4 so $\alpha(D'' - N(T^{C,4})) = \alpha(D'') \geq \alpha(R) \geq b$, again giving a contradiction. ■

Lemma 20. *Let $T^{C,1} \in \mathcal{H}_1(C) \cup \mathcal{H}_2(C)$. Then $\alpha(C - N(T^{C,1})) < \alpha(C)$.*

Proof. Assume $T^{C,1} \in \mathcal{H}_1(C) \cup \mathcal{H}_2(C)$. If $\alpha(C - N(T^{C,1})) = \alpha(C)$, then we color all edges between T and C red in Substep 1 of Step C. Since all edges are red, $T^{C,1} \in \mathcal{H}_0(C)$ which contradicts $T^{C,1} \in \mathcal{H}_1(C) \cup \mathcal{H}_2(C)$. ■

***Proof of Lemma* 17.** The proof works by induction on $|V(C)|$, where Step C is a step carried out by the algorithm. Fix an $\mathcal{I}_0 \in \mathrm{IND}_{G_b}(\mathcal{F})$ and a component C chosen by the algorithm and consider Step C. For the rest of this section, let $D_1, \ldots, D_k$ be the components of $G[A^5_C]$. We can then apply induction into each of the components D_i because at some future time in the algorithm D_i will be selected as a Step. Let $\mathcal{I} = \{T \in \mathcal{I}_0 : T \cap V(C) \neq \emptyset\}$ and $\mathcal{I}_i = \{T \in \mathcal{I} : T \cap V(D_i) \neq \emptyset\}$. We can apply induction into D_i with the independent subfamily $\mathcal{I}_i$.

If $\alpha(C) = 1$, we need to show $f(2\sqrt{2}) \leq |\mathcal{I}| + 2\,\mathrm{gap}(C, \mathcal{I}, \emptyset)$. If $\mathrm{gap}(C, \mathcal{I}, T) = 0$ then $|\mathcal{I}| = 1$. As f is non-decreasing, property P2 shows $f(2\sqrt{2}) \leq 1$.

Now assume $\alpha(C) > 1$ and consider the possibilities for $T \in \mathcal{I}$ with $T^{C,1} \neq \emptyset$. We cannot have two sets $T, R \in \mathcal{I}$ with $T^{C,1} \neq \emptyset$ and $R^{C,1} \neq \emptyset$, because this would contradict that $\mathcal{F}$ forms a partial simplicial elimination ordering since T and R touch C with blue edges ($G_b[T]$ is connected and $T \cap V(C) \neq \emptyset$) but all edges between T and R are red. Also, we cannot have two sets $T, R \in \mathcal{I}$ which were born in Substep 4 of Step C because the sets born in Substep 4 are pairwise touching using blue edges. Thus define T to be the set in $\mathcal{I}$ with $T^{C,1} \neq \emptyset$ if it exists and otherwise define $T = \emptyset$, and define R to be the set in $\mathcal{I}$ which was born in Substep 4 of

Step C, otherwise $R = \emptyset$. Note that $T \neq \emptyset$ implies that $T \in \mathcal{I}$ so that $T \cap V(C) \neq \emptyset$ which implies there are blue edges between T and C which implies $T^{C,1} \notin \mathcal{H}_0(C)$. Thus if $T \neq \emptyset$ we need to prove the inequality in either case (i) or (ii) of Lemma 17. If $T = \emptyset$ we need to prove the inequality in case (iii) of Lemma 17.

For each D_i, at most one of T or R can touch D_i using blue edges. (If both touch D_i using blue edges, then we contradict the partial simplicial elimination ordering.) Define Q_i to be T or R depending on which is contained in $\mathcal{I}_i$, and define $Q_i = \emptyset$ if neither is in $\mathcal{I}_i$. Define

$$\gamma_i = \begin{cases} 1 & Q_i^{D_i,1} \in \mathcal{H}_1(D_i), \\ \sqrt{2} & Q_i^{D_i,1} \in \mathcal{H}_2(D_i), \\ 2\sqrt{2} & Q_i = \emptyset. \end{cases}$$

Claim 21.

$$\sum_{1 \le i \le k} f(\gamma_i \alpha(D_i - N(Q_i^{C,5}))) - \sum_{1 \le i \le k} 2 \operatorname{gap}(D_i, \mathcal{I}_i, Q_i) \le |\mathcal{I} - \{T, R\}| .$$

