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SIMPLICIAL AND NONSIMPLICIAL COMPLETE SUBGRAPHS

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Abstract

Define a complete subgraph Q to be simplicial in a graph G when Q is contained in exactly one maximal complete subgraph ('maxclique') of G; otherwise, Q is nonsimplicial. Several graph classes—including strong p-Helly graphs and strongly chordal graphs—are shown to have pairs of peculiarly related new characterizations: (i) for every $k \geq 2$, a certain property holds for the complete subgraphs that are in k or more maxcliques of G, and (ii) in every induced subgraph H of G, that same property holds for the nonsimplicial complete subgraphs of H.

One example: G is shown to be hereditary clique-Helly if and only if, for every $k \geq 2$, every triangle whose edges are each in k or more maxcliques is itself in k or more maxcliques; equivalently, in every induced subgraph H of G, if each edge of a triangle is nonsimplicial in H, then the triangle itself is nonsimplicial in H.

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A *maxclique* of a graph is an inclusion-maximal complete subgraph. For each complete subgraph Q of a graph G, define $\mathrm{str}_G(Q)$ to be the number of maxcliques of G that contain Q. Notice that if H is an induced subgraph of G and Q is a complete subgraph of H, then $\mathrm{str}_H(Q) \leq \mathrm{str}_G(Q)$. As in [5], define Q to be *strength-k* in G if $\mathrm{str}_G(Q) \geq k$.

Define Q to be a *simplicial clique* of G if $\text{str}_G(Q) = 1$ and to be a *nonsimplicial clique* of G if Q is strength-2 in G. A *k-clique* is a complete subgraph of order k. When convenient, a complete subgraph Q will be identified with its vertex set $V(Q)$.

The distinguishing feature of each 'Theorem n' or 'Corollary n' below can be loosely described as the equivalence of two statements involving a parameterized graph property $\mathcal{P}(k)$ (defined in terms of the strengths of complete subgraphs):

(n.1) G satisfies $\mathcal{P}(k)$ for all $k \geq 2$.

(n.2) Every induced subgraph of G satisfies $\mathcal{P}(2)$.

Typically, there will also be equivalent statements (n.0), asserting G to be in a known graph class, and (n.3), expressed in terms of (non)simplicial cliques.

1. Clique Strength and Strong p-Helly Graphs

A graph is *strong p-Helly* if every family $\mathcal{Q}$ of maxcliques contains a subfamily $\mathcal{Q}'$ with $|\mathcal{Q}'| \leq p$ such that $\cap\mathcal{Q} = \cap\mathcal{Q}'$. Reference [2] proves that these are also precisely the graphs that are *hereditary p-clique-Helly* (meaning that, for every family $\mathcal{Q}$ of maxcliques, if every p members of $\mathcal{Q}$ have a vertex in common, then all the members of $\mathcal{Q}$ have a vertex in common). Theorem 1 will contain additional characterizations.

Theorem 1. *The following are equivalent for every graph G and $p \geq 2$:*

(1.0) *G is strong p-Helly.*

(1.1) *For every $k \geq 2$ and every p-clique Q of G, if each $(p-1)$-clique that is contained in Q is strength-k in G, then Q is also strength-k in G.*

(1.2) *For every p-clique Q of an induced subgraph H of G, if each $(p-1)$-clique that is contained in Q is strength-2 in H, then Q is also strength-2 in H.*

(1.3) *If a p-clique Q is simplicial in an induced subgraph H of G, then at least one $(p-1)$-clique that is contained in Q is simplicial in H.*

Proof. (1.1) $\Rightarrow$ (1.2): Suppose $p \geq 2$ and G satisfies condition (1.1). Suppose H is any proper induced subgraph of G and Q is a p-clique of H such that, if Q^- is a $(p-1)$-clique with $Q^- \subset Q$, then Q^- is strength-2 in H.

But assume Q itself is not strength-2 in H [arguing by contradiction]; so $\mathrm{str}_H(Q) = 1$. (Since each Q^- is also strength-2 in G, the $k = 2$ case of (1.1) implies Q is strength-2 in G.)

