# SIMPLICIAL AND NONSIMPLICIAL COMPLETE SUBGRAPHS 

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#### Abstract

Define a complete subgraph $Q$ to be simplicial in a graph $G$ when $Q$ is contained in exactly one maximal complete subgraph ('maxclique') of $G$; otherwise, $Q$ is nonsimplicial. Several graph classes-including strong $p$-Helly graphs and strongly chordal graphs - are shown to have pairs of peculiarly related new characterizations: (i) for every $k \geq 2$, a certain property holds for the complete subgraphs that are in $k$ or more maxcliques of $G$, and (ii) in every induced subgraph $H$ of $G$, that same property holds for the nonsimplicial complete subgraphs of $H$.

One example: $G$ is shown to be hereditary clique-Helly if and only if, for every $k \geq 2$, every triangle whose edges are each in $k$ or more maxcliques is itself in $k$ or more maxcliques; equivalently, in every induced subgraph $H$ of $G$, if each edge of a triangle is nonsimplicial in $H$, then the triangle itself is nonsimplicial in $H$.


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A maxclique of a graph is an inclusion-maximal complete subgraph. For each complete subgraph $Q$ of a graph $G$, $\operatorname{define~}_{\operatorname{str}}^{G}(Q)$ to be the number of maxcliques of $G$ that contain $Q$. Notice that if $H$ is an induced subgraph of $G$ and $Q$ is a complete subgraph of $H$, then $\operatorname{str}_{H}(Q) \leq \operatorname{str}_{G}(Q)$. As in [5], define $Q$ to be strength- $k$ in $G$ if $\operatorname{str}_{G}(Q) \geq k$.

Define $Q$ to be a simplicial clique of $G$ if $\operatorname{str}_{G}(Q)=1$ and to be a nonsimplicial clique of $G$ if $Q$ is strength- 2 in $G$. A $k$-clique is a complete subgraph of order $k$. When convenient, a complete subgraph $Q$ will be identified with its vertex set $V(Q)$.

The distinguishing feature of each 'Theorem $n$ ' or 'Corollary $n$ ' below can be loosely described as the equivalence of two statements involving a parameterized graph property $\mathcal{P}(k)$ (defined in terms of the strengths of complete subgraphs):
(n.1) $G$ satisfies $\mathcal{P}(k)$ for all $k \geq 2$.
(n.2) Every induced subgraph of $G$ satisfies $\mathcal{P}(2)$.

Typically, there will also be equivalent statements ( $n .0$ ), asserting $G$ to be in a known graph class, and (n.3), expressed in terms of (non)simplicial cliques.

## 1. Clique Strength and Strong $p$-Helly Graphs

A graph is strong $p$-Helly if every family $\mathcal{Q}$ of maxcliques contains a subfamily $\mathcal{Q}^{\prime}$ with $\left|\mathcal{Q}^{\prime}\right| \leq p$ such that $\cap \mathcal{Q}=\cap \mathcal{Q}^{\prime}$. Reference [2] proves that these are also precisely the graphs that are hereditary p-clique-Helly (meaning that, for every family $\mathcal{Q}$ of maxcliques, if every $p$ members of $\mathcal{Q}$ have a vertex in common, then all the members of $\mathcal{Q}$ have a vertex in common). Theorem 1 will contain additional characterizations.

Theorem 1. The following are equivalent for every graph $G$ and $p \geq 2$ :
(1.0) $G$ is strong $p$-Helly.
(1.1) For every $k \geq 2$ and every $p$-clique $Q$ of $G$, if each $(p-1)$-clique that is contained in $Q$ is strength-k in $G$, then $Q$ is also strength-k in $G$.
(1.2) For every $p$-clique $Q$ of an induced subgraph $H$ of $G$, if each $(p-1)$ clique that is contained in $Q$ is strength- 2 in $H$, then $Q$ is also strength2 in $H$.
(1.3) If a p-clique $Q$ is simplicial in an induced subgraph $H$ of $G$, then at least one ( $p-1$ )-clique that is contained in $Q$ is simplicial in $H$.

Proof. (1.1) $\Rightarrow$ (1.2): Suppose $p \geq 2$ and $G$ satisfies condition (1.1). Suppose $H$ is any proper induced subgraph of $G$ and $Q$ is a $p$-clique of $H$ such that, if $Q^{-}$is a $(p-1)$-clique with $Q^{-} \subset Q$, then $Q^{-}$is strength- 2 in $H$.