We actually use Claim 21 in the following form:

$$\begin{aligned} \sum_i f(\gamma_i \alpha(C - N(Q_i^{C,5}))) + 2 \operatorname{gap}(C, \mathcal{I}, \emptyset) - 2 \sum_i \operatorname{gap}(D_i, \mathcal{I}_i, \emptyset) \\ \le |\mathcal{I} - \{T, R\}| + 2 \operatorname{gap}(C, \mathcal{I}, \emptyset). \end{aligned} \tag{4}$$

Proof. Assume we have indexed the components so that for $1 \le i \le h_1$, $Q_i = T$ and for $h_1 < i \le h_2$, $Q_i = R$ and for $h_2 < i \le k$, $Q_i = \emptyset$. We consider Step D_i. Then $Q_i^{D_i,1}$ is the set of $\mathcal{I}_i$ which will touch D_i in blue and be considered in the statement of Lemma 17, and γ_i is the coefficient inside the function f. Thus for $1 \le i \le h_2$, we obtain

$$f(\gamma_i \alpha(D_i - N(Q_i^{D_i,1}))) \le |\mathcal{I}_i| - 1 + 2 \operatorname{gap}(D_i, \mathcal{I}_i, Q_i).$$

Note that $D_i \cap N(Q_i^{D_i,1}) = D_i \cap N(Q_i^{C,5})$ so we have

$$f(\gamma_i \alpha(D_i - N(Q_i^{C,5}))) \le |\mathcal{I}_i| - 1 + 2 \operatorname{gap}(D_i, \mathcal{I}_i, Q_i).$$

Here, we can think of $|\mathcal{I}_i| - 1$ as counting the number of sets in $\mathcal{I}_i$ besides Q_i. For $h_2 < i \le k$ we obtain

$$f(\gamma_i \alpha(D_i)) \le |\mathcal{I}_i| + 2 \operatorname{gap}(D_i, \mathcal{I}_i, \emptyset).$$

Again $|\mathcal{I}_i|$ is counting the sets of $\mathcal{I}_i$ besides Q_i. Thus

$$\sum_{i=1}^{h_2}(|\mathcal{I}_i|-1)+\sum_{i=h_2+1}^{k}|\mathcal{I}_i|=|\mathcal{I}-\{T,R\}|.$$

Thus summing the inductive bounds over all i we obtain (for $h_2 < i \le k$, $Q_i = \emptyset$ so that $\alpha(D_i - N(Q_i^{C,5})) = \alpha(D_i)$)

$$\sum_{1\le i\le k} f(\gamma_i\alpha(D_i-N(Q_i^{C,5}))) \le |\mathcal{I}-\{T,R\}| + \sum_{1\le i\le k} 2\operatorname{gap}(D_i,\mathcal{I}_i,Q_i).$$

∎

We finish the proof of Lemma 17 by showing that the inequality in Claim 21 simplifies in all cases to the inequalities in Lemma 17. For the simplification, we add $\operatorname{gap}(C,\mathcal{I},T)$ to both sides of Claim 21 and then use lower bounds on $\operatorname{gap}(C,\mathcal{I},T) - \sum_i \operatorname{gap}(D_i,\mathcal{I}_i,Q_i)$ and lower bounds on $\sum_i f(\gamma_i\alpha(D_i - N(Q_i^{C,5})))$. Define

$$\theta = \operatorname{ext}(C,T) - \sum_{1\le i\le k}\operatorname{ext}(D_i,T) = \text{the amount of extensions of } T \text{ during Step } C,$$

$$\lambda = \sum_{1\le i\le k}\alpha(D_i - N(Q_i^{C,5})),$$

$$J = \left\{i : \alpha(D_i - N(Q_i^{C,5})) > 0\right\}.$$

If $T = \emptyset$, we define $\theta = 0$.

We claim the following bounds.

Bound 1. If $T = R = \emptyset$, then $f(2\sqrt{2}\lambda) \le \sum_{i\in J} f(\gamma_i\alpha(D_i - N(Q_i^{C,5})))$,

Bound 2. If $|J| \ge 2$, then $f(2\sqrt{2}\lambda) \le \sum_{i\in J} f(\gamma_i\alpha(D_i - N(Q_i^{C,5})))$,

Bound 3. If $J = \{i\}$, then $f(\gamma_i\lambda) = \sum_{i\in J} f(\gamma_i\alpha(D_i - N(Q_i^{C,5})))$,

Bound 4. $f(\lambda) \le \sum_i f(\gamma_i\alpha(D_i - N(Q_i^{C,5})))$,

Bound 5. If $R = \emptyset$, then $\operatorname{gap}(C,\mathcal{I},T) - \sum_i \operatorname{gap}(D_i,\mathcal{I}_i,Q_i) = \alpha(C - N(T^{C,1})) - \theta - \lambda$,

Bound 6. If $R \ne \emptyset$, then $\operatorname{gap}(C,\mathcal{I},T) - \sum_i \operatorname{gap}(D_i,\mathcal{I}_i,Q_i) = \alpha(C - N(T^{C,1})) - b - \lambda$.