Let $g = \mathrm{str}_G(Q)$. Then Q will be in $g - 1$ more maxcliques in G than in H. Therefore, each of the $(p-1)$-cliques contained in Q will be strength-$(2 + [g-1])$ in G, and so strength-$(g+1)$ in G. But then (1.1) implies that Q is strength-$(g+1)$ in G [contradicting that $\mathrm{str}_G(Q) = g$].

(1.1) $\Leftarrow$ (1.2): Suppose $p \geq 2$ and G satisfies condition (1.2). Suppose Q is a p-clique and $Q_1, \ldots, Q_p$ are the $(p-1)$-cliques contained in Q. Suppose $k \geq 2$ and each Q_i is strength-k in G, but Q itself is not strength-k in G [arguing by contradiction].

Suppose Q is contained in the pairwise-distinct maxcliques $Q^1, \ldots, Q^g$ of G where $\mathrm{str}_G(Q) = g < k$, and suppose each Q_i is contained in the pairwise-distinct maxcliques $Q^1, \ldots, Q^g, Q_i^1, \ldots, Q_i^{k-g}$ of G where each $Q_i^j \cap Q = Q_i$. Let H be the subgraph of G induced by

$$Q \;\cup\; \bigcup_{i=1}^{p} \bigcup_{j=1}^{k-g} Q_i^j - \bigcup_{j=1}^{g} (Q^j - Q).$$

Then each Q_i is strength-2 in H, but $\mathrm{str}_H(Q) = 1$ [contradicting (1.2)].

(1.2) $\Leftrightarrow$ (1.3): Condition (1.3) simply restates (1.2) using that Q is strength-2 in H if and only if Q is nonsimplicial in H.

(1.0) $\Leftrightarrow$ (1.3): This follows from [2, Theorem 4]. ∎

Notice that the proof of (1.1) $\Leftrightarrow$ (1.2) in Theorem 1 did not use the characterization of strong-p Helly graphs from [2]. This enables the $p = 2$ and $p = 3$ cases of Theorem 1 to be presented separately as Corollaries 2 and 3.

Let C_k and P_k denote, respectively, a cycle or path on k vertices. For any graphs $G, H_1, \ldots, H_s$, say that G is *$\{H_1, \ldots, H_s\}$-free* (or simply *H_1-free* if $s = 1$) if G contains no induced subgraph isomorphic to any of the graphs $H_1, \ldots, H_s$. A graph is *trivially perfect* if it is $\{C_4, P_4\}$-free; see [1, 7] for additional characterizations (and additional names).

Corollary 2. *The following are equivalent for every graph G:*

(2.0) *G is trivially perfect.*

(2.1) *For every $k \geq 2$ and every edge xy of G, if both x and y are strength-k in G, then edge xy is strength-k in G.*

(2.2) *For every edge xy of an induced subgraph H of G, if both x and y are strength-2 in H, then edge xy is strength-2 in H.*

(2.3) *If an edge e is simplicial in an induced subgraph H of G, then at least one endpoint of e is simplicial in H.*

Proof. The $k = 2$ case of condition (2.1) implies that G is $\{C_4, P_4\}$-free—and so implies (2.0)—by letting xy be an edge of an induced C_4 or P_4 subgraph. Conversely, if (2.1) fails, suppose $xy \in E(G)$ where x is in a maxclique that does not contain y and y is in a maxclique that does not contain x. Then those maxcliques contain edges xx' and yy' where $\{x', x, y, y'\}$ induces either a P_4 or a C_4 subgraph, making (2.0) fail.

The equivalence of (2.1) and (2.2) is the $p = 2$ case of Theorem 1. Condition (2.3) simply restates (2.2). ∎

A graph is *clique-Helly* if, for every family $\mathcal{F}$ of maxcliques, if every two members of $\mathcal{F}$ have a vertex in common, then all the members of $\mathcal{F}$ have a vertex in common. A graph is *hereditary clique-Helly* if every induced subgraph is clique Helly. See [1, 5, 8, 9] for details. Reference [9] also proves that G is hereditary clique-Helly if and only if, for every maxclique Q of an induced subgraph H of G, at least one edge of Q is simplicial in H. The hereditary clique-Helly graphs are, of course, precisely the hereditary 2-clique-Helly graphs (and so are precisely the strong 2-Helly graphs).