But assume $Q$ itself is not strength-2 in $H$ [arguing by contradiction]; so $\operatorname{str}_{H}(Q)=1$. (Since each $Q^{-}$is also strength- 2 in $G$, the $k=2$ case of (1.1) implies $Q$ is strength-2 in $G$.)

Let $g=\operatorname{str}_{G}(Q)$. Then $Q$ will be in $g-1$ more maxcliques in $G$ than in $H$. Therefore, each of the $(p-1)$-cliques contained in $Q$ will be strength-$(2+[g-1])$ in $G$, and so strength- $(g+1)$ in $G$. But then (1.1) implies that $Q$ is strength- $(g+1)$ in $G$ [contradicting that $\left.\operatorname{str}_{G}(Q)=g\right]$.
$(1.1) \Leftarrow(1.2)$ : Suppose $p \geq 2$ and $G$ satisfies condition (1.2). Suppose $Q$ is a $p$-clique and $Q_{1}, \ldots, Q_{p}$ are the $(p-1)$-cliques contained in $Q$. Suppose $k \geq 2$ and each $Q_{i}$ is strength- $k$ in $G$, but $Q$ itself is not strength- $k$ in $G$ [arguing by contradiction].

Suppose $Q$ is contained in the pairwise-distinct maxcliques $Q^{1}, \ldots, Q^{g}$ of $G$ where $\operatorname{str}_{G}(Q)=g<k$, and suppose each $Q_{i}$ is contained in the pairwisedistinct maxcliques $Q^{1}, \ldots, Q^{g}, Q_{i}^{1}, \ldots, Q_{i}^{k-g}$ of $G$ where each $Q_{i}^{j} \cap Q=Q_{i}$. Let $H$ be the subgraph of $G$ induced by

$$
Q \cup \bigcup_{i=1}^{p} \bigcup_{j=1}^{k-g} Q_{i}^{j}-\bigcup_{j=1}^{g}\left(Q^{j}-Q\right)
$$

Then each $Q_{i}$ is strength-2 in $H$, but $\operatorname{str}_{H}(Q)=1$ [contradicting (1.2)].
$(1.2) \Leftrightarrow(1.3)$ : Condition (1.3) simply restates (1.2) using that $Q$ is strength-2 in $H$ if and only if $Q$ is nonsimplicial in $H$.
$(1.0) \Leftrightarrow(1.3)$ : This follows from $[2$, Theorem 4].

Notice that the proof of $(1.1) \Leftrightarrow(1.2)$ in Theorem 1 did not use the characterization of strong- $p$ Helly graphs from [2]. This enables the $p=2$ and $p=3$ cases of Theorem 1 to be presented separately as Corollaries 2 and 3 .

Let $C_{k}$ and $P_{k}$ denote, respectively, a cycle or path on $k$ vertices. For any graphs $G, H_{1}, \ldots, H_{s}$, say that $G$ is $\left\{H_{1}, \ldots, H_{s}\right\}$-free (or simply $H_{1}$ free if $s=1$ ) if $G$ contains no induced subgraph isomorphic to any of the graphs $H_{1}, \ldots, H_{s}$. A graph is trivially perfect if it is $\left\{C_{4}, P_{4}\right\}$-free; see $[1,7]$ for additional characterizations (and additional names).

Corollary 2. The following are equivalent for every graph $G$ :
(2.0) $G$ is trivially perfect.
(2.1) For every $k \geq 2$ and every edge $x y$ of $G$, if both $x$ and $y$ are strength- $k$ in $G$, then edge $x y$ is strength- $k$ in $G$.
(2.2) For every edge $x y$ of an induced subgraph $H$ of $G$, if both $x$ and $y$ are strength-2 in $H$, then edge $x y$ is strength-2 in $H$.
(2.3) If an edge $e$ is simplicial in an induced subgraph $H$ of $G$, then at least one endpoint of $e$ is simplicial in $H$.

Proof. The $k=2$ case of condition (2.1) implies that $G$ is $\left\{C_{4}, P_{4}\right\}$ free - and so implies (2.0) - by letting $x y$ be an edge of an induced $C_{4}$ or $P_{4}$ subgraph. Conversely, if (2.1) fails, suppose $x y \in E(G)$ where $x$ is in a maxclique that does not contain $y$ and $y$ is in a maxclique that does not contain $x$. Then those maxcliques contain edges $x x^{\prime}$ and $y y^{\prime}$ where $\left\{x^{\prime}, x, y, y^{\prime}\right\}$ induces either a $P_{4}$ or a $C_{4}$ subgraph, making (2.0) fail.