We now justify these bounds. For Bound 1, $T = R = \emptyset$ implies that $Q_i = \emptyset$ and $\gamma_i = 2\sqrt{2}$ for all i so the inequality follows by property P7. Bound 2 follows from property P4 since $|J| \geq 2$. Bound 3 is an equality by definition of λ. For Bound 4, property P7 and $\gamma_i \geq 1$ imply

$$f(\lambda) \leq \sum_{1\leq i\leq k} f(\alpha(D_i - N(Q_i^{C,5}))) \leq \sum_{1\leq i\leq k} f(\gamma_i\alpha(D_i - N(Q_i^{C,5}))).$$

Now we justify Bound 5; assume $R = \emptyset$. First, using the definition of Q_i we have

$$\begin{aligned}\sum_{1\leq i\leq k} \mathrm{ext}(D_i, Q_i) &= \sum_{1\leq i\leq h_1} \mathrm{ext}(D_i, T) + \sum_{h_1< i\leq k} \mathrm{ext}(D_i, \emptyset)\\ &= \sum_{1\leq i\leq h_1} \mathrm{ext}(D_i, T).\end{aligned} \tag{5}$$

For $h_1 < i \leq k$, we have $Q_i = \emptyset$ implying that $T \cap V(D_i) = \emptyset$ which implies $\mathrm{ext}(D_i, T) = 0$. Thus the equality in (5) expands to

$$\sum_{1\leq i\leq k} \mathrm{ext}(D_i, Q_i) = \sum_{1\leq i\leq k} \mathrm{ext}(D_i, T). \tag{6}$$

Then expanding the definition of gap,

$$\begin{aligned}\sum_{1\leq i\leq k} \mathrm{gap}(D_i, \mathcal{I}_i, Q_i) &= \sum_{1\leq i\leq k} \alpha(D_i - N(Q_i^{C,5})) - \sum_{1\leq i\leq k} \mathrm{ext}(D_i, Q_i)\\ &\quad - \sum_{1\leq i\leq k} \sum_{\substack{W\in\mathcal{I}_i\\ W\neq Q_i}} \mathrm{ext}(W)\\ &= \lambda - \sum_{1\leq i\leq k} \mathrm{ext}(D_i, T) - \sum_{W\in\mathcal{I}, W\neq T} \mathrm{ext}(W).\end{aligned} \tag{7}$$

Then expanding the definition of $\mathrm{gap}(C, \mathcal{I}, T)$ and combining with the equality in (7) gives

$$\begin{aligned}&\mathrm{gap}(C, \mathcal{I}, T) - \sum_{1\leq i\leq k} \mathrm{gap}(D_i, \mathcal{I}_i, Q_i) =\\ &= \alpha(C - N(T^{C,1})) - \mathrm{ext}(T, C) - \sum_{\substack{W\in\mathcal{I}\\ W\neq T}} \mathrm{ext}(W) - \sum_{1\leq i\leq k} \mathrm{gap}(D_i, \mathcal{I}_i, Q_i)\end{aligned}$$

$$= \alpha(C - N(T^{C,1})) - \operatorname{ext}(T, C) - \lambda + \sum_{1 \le i \le k} \operatorname{ext}(D_i, T)$$
$$= \alpha(C - N(T^{C,1})) - \lambda - \theta.$$

This completes the proof of Bound 5.