Corollary 3. *The following are equivalent for every graph G:*

(3.0) *G is hereditary clique-Helly.*

(3.1) *For every $k \geq 2$ and every triangle xyz of G, if each edge xy, xz, and yz is strength-k in G, then triangle xyz is strength-k in G.*

(3.2) *For every triangle xyz of an induced subgraph H of G, if each edge xy, xz, and yz is strength-2 in H, then triangle xyz is strength-2 in H.*

(3.3) *If a triangle Δ is simplicial in an induced subgraph H of G, then at least one edge of Δ is simplicial in H.*

Proof. The equivalence of (3.0) and (3.1) restates [5, Theorem 2]. The equivalence of (3.1) and (3.2) is the $p = 3$ case of Theorem 1. Condition (3.3) simply restates (3.2). ∎

Sections 2 and 3 go in a different direction, generalizing Corollary 3 by replacing triangles with arbitrary cycles.

2. Edge Strength and Chordal Graphs

A *chord* of a cycle is an edge that joins two nonconsecutive vertices of the cycle (only cycles of length four or more can have chords). A graph is *chordal* if and only if every cycle of length four or more has a chord; see [1, 7] for thorough discussions. Define a graph to be *strength-k chordal* if every cycle of strength-k edges either has a strength-k chord or is a strength-k triangle. Being strength-1 chordal is equivalent to being chordal, and Corollary 6 will characterize being strength-k chordal for all $k \geq 1$.

The graph G_1 in Figure 1 is the smallest chordal graph that is not strength-2 chordal—the three edges between the vertices 2, 3, and 5 are each strength-2, but the triangle they form is simplicial in G. The graph G_2 is strength-2 chordal—the nine edges incident to vertices 3 or 4 are each strength-2 (indeed, the edge 34 is strength-4), as are the four triangles that contain edge 34—yet G_2 is not chordal because of the chordless cycle $1, 2, 6, 5, 1$. (The graph G_2 is vacuously strength-k chordal for all $k > 2$.)

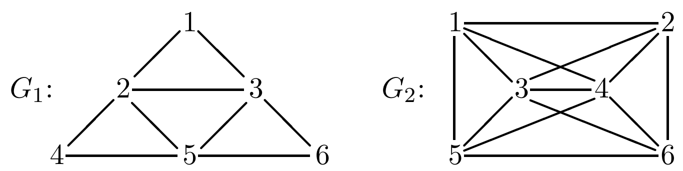

Figure 1. Graph G_1 is chordal, but not strength-2 chordal; G_2 is strength-k chordal for all $k \geq 2$, but not chordal.

As is common when working with cycle spaces, a *sum of cycles* will mean the symmetric difference of the edge sets of those cycles—in other words, an edge e is in the sum (denoted) $C_1 \oplus \cdots \oplus C_k$ if and only if e is in an odd number of the cycles $C_1, \ldots, C_k$. The notation $|C|$ will be used to denote the length of a cycle C, and C is a *k-cycle* if $|C| = k$. Lemma 4 will generalize the following simple fact from [4, Lemma 3.2] (also see [6, Corollary 1]): *A graph is chordal if and only if every cycle C is the sum of $|C| - 2$ triangles.*

Lemma 4. *A graph is strength-k chordal if and only if every cycle C of strength-k edges is the sum of $|C| - 2$ strength-k triangles.*

Proof. First suppose G is a chordal graph in which every cycle C of strength-k edges with $|C| = l \geq 3$ is the sum of $l - 2$ strength-k triangles

$\Delta_1, \ldots, \Delta_{l-2}$. If $l = 3$, then C itself is a strength-k triangle Δ_1. Suppose $l \geq 4$ [toward showing that C has a strength-k chord]. Because G is chordal, each edge of C must be in some triangle Δ_i. The pigeon-hole principle implies that some Δ_i must contain two (necessarily consecutive) edges of C. Then the third side of Δ_i is a chord of C. Since Δ_i is strength-k, that third side is a strength-k chord of C.

