Note

# FORBIDDEN-MINOR CHARACTERIZATION FOR THE CLASS OF COGRAPHIC ELEMENT SPLITTING MATROIDS 

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#### Abstract

In this paper, we prove that an element splitting operation by every pair of elements on a cographic matroid yields a cographic matroid if and only if it has no minor isomorphic to $M\left(K_{4}\right)$. Keywords: binary matroid, graphic matroid, cographic matroid, minor. 2010 Mathematics Subject Classifications: 05B35.


## 1. Introduction

The element splitting operation for binary matroid is defined in [3] as follows: Let $A$ be a matrix over $G F(2)$ that represents the matroid $M$. Suppose that $x$ and $y$ are distinct elements of $M$. Let $A_{x, y}^{\prime}$ be the matrix that is obtained

[^0]by adjoining an extra row to $A$ with this row being zero everywhere except in the columns corresponding to $x$ and $y$ where it takes the value 1 and then adjoining an extra column (corresponding to $a$ ) with this column being zero everywhere except in the last row where it takes the value 1 . Suppose $M_{x, y}^{\prime}$ is the matroid represented by the matrix $A_{x, y}^{\prime}$. Then $M_{x, y}^{\prime}$ is said to be obtained from $M$ by element splitting the pair of elements $x$ and $y$. The transition from $M$ to $M_{x, y}^{\prime}$ is called an element splitting operation. The matroid $M_{x, y}^{\prime}$ is called the element splitting matroid.

If $M$ is the cycle matroid of a graph $G$ of Figure $1, M_{x, y}^{\prime}$ is the cycle matroid of the graph $G_{x, y}^{\prime}$ of Figure 1.


Figure 1
Alternatively, the element splitting operation can be defined in terms of circuits of binary matroids as follows: Let $M=(S, \mathcal{C})$ be a binary matroid, $\{x, y\} \subseteq S$, and $a \notin S$. Let $\mathcal{C}_{0}=\{C \in \mathcal{C}: x, y \in C$ or $x, y \notin C\} ; \mathcal{C}_{1}=$ set of minimal members of $\left\{C_{1} \cup C_{2}: C_{1}, C_{2} \in \mathcal{C}, C_{1} \cap C_{2}=\phi\right.$ and $x \in C_{1}, y \in C_{2}$ such that $C_{1} \cup C_{2}$ contains no member of $\left.\mathcal{C}_{0}\right\}$; and $\mathcal{C}_{2}=\{C \cup\{a\}: C \in \mathcal{C}$ and $C$ contains exactly one of $x$ and $y\}$. Let $\mathcal{C}^{\prime}=\mathcal{C}_{0} \cup \mathcal{C}_{1} \cup \mathcal{C}_{2}$. Then $M_{x, y}^{\prime}=\left(S \cup\{a\}, \mathcal{C}^{\prime}\right)$ is the element splitting matroid.

The element splitting operation arises in the following way also [1]: Consider the unique binary extension of $M$ by the element $a$ so that $\{x, y, a\}$ is a triangle. Perform a Delta- $Y$ exchange on the triangle $\{x, y, a\}$. The resulting matroid is produced by performing an element splitting on the pair $x, y$.

The splitting operation for binary matroid is defined as follows [6]: Let $A$ be a matrix over $G F(2)$ that represents the matroid $M$. Consider distinct elements $x$ and $y$ of $M$. Let $A_{x, y}$ be the matrix that is obtained by adjoining an extra row to $A$ with this row being zero everywhere except in the columns corresponding to $x$ and $y$ where it takes the value 1 . Suppose $M_{x, y}$ is the matroid represented by the matrix $A_{x, y}$. Then $M_{x, y}$ is said to be obtained from $M$ by splitting away the pair $x, y$. The relation between the splitting operation and the element splitting operation is that $M_{x, y}^{\prime} \backslash\{a\}=M_{x, y}$.

Dalvi, Borse and Shikare [3] characterized graphic matroids whose element splitting matroids are also graphic as follows.

Theorem 1.1. The element splitting operation, by any pair of elements, on a graphic matroid yields a graphic matroid if and only if it has no minor isomorphic to $M\left(K_{4}\right)$, where $K_{4}$ is the complete graph on 4 vertices.

The element splitting operation on a cographic matroid may not yield a cographic matroid. In this paper, we characterize those cographic matroids $M$ for which the matroid $M_{x, y}^{\prime}$ is cographic for every pair of elements $\{x, y\}$ of $M$. The main result in this paper is the following theorem.