Finally, consider Bound 6 and assume $R \neq \emptyset$. Using the definition of Q_i we have

$$\sum_{1 \le i \le k} \operatorname{ext}(D_i, Q_i) = \sum_{1 \le i \le h_1} \operatorname{ext}(D_i, T) + \sum_{h_1 < i \le h_2} \operatorname{ext}(D_i, R). \tag{8}$$

By Lemma 18 and Lemma 19 we did not extend T during Step C so that

$$\sum_{1 \le i \le h_1} \operatorname{ext}(D_i, T) = \operatorname{ext}(T, C). \tag{9}$$

Also, if we have an index i with $Q_i \neq R$, this implies $R \cap V(D_i) = \emptyset$ so $\operatorname{ext}(D_i, R) = 0$. Combining (8) and (9) gives

$$\sum_{1 \le i \le k} \operatorname{ext}(D_i, Q_i) = \operatorname{ext}(C, T) + \sum_{1 \le i \le k} \operatorname{ext}(D_i, R).$$

Then expanding the definition of gap, we have

$$\begin{aligned}
&\sum_{1 \le i \le k} \operatorname{gap}(D_i, \mathcal{I}_i, Q_i) = \\
&= \sum_{1 \le i \le k} \alpha(D_i - N(Q_i^{C,5})) - \sum_{1 \le i \le k} \operatorname{ext}(D_i, Q_i) - \sum_{1 \le i \le k} \sum_{\substack{W \in \mathcal{I}_i \\ W \neq Q_i}} \operatorname{ext}(W) \\
&= \lambda - \operatorname{ext}(C, T) - \sum_{1 \le i \le k} \operatorname{ext}(D_i, R) - \sum_{W \in \mathcal{I}, W \neq T, W \neq R} \operatorname{ext}(W) \\
&= \lambda - \operatorname{ext}(C, T) + \operatorname{ext}(R) - \sum_{1 \le i \le k} \operatorname{ext}(D_i, R) - \sum_{W \in \mathcal{I}, W \neq T} \operatorname{ext}(W) \\
&= \lambda - \operatorname{ext}(C, T) + b - \sum_{W \in \mathcal{I}, W \neq T} \operatorname{ext}(W).
\end{aligned} \tag{10}$$

The last inequality holds because $\operatorname{ext}(R) - \sum_i \operatorname{ext}(D_i, R)$ is one plus the number of extensions of R during Substep 4 of Step C which is b. Then expanding the definition of $\mathrm{gap}(C,\mathcal{I},T)$ and combining with the equality in (10) gives

$$\begin{aligned}
&\mathrm{gap}(C,\mathcal{I},T) - \sum_{1\le i\le k} \mathrm{gap}(D_i,\mathcal{I}_i,Q_i) = \\
&= \alpha(C - N(T^{C,1})) - \mathrm{ext}(C,T) - \sum_{\substack{W\in\mathcal{I}\\ W\neq T}} \mathrm{ext}(W) - \sum_{1\le i\le k} \mathrm{gap}(D_i,\mathcal{I}_i,Q_i) \\
&= \alpha(C - N(T^{C,1})) - \lambda - b.
\end{aligned}$$

This finishes the proof of all the bounds.

We now just need to show that in all the different cases, the inequality in Claim 21 simplifies to the inequalities in Lemma 17.

Case 1. $R = T = \emptyset$.
We apply Bounds 1 and 5 to simplify (4)

$$f(2\sqrt{2}\lambda) + 2\alpha(C) - 2\lambda \le |\mathcal{I}| + 2\,\mathrm{gap}(C,\mathcal{I},\emptyset). \tag{11}$$

Since $\lambda \le \alpha(C)$ we use property P6 with $r = 2\sqrt{2}$ to obtain

$$f(2\sqrt{2}\alpha(C)) \le f(2\sqrt{2}\alpha(C)) + \alpha(C) - \lambda \le f(2\sqrt{2}\lambda) + 2\alpha(C) - 2\lambda. \tag{12}$$

Combining (11) with (12) proves the inequality in case (iii) of Lemma 17.

Case 2. $R = \emptyset$, $T \neq \emptyset$, T was extended during Step C, and $|J| \ge 2$.
We apply Bounds 2 and 5 to simplify (4):

$$f(2\sqrt{2}\lambda) + 2\alpha(C - N(T^{C,1})) - 2\theta - 2\lambda \le |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C,\mathcal{I},T). \tag{13}$$

We have $\alpha(C - N(T^{C,1})) \ge \theta + \lambda$ since $\lambda = \sum_i \alpha(D_i - N(T^{C,5}))$ and $\alpha(T)$ increased by at least θ during Step C by adding vertices from $C - N(T^{C,1})$. Also, $\lambda \ge 1$ since $J \neq \emptyset$. Thus we can apply property P6 with $r = \sqrt{2}$ to get

$$\begin{aligned}
&f(2\sqrt{2}\lambda) + 2\alpha(C - N(T^{C,1})) - 2\theta - 2\lambda \\
&\ge f(2\sqrt{2}\alpha(C - N(T^{C,1})) - 2\sqrt{2}\theta) \\
&\ge f(2\sqrt{2}(b-1)).
\end{aligned} \tag{14}$$