Conversely, suppose G is a strength-k chordal graph and C is a cycle of strength-k edges. Argue by induction on $|C| = l \geq 3$. If $l = 3$, then C is a strength-k triangle and so C is trivially the sum of $l - 2 = 1$ strength-k triangles. Now suppose $l \geq 4$. Since G is strength-k chordal, cycle C has a strength-k chord e. Then $C = C_a \oplus C_b$ where C_a and C_b are cycles of strength-k edges from $E(C_a) \cup E(C_b) \cup \{e\}$, with $\{e\} = C_a \cap C_b$, $|C_a| = a$, $|C_b| = b$, and $a + b = l + 2$. The induction hypothesis implies that C_a [respectively, C_b] is the sum of $a - 2$ [or $b - 2$] strength-k triangles. This makes C the sum of $(a - 2) + (b - 2) = l - 2$ strength-k triangles. ■

Theorem 5. *The following are equivalent for every graph G:*

(5.1) *G is strength-k chordal for all $k \geq 2$.*

(5.2) *Every induced subgraph of G is strength-2 chordal.*

(5.3) *Every cycle of nonsimplicial edges in an induced subgraph H of G either has a chord that is nonsimplicial in H or is a nonsimplicial triangle of H.*

(5.4) *Every cycle C of nonsimplicial edges in an induced subgraph H of G is the sum of $|C| - 2$ nonsimplicial triangles of H.*

Proof. (5.1) $\Rightarrow$ (5.2): Suppose G satisfies condition (5.1). Suppose H is any induced subgraph of G and C is a cycle of edges that are strength-2 in H, but C is not the sum of $|C| - 2$ triangles that are strength-2 in H [arguing by contradiction, using Lemma 4]; further suppose $|C|$ is minimum with respect to all that. By the minimality of $|C|$, every chord of C is simplicial in H. This implies that every triangle Δ with $V(\Delta) \subseteq V(C)$ is simplicial in H. Thus, for every edge e and triangle Δ, if $e \in E(C) \cap E(\Delta)$ and $V(\Delta) \subseteq V(C)$, then $\mathrm{str}_H(e) > \mathrm{str}_H(\Delta)$. But since every maxclique of G that contains such a Δ also contains such edges e, the same inequality holds with H replaced by G [contradicting (5.1), using Lemma 4 with $k = \min\{\mathrm{str}_G(e) : e \in E(C)\}$].

(5.1) $\Leftarrow$ (5.2): Suppose G satisfies condition (5.2), toward proving G is strength-k chordal by induction on $k \geq 2$. The $k = 2$ basis step is immediate.

For the inductive step, suppose G is strength-k chordal and C is a cycle of edges that are strength-$(k+1)$ in G, but C is not the sum of $|C|-2$ triangles that are strength-$(k+1)$ in G [arguing by contradiction, using Lemma 4]; further suppose $|C| = l \geq 3$ is minimal with respect to all that. By the minimality of l, cycle C has no chords that are strength-$(k+1)$ in G. Since G is strength-k chordal, C is the sum of triangles $\Delta_1, \ldots, \Delta_{l-2}$ of G that are strength-k in G, where each Δ_i is made from edges of C that are strength-$(k+1)$ in G together with chords e of C with $\mathrm{str}_G(e) = k$. Therefore if Δ_i and Δ_j share a chord of C, then $V(\Delta_i) \cup V(\Delta_j)$ must induce a complete subgraph Q_1 that is strength-k in G. Performing similar consolidations of complete subgraphs $n \leq l-3$ times partitions $\{\Delta_1, \ldots, \Delta_{l-2}\}$ into $l-2-n$ parts that are sets of contiguous triangles that are strength-k in G and whose vertices induce $l-2-n$ complete subgraphs that are strength-k in G and that cover $V(C)$. Performing this consolidation $n = l-3$ times shows that $V(C)$ induces a complete subgraph Q_n that is strength-k in G. Since C has no chords that are strength-$(k+1)$ in G, it follows that $\mathrm{str}_G(Q_n) = k$. Yet each $e \in E(C)$ is strength-$(k+1)$ in G and so is in a maxclique Q_e of G that has $E(Q_e) \cap E(Q_n) = \{e\}$ (again using that C has no chords that are strength-$(k+1)$ in G). But then $V(Q_n)$ together with one vertex from $V(Q_e) - V(Q_n)$ for each $e \in E(C)$ would induce a subgraph H of G such that each edge of C is strength-2 in H while $\mathrm{str}_H(Q_n) = 1$ and each chord e of C has $\mathrm{str}_H(e) = 1$ [contradicting (5.2)].