The equivalence of (2.1) and (2.2) is the $p=2$ case of Theorem 1. Condition (2.3) simply restates (2.2).

A graph is clique-Helly if, for every family $\mathcal{F}$ of maxcliques, if every two members of $\mathcal{F}$ have a vertex in common, then all the members of $\mathcal{F}$ have a vertex in common. A graph is hereditary clique-Helly if every induced subgraph is clique Helly. See [1, 5, 8, 9] for details. Reference [9] also proves that $G$ is hereditary clique-Helly if and only if, for every maxclique $Q$ of an induced subgraph $H$ of $G$, at least one edge of $Q$ is simplicial in $H$. The hereditary clique-Helly graphs are, of course, precisely the hereditary 2-clique-Helly graphs (and so are precisely the strong 2-Helly graphs).
Corollary 3. The following are equivalent for every graph $G$ :
(3.0) $G$ is hereditary clique-Helly.
(3.1) For every $k \geq 2$ and every triangle $x y z$ of $G$, if each edge $x y, x z$, and $y z$ is strength-k in $G$, then triangle $x y z$ is strength- $k$ in $G$.
(3.2) For every triangle $x y z$ of an induced subgraph $H$ of $G$, if each edge $x y$, $x z$, and $y z$ is strength-2 in $H$, then triangle $x y z$ is strength-2 in $H$.
(3.3) If a triangle $\Delta$ is simplicial in an induced subgraph $H$ of $G$, then at least one edge of $\Delta$ is simplicial in $H$.

Proof. The equivalence of (3.0) and (3.1) restates [5, Theorem 2]. The equivalence of (3.1) and (3.2) is the $p=3$ case of Theorem 1. Condition (3.3) simply restates (3.2).

Sections 2 and 3 go in a different direction, generalizing Corollary 3 by replacing triangles with arbitrary cycles.

## 2. Edge Strength and Chordal Graphs

A chord of a cycle is an edge that joins two nonconsecutive vertices of the cycle (only cycles of length four or more can have chords). A graph is chordal if and only if every cycle of length four or more has a chord; see [1, 7] for thorough discussions. Define a graph to be strength-k chordal if every cycle of strength- $k$ edges either has a strength- $k$ chord or is a strength- $k$ triangle. Being strength- 1 chordal is equivalent to being chordal, and Corollary 6 will characterize being strength- $k$ chordal for all $k \geq 1$.

The graph $G_{1}$ in Figure 1 is the smallest chordal graph that is not strength-2 chordal-the three edges between the vertices 2,3 , and 5 are each strength-2, but the triangle they form is simplicial in $G$. The graph $G_{2}$ is strength-2 chordal-the nine edges incident to vertices 3 or 4 are each strength-2 (indeed, the edge 34 is strength-4), as are the four triangles that contain edge 34 - yet $G_{2}$ is not chordal because of the chordless cycle $1,2,6,5,1$. (The graph $G_{2}$ is vacuously strength- $k$ chordal for all $k>2$.)


Figure 1. Graph $G_{1}$ is chordal, but not strength-2 chordal; $G_{2}$ is strength- $k$ chordal for all $k \geq 2$, but not chordal.

As is common when working with cycle spaces, a sum of cycles will mean the symmetric difference of the edge sets of those cycles-in other words, an edge $e$ is in the sum (denoted) $C_{1} \oplus \cdots \oplus C_{k}$ if and only if $e$ is in an odd number of the cycles $C_{1}, \ldots, C_{k}$. The notation $|C|$ will be used to denote the length of a cycle $C$, and $C$ is a $k$-cycle if $|C|=k$. Lemma 4 will generalize the following simple fact from [4, Lemma 3.2] (also see [6, Corollary 1]): $A$ graph is chordal if and only if every cycle $C$ is the sum of $|C|-2$ triangles.

Lemma 4. A graph is strength-k chordal if and only if every cycle $C$ of strength- $k$ edges is the sum of $|C|-2$ strength- $k$ triangles.