Theorem 1.2. The element splitting operation, by any pair of elements, on a cographic matroid yields a cographic matroid if and only if it has no minor isomorphic to $M\left(K_{4}\right)$, where $K_{4}$ is the complete graph on 4 vertices.

## 2. Proof of the Main Theorem

In this section, firstly we provide necessary lemmas.
Lemma 2.1 [5]. A binary matroid is cographic if and only if it has no minor isomorphic to $F_{7}, F_{7}^{*}, M\left(K_{5}\right)$, or $M\left(K_{3,3}\right)$.

Lemma 2.2 [5]. A binary matroid is graphic if and only if it has no minor isomorphic to $F_{7}, F_{7}^{*}, M^{*}\left(K_{5}\right)$, or $M^{*}\left(K_{3,3}\right)$.

Lemma 2.3 [5]. Every 3-connected binary matroid having at least four elements has a minor isomorphic to $M\left(K_{4}\right)$.

Lemma 2.4. Every binary matroid having no $M\left(K_{4}\right)$ minor is graphic and cographic.

Proof. Suppose that $M$ be a binary matroid without $M\left(K_{4}\right)$ as a minor. If $M$ is not graphic or cographic, then by Lemmas 2.1 and $2.2, M$ contains $F_{7}, F_{7}^{*}, M\left(K_{5}\right), M\left(K_{3,3}\right), M^{*}\left(K_{5}\right)$ or $M^{*}\left(K_{3,3}\right)$ as a minor. Since all the six matroids are binary and 3 -connected, by Lemma 2.3, each of these have $M\left(K_{4}\right)$ as a minor and hence $M$ has $M\left(K_{4}\right)$ as a minor, a contradiction.

Lemma 2.5. Let $M$ be a graphic matroid having no $M\left(K_{4}\right)$ minor and let $x, y \in E(M)$ be such that $M_{x, y}^{\prime}$ is not cographic. Then there is a minor $N$
of $M$ such that no two elements of $N$ are in series and $N_{x, y}^{\prime} \backslash\{a\} /\{x\} \cong F$ or $N_{x, y}^{\prime} \backslash\{a\} /\{x, y\} \cong F$ or $N_{x, y}^{\prime} \cong F$ or $N_{x, y}^{\prime} /\{x\} \cong F$ or $N_{x, y}^{\prime} /\{y\} \cong F$ or $N_{x, y}^{\prime} /\{x, y\} \cong F$ for some $F \in\left\{M\left(K_{5}\right), M\left(K_{3,3}\right)\right\}$.

Proof. The proof is similar to the proof of Lemma 2.3 of [3].

Proof of Theorem 1.2. Let $M$ be a cographic matroid. Suppose that $M$ has a minor $N$ isomorphic to $M\left(K_{4}\right)$. Then $N_{x, y}^{\prime} \cong F_{7}^{*}$ for $x, y$ corresponding to any pair of non-adjacent edges of $K_{4}$. So $N_{x, y}^{\prime}$ and hence $M_{x, y}^{\prime}$ is not cographic.

Suppose $M$ has no minor isomorphic to $M\left(K_{4}\right)$. Then, by Lemma 2.4, $M$ is graphic. We claim that $M_{x, y}^{\prime}$ is cographic. Suppose that $M_{x, y}^{\prime}$ is not cographic for some $x, y \in E(M)$. By, Theorem 1.1, $M_{x, y}^{\prime}$ is graphic. Hence $M_{x, y}^{\prime}$ does not contain $F_{7}$ and $F_{7}^{*}$ as minors. By, Lemmas 2.1 and 2.5, it is enough to prove that $M$ does not have a minor $N$ such that no two elements of $N$ are in series and $N_{x, y}^{\prime} \backslash\{a\} /\{x\} \cong F$ or $N_{x, y}^{\prime} \backslash\{a\} /\{x, y\} \cong F$ or $N_{x, y}^{\prime} \cong F$ or $N_{x, y}^{\prime} /\{x\} \cong F$ or $N_{x, y}^{\prime} /\{y\} \cong F$ or $N_{x, y}^{\prime} /\{x, y\} \cong F$ for some $F \in\left\{M\left(K_{5}\right), M\left(K_{3,3}\right)\right\}$. Since $M$ is graphic, $N$ is graphic. Let $G$ be a graph corresponding to $N$. Then $G$ is planar and has minimum degree at least three. Considering circuits of $M_{x, y}^{\prime}$, we note that every 1-cycle or 2 -cycle of $G$ must contain exactly one of $x$ and $y$. This implies that $G$ is loopless.