(5.2) $\Leftrightarrow$ (5.3) $\Leftrightarrow$ (5.4) follows since (5.3) and (5.4) simply restate (5.2) (using Lemma 4). ∎

For $k \geq 3$, a *k-sun*—sometimes called a *complete k-sun* or *trampoline*, see [1, 3, 5, 7]—is a graph that consists of an even-length cycle $v_1, \ldots, v_{2k}, v_1$, together with all of the $\binom{k}{2}$ chords between even-subscripted vertices. (The graph G_1 in Figure 1 is a 3-sun, and the subgraph H constructed in the (5.1) $\Leftarrow$ (5.2) proof of Theorem 5 is an l-sun.) A graph is *strongly chordal* if it is chordal and no induced subgraph is isomorphic to any k-sun; see [1, 3, 5, 7] for other characterizations of this widely-studied concept.

Corollary 6. *The following are equivalent for every graph G:*

(6.0) *G is strongly chordal.*

(6.1) *G is strength-k chordal for all $k \geq 1$.*

(6.2) *G is chordal and every induced subgraph of G is strength-2 chordal.*

(6.3) *G is chordal and every cycle of nonsimplicial edges in an induced subgraph H of G either has a chord that is nonsimplicial in H or is a nonsimplicial triangle of H.*

(6.4) *G is chordal and every cycle C of nonsimplicial edges in an induced subgraph H of G is the sum of $|C| - 2$ nonsimplicial triangles of H.*

Proof. The equivalence of (6.0) and (6.1) restates [5, Theorem 1]. The equivalence of conditions (6.i) and (6.j) when $1 \leq i < j \leq 4$ follows immediately from the equivalence of conditions (5.i) and (5.j). ∎

3. Vertex Strength and Chordal Graphs

Recognizing that cycles are determined by their vertices just as well as by their edges, define a graph to be *vertex strength-k chordal* if every cycle of strength-k vertices either has a strength-k chord or is a strength-k triangle. (Strength-k chordal graphs could have been called 'edge strength-k chordal' graphs.) Being vertex strength-1 chordal is equivalent to being chordal. Clearly, every cycle of strength-k edges is a cycle of strength-k vertices, and so every vertex strength-k chordal graph is strength-k chordal. The three graphs in Figure 2 are strength-2 chordal but not vertex strength-2 chordal (because each vertex shown as 'hollow' is a strength-2 vertex).

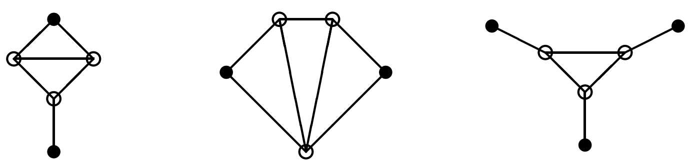

Figure 2. From left to right, the *kite*, *gem*, and *net* graphs.

Lemma 7. *A graph is vertex strength-k chordal if and only if every cycle C of strength-k vertices is the sum of $|C| - 2$ strength-k triangles.*

Proof. This is proved by a straightforward modification of the proof of Lemma 4 (observing that every strength-k edge has strength-k endpoints). ∎

Theorem 8. *The following are equivalent for every graph G:*

(8.0) *G is $\{$kite, gem, net$\}$-free strongly chordal.*

(8.1) *G is vertex strength-k chordal for all $k \geq 2$.*

(8.2) *Every induced subgraph of G is vertex strength-2 chordal.*

(8.3) *Every cycle of nonsimplicial vertices in an induced subgraph H of G either has a chord that is nonsimplicial in H or is a nonsimplicial triangle of H.*

(8.4) *Every cycle C of nonsimplicial vertices in an induced subgraph H of G is the sum of $|C| - 2$ nonsimplicial triangles of H.*