Proof. First suppose $G$ is a chordal graph in which every cycle $C$ of strength- $k$ edges with $|C|=l \geq 3$ is the sum of $l-2$ strength- $k$ triangles
$\Delta_{1}, \ldots, \Delta_{l-2}$. If $l=3$, then $C$ itself is a strength- $k$ triangle $\Delta_{1}$. Suppose $l \geq 4$ [toward showing that $C$ has a strength- $k$ chord]. Because $G$ is chordal, each edge of $C$ must be in some triangle $\Delta_{i}$. The pigeon-hole principle implies that some $\Delta_{i}$ must contain two (necessarily consecutive) edges of $C$. Then the third side of $\Delta_{i}$ is a chord of $C$. Since $\Delta_{i}$ is strength- $k$, that third side is a strength- $k$ chord of $C$.

Conversely, suppose $G$ is a strength- $k$ chordal graph and $C$ is a cycle of strength- $k$ edges. Argue by induction on $|C|=l \geq 3$. If $l=3$, then $C$ is a strength- $k$ triangle and so $C$ is trivially the sum of $l-2=1$ strength- $k$ triangles. Now suppose $l \geq 4$. Since $G$ is strength- $k$ chordal, cycle $C$ has a strength- $k$ chord $e$. Then $C=C_{a} \oplus C_{b}$ where $C_{a}$ and $C_{b}$ are cycles of strength- $k$ edges from $E\left(C_{a}\right) \cup E\left(C_{b}\right) \cup\{e\}$, with $\{e\}=C_{a} \cap C_{b},\left|C_{a}\right|=a$, $\left|C_{b}\right|=b$, and $a+b=l+2$. The induction hypothesis implies that $C_{a}$ [respectively, $C_{b}$ ] is the sum of $a-2$ [or $b-2$ ] strength- $k$ triangles. This makes $C$ the sum of $(a-2)+(b-2)=l-2$ strength- $k$ triangles.

Theorem 5. The following are equivalent for every graph $G$ :
(5.1) $G$ is strength $-k$ chordal for all $k \geq 2$.
(5.2) Every induced subgraph of $G$ is strength-2 chordal.
(5.3) Every cycle of nonsimplicial edges in an induced subgraph $H$ of $G$ either has a chord that is nonsimplicial in $H$ or is a nonsimplicial triangle of $H$.
(5.4) Every cycle $C$ of nonsimplicial edges in an induced subgraph $H$ of $G$ is the sum of $|C|-2$ nonsimplicial triangles of $H$.

Proof. (5.1) $\Rightarrow$ (5.2): Suppose $G$ satisfies condition (5.1). Suppose $H$ is any induced subgraph of $G$ and $C$ is a cycle of edges that are strength-2 in $H$, but $C$ is not the sum of $|C|-2$ triangles that are strength- 2 in $H$ [arguing by contradiction, using Lemma 4]; further suppose $|C|$ is minimum with respect to all that. By the minimality of $|C|$, every chord of $C$ is simplicial in $H$. This implies that every triangle $\Delta$ with $V(\Delta) \subseteq V(C)$ is simplicial in $H$. Thus, for every edge $e$ and triangle $\Delta$, if $e \in E(C) \cap E(\Delta)$ and $V(\Delta) \subseteq V(C)$, then $\operatorname{str}_{H}(e)>\operatorname{str}_{H}(\Delta)$. But since every maxclique of $G$ that contains such a $\Delta$ also contains such edges $e$, the same inequality holds with $H$ replaced by $G\left[\right.$ contradicting (5.1), using Lemma 4 with $k=\min \left\{\operatorname{str}_{G}(e): e \in E(C)\right]$.
$(5.1) \Leftarrow(5.2)$ : Suppose $G$ satisfies condition (5.2), toward proving $G$ is strength- $k$ chordal by induction on $k \geq 2$. The $k=2$ basis step is immediate.