The last inequality holds because $\theta \le \alpha(C - N(T^{C,1})) - b + 1$ and f is non-decreasing. Now we combine (13) with (14) to obtain

$$f(2\sqrt{2}(b-1)) \le |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T). \tag{15}$$

Then by definition of b we have for $\alpha(C) \ge 2$

$$2\sqrt{2}(b-1) = 2\sqrt{2}\left\lceil \frac{\alpha(C)-1}{2} \right\rceil \ge \sqrt{2}(\alpha(C)-1). \tag{16}$$

Since we extended $T^{C,1}$ we have $T^{C,1} \in \mathcal{H}_1(C) \cup \mathcal{H}_2(C)$ so by Lemma 20, $\alpha(C - N(T^{C,1})) \le \alpha(C) - 1$. Then since f is non-decreasing, (15) and (16) simplify to

$$f(\sqrt{2}\alpha(C - N(T^{C,1}))) \le f(\sqrt{2}(\alpha(C)-1)) \le |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T).$$

This proves the inequality in case (i) and (ii) of Lemma 17.

Case 3. $R = \emptyset$, $T \ne \emptyset$, T was extended during Step C, $J = \{i\}$, and we continued to Substep 3.

Then we use Bounds 3 and 5 to simplify (4):

$$f(\gamma_i\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda - 2\theta \le |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T). \tag{17}$$

Since $J \ne \emptyset$, we have $\lambda \ge 1$. Also, $\lambda + \theta \le \alpha(C - N(T^{C,1}))$ since $\alpha(T)$ increased by at least θ during Step C and $\lambda = \sum_i \alpha(C - N(T^{C,5}))$. Thus we can apply property P6 with $r = \gamma_i$ to obtain

$$\begin{aligned} &f(\gamma_i\lambda) + 2\alpha(C - N(T^{C,1})) - 2\theta - 2\lambda \\ &\ge f(\gamma_i\alpha(C - N(T^{C,1})) - \gamma_i\theta) + \alpha(C - N(T^{C,1})) - \theta - \lambda \\ &\ge f(\gamma_i(b-1)) + b - 1 - \lambda. \end{aligned} \tag{18}$$

The last inequality holds because $\theta \le \alpha(C - N(T^{C,1})) - b + 1$ and f is non-decreasing. If $\lambda < b - 1$, then using properties P10 and P3 and $\gamma_i \ge 1$ we obtain

$$f(\gamma_i(b-1)) + b - 1 - \lambda \ge f(b-1) + 1 \ge f(\tau(b-1)) \ge f(\sqrt{2}\alpha(C)). \tag{19}$$

Since $\alpha(C-N(T^{C,1}))\leq\alpha(C)$, we combine (17), (18), and (19) to prove the inequality in case (i) and (ii) of Lemma 17.

Now assume $\lambda\geq b-1$. Since $|J|=1$ we have $\lambda=\alpha(D_i-N(Q_i^{C,5}))\leq\alpha(D_i)$ and since we ran Substep 4 we have $\alpha(D_i)\leq b-1$ so $\lambda=b-1$. We are also forced to have $\alpha(D_i-N(Q_i^{C,5}))=\alpha(D_i)$. Assume $Q_i\neq\emptyset$. Since $D_i\cap N(Q_i^{C,5})=D_i\cap N(Q_i^{D_i,1})$ we will color all edges between $Q_i^{D_i,1}$ and D_i red in Substep 1 of Step D_i, which implies $Q_i^{C,5}\cap V(D_i)=\emptyset$. This contradicts that $Q_i\in\mathcal{I}_i$. Thus $Q_i=\emptyset$ and so by definition of γ_i, we have $\gamma_i=2\sqrt{2}$. Then we combine (17) with (18) to obtain ($\lambda=b-1$)

$$f(2\sqrt{2}(b-1))\leq|\mathcal{I}|-1+2\operatorname{gap}(C,\mathcal{I},T). \tag{20}$$

By the inequality in (16) we have $2\sqrt{2}(b-1)\geq\sqrt{2}(\alpha(C)-1)$. Since f is non-decreasing, (20) simplifies to

$$f(\sqrt{2}(\alpha(C-1)))\leq|\mathcal{I}|-1+2\operatorname{gap}(C,\mathcal{I},T).$$

Since $T^{C,1}\in\mathcal{H}_1(C)\cup\mathcal{H}_2(C)$ we have by Lemma 20 that $\alpha(C-N(T^{C,1}))<\alpha(C)$. Thus we have proved the inequality in case (i) and (ii) of Lemma 17.