Case (i). Suppose that $F=M\left(K_{3,3}\right)$.
Note that $N_{x, y}^{\prime} \backslash\{a\}=N_{x, y}$. If $N_{x, y}^{\prime} \backslash\{a\} /\{x\} \cong M\left(K_{3,3}\right)$, then $N_{x, y} /\{x\} \cong$ $M\left(K_{3,3}\right)$. Hence, by Case (i) of Lemma 3.3 of [2], $N$ is isomorphic to the cycle matroid of graphs (i) or (ii) of Figure 2. As $K_{4}$ is a minor of each of these graphs, we obtain a contradiction. If $N_{x, y}^{\prime} \backslash\{a\} /\{x, y\} \cong M\left(K_{3,3}\right)$, then $N_{x, y} /\{x, y\} \cong M\left(K_{3,3}\right)$. So, by Case (ii) of Lemma 3.3 of [2], $N$ is isomorphic to the cycle matroid of graph (iii) of Figure 2 and thus has $K_{4}$ as a minor, a contradiction.


Figure 2

Suppose that $N_{x, y}^{\prime} \cong M\left(K_{3,3}\right)$. Then $G$ has 5 vertices, 8 edges and the minimum vertex degree at least three. If $G$ has a 2 -cycle then we get a 3 circuit in $M_{x, y}^{\prime}$ containing $a$, a contradiction. This implies that $G$ is simple. Therefore, by Appendix 1 of [4], $G$ is isomorphic to the graph (iv) of Figure 2 and has $K_{4}$ as a minor, a contradiction. Suppose that $N_{x, y}^{\prime} /\{x\} \cong M\left(K_{3,3}\right)$. Then $G$ has 6 vertices, 9 edges. Further, $G$ is simple. Since minimum degree in $G$ is at least 3, $G$ is isomorphic to the graph (v) of Figure 2 (see Appendix 1 of [4]) and hence has $K_{4}$ as a minor, a contradiction. Finally, suppose that $N_{x, y}^{\prime} /\{x, y\} \cong M\left(K_{3,3}\right)$. Then a graph corresponding to $M$ has 7 vertices and 10 edges. This implies that $G$ has at least one vertex of degree two, which is a contradiction.

Case (ii). Suppose that $F=M\left(K_{5}\right)$.
If $N_{x, y}^{\prime} \backslash\{a\} /\{x\} \cong M\left(K_{5}\right)$ or $N_{x, y}^{\prime} \backslash\{a\} /\{x, y\} \cong M\left(K_{5}\right)$, then $N_{x, y} /\{x\} \cong$ $M\left(K_{5}\right)$ or $N_{x, y} /\{x, y\} \cong M\left(K_{5}\right)$. So, by Cases (i) and (ii), respectively of Lemma 3.4 of [2], $N$ is isomorphic to the cycle matroid of one of the graphs of Figure 3. As all each of these graphs has $K_{4}$ as a minor, we obtain a contradiction.


Figure 3
Since $N_{x, y}^{\prime}$ is not Eulerian, $N_{x, y}^{\prime} \neq M\left(K_{5}\right)$. Suppose that $N_{x, y}^{\prime} /\{x\} \cong M\left(K_{5}\right)$. Then $G$ has 5 vertices, 10 edges. By Appendix 1 of [4], $G$ must be non-simple. If $x$ belongs to a 2 -cycle of $G$, then $N_{x, y}^{\prime}$ has a 3 -circuit containing $x$ and consequently $M\left(K_{5}\right)$ has a 2 -circuit, a contradiction. This implies that $G$ has exactly one 2 -cycle. Hence $G$ can be obtained from a simple planar graph with 5 vertices and 9 edges by adding an edge in parallel. By Appendix 1 of [4], there is only one graph with 5 vertices and 9 edges which has $M\left(K_{4}\right)$ as a minor, a contradiction. Finally, suppose that $N_{x, y}^{\prime} /\{x, y\} \cong M\left(K_{5}\right)$. Then $G$ is a planar graph with 6 vertices, 11 edges and has minimum degree at least 3. It follows that $G$ is simple. There are 3 such non-isomorphic graphs (see Appendix 1 of [4]). As each of these graphs has $M\left(K_{4}\right)$ as a minor,
$G$ cannot be isomorphic to any one of them. This completes the proof of the theorem.

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