For the inductive step, suppose $G$ is strength- $k$ chordal and $C$ is a cycle of edges that are strength- $(k+1)$ in $G$, but $C$ is not the sum of $|C|-2$ triangles that are strength- $(k+1)$ in $G$ [arguing by contradiction, using Lemma 4]; further suppose $|C|=l \geq 3$ is minimal with respect to all that. By the minimality of $l$, cycle $C$ has no chords that are strength- $(k+1)$ in $G$. Since $G$ is strength- $k$ chordal, $C$ is the sum of triangles $\Delta_{1}, \ldots, \Delta_{l-2}$ of $G$ that are strength- $k$ in $G$, where each $\Delta_{i}$ is made from edges of $C$ that are strength$(k+1)$ in $G$ together with chords $e$ of $C$ with $\operatorname{str}_{G}(e)=k$. Therefore if $\Delta_{i}$ and $\Delta_{j}$ share a chord of $C$, then $V\left(\Delta_{i}\right) \cup V\left(\Delta_{j}\right)$ must induce a complete subgraph $Q_{1}$ that is strength- $k$ in $G$. Performing similar consolidations of complete subgraphs $n \leq l-3$ times partitions $\left\{\Delta_{1}, \ldots, \Delta_{l-2}\right\}$ into $l-2-n$ parts that are sets of contiguous triangles that are strength- $k$ in $G$ and whose vertices induce $l-2-n$ complete subgraphs that are strength- $k$ in $G$ and that cover $V(C)$. Performing this consolidation $n=l-3$ times shows that $V(C)$ induces a complete subgraph $Q_{n}$ that is strength- $k$ in $G$. Since $C$ has no chords that are strength- $(k+1)$ in $G$, it follows that $\operatorname{str}_{G}\left(Q_{n}\right)=k$. Yet each $e \in E(C)$ is strength- $(k+1)$ in $G$ and so is in a maxclique $Q_{e}$ of $G$ that has $E\left(Q_{e}\right) \cap E\left(Q_{n}\right)=\{e\}$ (again using that $C$ has no chords that are strength- $(k+1)$ in $G$ ). But then $V\left(Q_{n}\right)$ together with one vertex from $V\left(Q_{e}\right)-V\left(Q_{n}\right)$ for each $e \in E(C)$ would induce a subgraph $H$ of $G$ such that each edge of $C$ is strength-2 in $H$ while $\operatorname{str}_{H}\left(Q_{n}\right)=1$ and each chord $e$ of $C$ has $\operatorname{str}_{H}(e)=1$ [contradicting (5.2)].
(5.2) $\Leftrightarrow(5.3) \Leftrightarrow$ (5.4) follows since (5.3) and (5.4) simply restate (5.2) (using Lemma 4).

For $k \geq 3$, a $k$-sun-sometimes called a complete $k$-sun or trampoline, see $[1,3,5,7]$-is a graph that consists of an even-length cycle $v_{1}, \ldots, v_{2 k}, v_{1}$, together with all of the $\binom{k}{2}$ chords between even-subscripted vertices. (The graph $G_{1}$ in Figure 1 is a 3 -sun, and the subgraph $H$ constructed in the (5.1) $\Leftarrow(5.2)$ proof of Theorem 5 is an $l$-sun.) A graph is strongly chordal if it is chordal and no induced subgraph is isomorphic to any $k$-sun; see $[1,3,5,7]$ for other characterizations of this widely-studied concept.

Corollary 6. The following are equivalent for every graph $G$ :
(6.0) $G$ is strongly chordal.
(6.1) $G$ is strength-k chordal for all $k \geq 1$.
(6.2) $G$ is chordal and every induced subgraph of $G$ is strength-2 chordal.
(6.3) $G$ is chordal and every cycle of nonsimplicial edges in an induced subgraph $H$ of $G$ either has a chord that is nonsimplicial in $H$ or is a nonsimplicial triangle of $H$.
(6.4) $G$ is chordal and every cycle $C$ of nonsimplicial edges in an induced subgraph $H$ of $G$ is the sum of $|C|-2$ nonsimplicial triangles of $H$.

Proof. The equivalence of (6.0) and (6.1) restates [5, Theorem 1]. The equivalence of conditions (6.i) and (6.j) when $1 \leq i<j \leq 4$ follows immediately from the equivalence of conditions (5.i) and (5.j).

## 3. Vertex Strength and Chordal Graphs

Recognizing that cycles are determined by their vertices just as well as by their edges, define a graph to be vertex strength-k chordal if every cycle of strength- $k$ vertices either has a strength- $k$ chord or is a strength- $k$ triangle. (Strength- $k$ chordal graphs could have been called 'edge strength- $k$ chordal' graphs.) Being vertex strength-1 chordal is equivalent to being chordal. Clearly, every cycle of strength- $k$ edges is a cycle of strength- $k$ vertices, and so every vertex strength- $k$ chordal graph is strength- $k$ chordal. The three graphs in Figure 2 are strength-2 chordal but not vertex strength-2 chordal (because each vertex shown as 'hollow' is a strength-2 vertex).


Figure 2. From left to right, the kite, gem, and net graphs.
Lemma 7. A graph is vertex strength-k chordal if and only if every cycle $C$ of strength- $k$ vertices is the sum of $|C|-2$ strength- $k$ triangles.