Case 4. $R=\emptyset$, $T\neq\emptyset$, T was extended during Step C, $J=\{i\}$, and we did not continue to Substep 3 after Substep 2.

In this case, we must have $T^{C,1}\in\mathcal{H}_1(C)$ because we did not run Substep 3. We apply Bounds 3 and 5 to simplify (4):

$$f(\gamma_i\lambda)+2\alpha(C-N(T^{C,1}))-2\lambda-2\theta\leq|\mathcal{I}|-1+2\operatorname{gap}(C,\mathcal{I},T). \tag{21}$$

First assume that $\lambda<b-1$. We use $\gamma_i\geq1$ and that f is non-decreasing to simplify (21) to

$$f(\lambda)+2\alpha(C-N(T^{C,1}))-2\lambda-2\theta\leq|\mathcal{I}|-1+2\operatorname{gap}(C,\mathcal{I},T). \tag{22}$$

Because $\alpha(T)$ increased by at least θ during Step C, we have $\lambda+\theta\leq\alpha(C-N(T^{C,1}))$. We use properties P6 ($J\neq\emptyset$ so $\lambda\geq1$), P3, and P10 to obtain

$$\begin{aligned}&f(\lambda)+2\alpha(C-N(T^{C,1}))-2\lambda-2\theta\\ &\geq f(\alpha(C-N(T^{C,1}))-\theta)+\alpha(C-N(T^{C,1}))-\lambda-\theta \qquad (23)\\ &\geq f(b-1)+b-1-\lambda\geq f(b-1)+1\geq f(\tau(b-1))\geq f(\sqrt{2}\alpha(C)).\end{aligned}$$

Since $\alpha(C-N(T^{C,1})) \leq \alpha(C)$, combining (22) with (23) proves the inequality in both case (i) and (ii) of Lemma 17.

Now assume $\lambda \geq b-1$ so that $\alpha(D_i - N(Q_i^{C,5})) \geq b-1$. Since we did not continue to Substep 3 after Substep 2 we must have some component D_j and some $S^{C,1} \in \mathcal{H}_1(C)$ with $b \leq \alpha(D_j) = \alpha(D_j - N(S^{C,1})) \leq \sqrt{2}(b-1)$. If $i = j$ then we have $\alpha(D_i) \leq \sqrt{2}(b-1)$ and if $i \neq j$ then $\alpha(D_i) \leq \alpha(C) - \alpha(D_j) \leq \alpha(C) - b \leq \sqrt{2}(b-1)$. Thus

$$\alpha(D_i - N(Q_i^{C,5})) \geq b-1 = \frac{1}{\sqrt{2}}\sqrt{2}(b-1) \geq \frac{\alpha(D_i)}{\sqrt{2}}. \tag{24}$$

Since

$$\alpha(D_i) \geq 2\left\lceil \frac{\alpha(D_i)-1}{2} \right\rceil,$$

(24) implies that

$$\alpha(D_i - N(Q_i^{D_i,1})) \geq \frac{\alpha(D_i)}{\sqrt{2}} \geq \sqrt{2}\left(\left\lceil \frac{\alpha(D_i)+1}{2} \right\rceil - 1\right).$$

This shows that either $Q_i = \emptyset$ or $Q_i^{D_i,1} \in \mathcal{H}_0(D_i) \cup \mathcal{H}_2(D_i)$. If $Q_i \neq \emptyset$ and $Q_i^{D_i,1} \in \mathcal{H}_0(D_i)$ then we must have colored all edges between $Q_i^{D_i,1}$ and D_i red in Substep 1 of Step D_i which contradicts $Q_i \cap V(D_i) \neq \emptyset$. Thus either $Q_i = \emptyset$ so $\gamma_i = 2\sqrt{2}$ or $Q_i \neq \emptyset$ and $Q_i^{D_i,1} \in \mathcal{H}_2(D_i)$ so that $\gamma_i = \sqrt{2}$. Then (21) simplifies to

$$f(\sqrt{2}\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda - 2\theta \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T). \tag{25}$$