Proof. (8.0) $\Rightarrow$ (8.1): Suppose $k \geq 2$ and G satisfies condition (8.0), and C is a cycle of strength-k vertices. Since G is chordal, C is the sum of $|C| - 2$ triangles. Suppose any of those triangles—say triangle $v_1v_2v_3$—has $\mathrm{str}_G(v_1v_2v_3) < k$ [arguing by contradiction, using Lemma 7, showing that G would contain an induced kite, gem, net, or 3-sun]. Then for each $i \in \{1, 2, 3\}$, there exists a vertex w_i with each $w_i \sim v_i$ and $w_i \notin \{v_1, v_2, v_3\}$ and w_i not adjacent to some v_j. Thus $|\{w_1, w_2, w_3\}| > 1$. Let H be the subgraph of G that is induced by $\{v_1, v_2, v_3, w_1, w_2, w_3\}$. If $|\{w_1, w_2, w_3\}| = 2$, then H is an induced kite or gem [a contradiction]. If $|\{w_1, w_2, w_3\}| = 3$, then either H is an induced net or 3-sun or H contains an induced kite or gem [a contradiction].

(8.1) $\Rightarrow$ (8.2): Suppose G satisfies condition (8.1). Suppose H is any induced subgraph of G and C is a cycle of vertices that are strength-2 in H, but C is not the sum of $|C| - 2$ triangles that are strength-2 in H [arguing by contradiction, using Lemma 7]; further suppose $|C|$ is minimum with respect to all that. By the minimality of $|C|$, the cycle C is chordless and so (since vertex strength-k chordal implies chordal) C is a triangle Δ where $\mathrm{str}_H(\Delta) = 1$. Thus $\mathrm{str}_H(v) > \mathrm{str}_H(\Delta)$ for every $v \in V(\Delta)$. But since every maxclique of G that contains Δ also contains every $v \in V(\Delta)$, it follows that $\mathrm{str}_G(v) > \mathrm{str}_G(\Delta)$ [contradicting (8.1) with $k = \min\{\mathrm{str}_G(v) : v \in V(\Delta)\}$].

(8.0) $\Leftarrow$ (8.2): Suppose G satisfies condition (8.2). Then no induced subgraph H of G can be isomorphic to C_k with $k \geq 4$ (a chordless cycle of vertices that are strength-2 in G), or to a k-sun, kite, gem, or net graph (each containing a triangle Δ of vertices that are strength-2 in G while $\mathrm{str}_G(\Delta) = 1$). Thus (8.0) holds.

(8.2) $\Leftrightarrow$ (8.3) $\Leftrightarrow$ (8.4) follows since (8.3) and (8.4) simply restate (8.2). ∎

If a graph G is vertex strength-k chordal for all $k \geq 2$, then G is strongly chordal and so is certainly vertex strength-1 chordal. Therefore, the $k \geq 2$

restriction in condition (8.1) could just as well be replaced with $k \geq 1$, and no 'Corollary 9' is needed to parallel Corollary 6.

References

[1] A. Brandstädt, V.B. Le and J.P. Spinrad, Graph Classes: A Survey, Society for Industrial and Applied Mathematics (Philadelphia, 1999).

[2] M.C. Dourado, F. Protti and J.L. Szwarcfiter, *On the strong p-Helly property*, Discrete Appl. Math. **156** (2008) 1053–1057.

[3] M. Farber, *Characterizations of strongly chordal graphs*, Discrete Math. **43** (1983) 173–189.

[4] R.E. Jamison, *On the null-homotopy of bridged graphs*, European J. Combin. **8** (1987) 421–428.

[5] T.A. McKee, *A new characterization of strongly chordal graphs*, Discrete Math. **205** (1999) 245–247.

[6] T.A. McKee, *Requiring chords in cycles*, Discrete Math. **297** (2005) 182–189.

[7] T.A. McKee and F.R. McMorris, Topics in Intersection Graph Theory, Society for Industrial and Applied Mathematics (Philadelphia, 1999).

[8] E. Prisner, *Hereditary clique-Helly graphs*, J. Combin. Math. Combin. Comput. **14** (1993) 216–220.

[9] W.D. Wallis and G.-H. Zhang, *On maximal clique irreducible graphs*, J. Combin. Math. Combin. Comput. **8** (1993) 187–193.

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