Proof. This is proved by a straightforward modification of the proof of Lemma 4 (observing that every strength- $k$ edge has strength- $k$ endpoints).

Theorem 8. The following are equivalent for every graph $G$ :
(8.0) $G$ is $\{$ kite, gem, net $\}$-free strongly chordal.
(8.1) $G$ is vertex strength $-k$ chordal for all $k \geq 2$.
(8.2) Every induced subgraph of $G$ is vertex strength-2 chordal.
(8.3) Every cycle of nonsimplicial vertices in an induced subgraph $H$ of $G$ either has a chord that is nonsimplicial in $H$ or is a nonsimplicial triangle of $H$.
(8.4) Every cycle $C$ of nonsimplicial vertices in an induced subgraph $H$ of $G$ is the sum of $|C|-2$ nonsimplicial triangles of $H$.

Proof. (8.0) $\Rightarrow$ (8.1): Suppose $k \geq 2$ and $G$ satisfies condition (8.0), and $C$ is a cycle of strength- $k$ vertices. Since $G$ is chordal, $C$ is the sum of $|C|-2$ triangles. Suppose any of those triangles - say triangle $v_{1} v_{2} v_{3}$-has $\operatorname{str}_{G}\left(v_{1} v_{2} v_{3}\right)<k$ [arguing by contradiction, using Lemma 7 , showing that $G$ would contain an induced kite, gem, net, or 3 -sun]. Then for each $i \in$ $\{1,2,3\}$, there exists a vertex $w_{i}$ with each $w_{i} \sim v_{i}$ and $w_{i} \notin\left\{v_{1}, v_{2}, v_{3}\right\}$ and $w_{i}$ not adjacent to some $v_{j}$. Thus $\left|\left\{w_{1}, w_{2}, w_{3}\right\}\right|>1$. Let $H$ be the subgraph of $G$ that is induced by $\left\{v_{1}, v_{2}, v_{3}, w_{1}, w_{2}, w_{3}\right\}$. If $\left|\left\{w_{1}, w_{2}, w_{3}\right\}\right|=2$, then $H$ is an induced kite or gem [a contradiction]. If $\left|\left\{w_{1}, w_{2}, w_{3}\right\}\right|=3$, then either $H$ is an induced net or 3 -sun or $H$ contains an induced kite or gem [a contradiction].
$(8.1) \Rightarrow(8.2)$ : Suppose $G$ satisfies condition (8.1). Suppose $H$ is any induced subgraph of $G$ and $C$ is a cycle of vertices that are strength- 2 in $H$, but $C$ is not the sum of $|C|-2$ triangles that are strength- 2 in $H$ [arguing by contradiction, using Lemma 7]; further suppose $|C|$ is minimum with respect to all that. By the minimality of $|C|$, the cycle $C$ is chordless and so (since vertex strength- $k$ chordal implies chordal) $C$ is a triangle $\Delta$ where $\operatorname{str}_{H}(\Delta)=1$. Thus $\operatorname{str}_{H}(v)>\operatorname{str}_{H}(\Delta)$ for every $v \in V(\Delta)$. But since every maxclique of $G$ that contains $\Delta$ also contains every $v \in V(\Delta)$, it follows that $\operatorname{str}_{G}(v)>\operatorname{str}_{G}(\Delta)\left[\right.$ contradicting (8.1) with $\left.k=\min \left\{\operatorname{str}_{G}(v): v \in V(\Delta)\right\}\right]$.
$(8.0) \Leftarrow(8.2)$ : Suppose $G$ satisfies condition (8.2). Then no induced subgraph $H$ of $G$ can be isomorphic to $C_{k}$ with $k \geq 4$ (a chordless cycle of vertices that are strength- 2 in $G$ ), or to a $k$-sun, kite, gem, or net graph (each containing a triangle $\Delta$ of vertices that are strength-2 in $G$ while $\operatorname{str}_{G}(\Delta)=1$ ). Thus (8.0) holds.
$(8.2) \Leftrightarrow(8.3) \Leftrightarrow(8.4)$ follows since (8.3) and (8.4) simply restate (8.2).

If a graph $G$ is vertex strength- $k$ chordal for all $k \geq 2$, then $G$ is strongly chordal and so is certainly vertex strength-1 chordal. Therefore, the $k \geq 2$
restriction in condition (8.1) could just as well be replaced with $k \geq 1$, and no 'Corollary 9 ' is needed to parallel Corollary 6.

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