Because $\alpha(T)$ increased by at least θ during Step C, we have $\lambda + \theta \leq \alpha(C - N(T^{C,1}))$. Using property P6 ($J \neq \emptyset$ so $\lambda \geq 1$) we obtain

$$\begin{aligned} &f(\sqrt{2}\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda - 2\theta \\ &\geq f(\sqrt{2}\alpha(C - N(T^{C,1})) - \sqrt{2}\theta) \geq f(\sqrt{2}(b-1)). \end{aligned} \tag{26}$$

Combining (25) with (26) gives

$$f(\sqrt{2}(b-1)) \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T).$$

Since $T^{C,1} \in \mathcal{H}_1(C)$ we have $\sqrt{2}(b-1) \geq \alpha(C - N(T^{C,1}))$ so we have proved the bound in case (i) of Lemma 17.

Case 5. $R = \emptyset$, $T \neq \emptyset$, T was extended during Step C, and $J = \emptyset$. We apply Bounds 4 and 5 to (4) and then use $\lambda = 0$ and property P1 to obtain

$$2\alpha(C - N(T^{C,1})) - 2\theta \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T).$$

Then $\theta \leq \alpha(C - N(T^{C,1})) - b + 1$ so

$$2(b-1) \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T). \tag{27}$$

By property P11, $f(\sqrt{2}\alpha(C)) \leq 2(b-1)$. Since $\alpha(C - N(T^{C,1})) \leq \alpha(C)$ we have that (27) simplifies to

$$f(\sqrt{2}\alpha(C - N(T^{C,1}))) \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T).$$

This proves the inequality in cases (i) and (ii) in Lemma 17.

Case 6. $R = \emptyset$, $T \neq \emptyset$ and $T^{C,1} = T^{C,5}$.
Since $T^{C,1} = T^{C,5}$, we have $\theta = 0$.

We apply Bounds 4 and 5 to simplify (4):

$$f(\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T). \tag{28}$$

Since $\lambda \leq \alpha(C - N(T^{C,1}))$, we use property P5 to obtain

$$f(\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda \geq f(\alpha(C - N(T^{C,1}))). \tag{29}$$

If $T^{C,1} \in \mathcal{H}_1(C)$, then (29) and (28) prove the inequality in case (i) of Lemma 17.

So assume $T^{C,1} \in \mathcal{H}_2(C)$. If $\lambda < \alpha(C - N(T^{C,1}))$ we can apply property P5 and P3 to obtain

$$\begin{aligned} &f(\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda \\ &\geq f(\alpha(C - N(T^{C,1}))) + 1 \geq f(\tau\alpha(C - N(T^{C,1}))). \end{aligned} \tag{30}$$

Because $\tau \geq \sqrt{2}$, we can combine (30) with (28) to prove the inequality in case (ii) of Lemma 17.

So assume $\lambda = \alpha(C - N(T^{C,1}))$. Since $T^{C,1} \in \mathcal{H}_2(C)$ we have $\lambda = \alpha(C - N(T^{C,1})) \geq \sqrt{2}(b-1) > \alpha(C)/2$. Then $|J| \geq 2$ since each component of $G[A_C^4]$ has independence number at most $\alpha(C)/2$ and $\lambda > \alpha(C)/2$. Using $|J| \geq 2$ we can apply Bounds 2 and 5 to Claim 21 to get

$$f(2\sqrt{2}\lambda) \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T).$$

Since $\lambda = \alpha(C - N(T^{C,1}))$, we have proved the bound in case (ii) of Lemma 17.

Case 7. $R \neq \emptyset$ and $T \neq \emptyset$.
First, $T^{C,1} \in \mathcal{H}_2(C)$ by Lemma 18 and $T^{C,4} = T^{C,1}$ by Lemma 19. Then we apply Bounds 4 and 6 to simplify (4):

$$f(\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda - 2b \leq |\mathcal{I}| - 2 + 2\,\mathrm{gap}(C, \mathcal{I}, T). \tag{31}$$

Because $b \leq \alpha(R^{C,5})$ and Q_i is T or R we have $\cup_i(D_i - N(Q_i^{C,5})) \bigcup R^{C,5} \subseteq C - N(T^{C,1})$ so that $\lambda + b \leq \alpha(C - N(T^{C,1}))$. (Note that $T^{C,1} \notin \mathcal{H}_0(C)$ so there are no red edges between $T^{C,1}$ and $R^{C,5}$.)

Assume $\alpha(C - N(T^{C,1})) = b$ so that $\lambda = 0$. Since $T^{C,1} \in \mathcal{H}_2(C)$ we know $b = \alpha(C-N(T^{C,1})) \geq \sqrt{2}(b-1)$ so b is one or two so $\alpha(C-N(T^{C,1})) \leq \alpha(C) \leq 2b \leq 4$. Thus $f(\sqrt{2}\alpha(C - N(T^{C,1}))) \leq f(4\sqrt{2}) \leq 1$ by property P2. The left side of (31) is zero (using property P1) so adding one to both sides of (31) simplifies to

$$f(\sqrt{2}\alpha(C - N(T^{C,1}))) \leq 1 \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T)$$

which is the inequality in case (ii) of Lemma 17.

We now assume $\alpha(C - N(T^{C,1})) > b$ and we use property P5 to obtain

$$f(\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda - 2b \geq f(\alpha(C - N(T^{C,1})) - b). \tag{32}$$

Combining (31) with (32) and adding 1 to both sides we obtain

$$f(\alpha(C - N(T^{C,1})) - b) + 1 \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T). \tag{33}$$

Using property P3 this transforms into

$$f(\tau\alpha(C - N(T^{C,1})) - \tau b) \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T). \tag{34}$$

We now apply property P8 with $x = b$. Assume $f(2\sqrt{2}b) \leq f(\tau)$. Then $\alpha(C - N(T^{C,1})) - b \geq 1$ and $\alpha(C - N(T^{C,1})) \leq \alpha(C) \leq 2b$ imply that

$$\begin{aligned} f(\sqrt{2}\alpha(C - N(T^{C,1}))) &\leq f(2\sqrt{2}b) \leq f(\tau) \\ &\leq f(\tau\alpha(C - N(T^{C,1})) - \tau b). \end{aligned} \tag{35}$$

Combining (34) and (35) we obtain

$$f(\sqrt{2}\alpha(C - N(T^{C,1}))) \leq |\mathcal{I}| - 1 + 2\operatorname{gap}(C, \mathcal{I}, T),$$

which is the inequality in case (ii) of Lemma 17.

Therefore, we can assume the other case of property P8 holds, namley that $\lceil \sqrt{2}(b-1) \rceil \geq \frac{\tau}{\tau - \sqrt{2}} b$. Since $T^{C,1} \in \mathcal{H}_2(C)$ we have $\alpha(C - N(T^{C,1})) \geq \sqrt{2}(b-1)$ so

$$\alpha(C - N(T^{C,1})) \geq \frac{\tau}{\tau - \sqrt{2}} b.$$

Manipulating this inequality, we find

$$\tau\alpha(C - N(T^{C,1})) - \tau b \geq \sqrt{2}\alpha(C - N(T^{C,1}))$$

so that

$$f(\tau\alpha(C - N(T^{C,1})) - \tau b) \geq f(\sqrt{2}\alpha(C - N(T^{C,1}))).$$

Combining this with (34) we obtain

$$f(\sqrt{2}\alpha(C - N(T^{C,1}))) \leq |\mathcal{I}| - 1 + 2\operatorname{gap}(C, \mathcal{I}, T),$$

which is the inequality in case *(ii)* of Lemma 17

Case 8. $R \neq \emptyset$ and $T = \emptyset$.
Using $T = \emptyset$ and $\theta = 0$, we apply Bounds 4 and 6 to simplify (4)

$$f(\lambda) + 2\alpha(C) - 2\lambda - 2b \leq |\mathcal{I}| - 1 + 2\operatorname{gap}(C, \mathcal{I}, \emptyset). \tag{36}$$

Since $\lambda = \sum_i \alpha(D_i - N(R^{C,5}))$ and $b \leq \alpha(R^{C,5})$ we have $\lambda + b \leq \alpha(C)$. We use property P5 to obtain

$$\begin{aligned} f(\alpha(C) - b) &\leq f(\alpha(C) - b) + \alpha(C) - b - \lambda \\ &\leq f(\lambda) + 2\alpha(C) - 2\lambda - 2b. \end{aligned} \tag{37}$$

We then combine (36) with (37) and add 1 to both sides to obtain

$$f(\alpha(C) - b) + 1 \le |\mathcal{I}| + 2\,\mathrm{gap}(C, \mathcal{I}, T).$$

Then property P9 shows $f(2\sqrt{2}\alpha(C)) \le f(\alpha(C) - b) + 1$ so we have proved the inequality in case (iii) of Lemma 17. ∎

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