*Discussiones Mathematicae*
*Graph Theory 31 (2011) 559–575*

# UNIQUE FACTORIZATION THEOREM FOR OBJECT-SYSTEMS

Peter Mihók

*Department of Applied Mathematics*
*Faculty of Economics, Technical University*
*B. Němcovej, 040 01 Košice, Slovak Republic*
*and*
*Mathematical Institute, Slovak Academy of Science*
*Grešákova 6, 040 01 Košice, Slovak Republic*

**e-mail:** peter.mihok@tuke.sk

and

Gabriel Semanišin

*Institute of Computer Science*
*P.J. Šafárik University, Faculty of Science*
*Jesenná 5, 041 54 Košice, Slovak Republic*

**e-mail:** gabriel.semanisin@upjs.sk

**Abstract**

The concept of an object-system is a common generalization of simple graph, digraph and hypergraph. In the theory of generalised colourings of graphs, the Unique Factorization Theorem (UFT) for additive induced-hereditary properties of graphs provides an analogy of the well-known Fundamental Theorem of Arithmetics. The purpose of this paper is to present UFT for object-systems. This result generalises known UFT for additive induced-hereditary and hereditary properties of graphs and digraphs. Formal Concept Analysis is applied in the proof.

Research supported in part by the Slovak research grants VEGA 1/0131/09 and 2/0194/10, by Slovak APVV grant APVV-0007-07 and by the Agency of the Slovak Ministry of Education for the Structural Funds of the EU, under project ITMS:26220120007.

**Keywords:** object-system, unique factorization, graph, hypergraph, formal concept analysis.

**2010 Mathematics Subject Classification:** 05C15, 05C75.

## 1. Introduction and Motivation

In the previous years a few results related to a factorisation of properties of various combinatorial structures were presented (see [8, 9, 10, 11, 15, 18, 20, 22, 27]). The methods that were developed have many common features. We try to extract them and present a common generalisation of these results. Since the original motivation for our study comes from graph theory, we start with introducing the concepts from this area.

Let $\mathcal{I}$ denotes the class of all finite simple graphs. A *graph property* $\mathcal{P}$ is any nonempty proper isomorphism closed subclass of $\mathcal{I}$. A graph property is said to be *induced-hereditary* if it is closed under taking induced subgraphs and *additive* if it is closed under taking disjoint union of graphs (see [2, 3]). Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be graph properties, a vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$*-partition* of a graph $G$ is a partition $\{V_1, V_2, \ldots, V_n\}$ of $V(G)$ such that each partition class $V_i$ induces a subgraph $G[V_i]$ of property $\mathcal{P}_i$, $i = 1, 2, \ldots, n$.

A graph $G$ is said to be $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$*-partitionable* if $G$ has a vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition. Let us denote by $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ the set of all vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graphs. A property $\mathcal{P}$ is said to be *reducible* if there exist properties $\mathcal{P}_1, \mathcal{P}_2$ such that $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$. Otherwise $\mathcal{P}$ is called *irreducible*. For example, the property $\mathcal{O}$ - "to be an edgeless graph", related to regular proper colouring, is irreducible and the smallest additive induced-hereditary reducible property is the class $\mathcal{O}^2$ of all bipartite graphs. The notion of reducible property has been introduced in [19]. The problem of unique factorization of a reducible induced-hereditary graph property into induced-hereditary factors was introduced in connection with the study of reducible graph properties (see [2, 3, 8, 10, 18, 22] and Problem 17.9. in the [17]). In [22] it is showed that every reducible additive hereditary property of finite graphs is uniquely factorisable into irreducible additive hereditary factors. An analogous result was obtained in [10, 20] for additive induced-hereditary properties of finite graphs. Following [3] let us denote by $\mathbb{M}^a$ the set of all additive induced-hereditary properties of finite graphs. Then UFT can be stated as follows.

**Theorem 1** (Unique Factorization Theorem—UFT, [10, 20]). *Every additive induced-hereditary property of finite graphs is in* $\mathbb{M}^a$ *uniquely factorisable into a finite number of irreducible additive induced-hereditary properties, up to the order of factors.*

The extension of the Unique Factorization Theorem for digraphs have been presented in [27]. Theorem 1 has several deep applications related to the existence of uniquely partitionable graphs (see [4, 5]) and consequently the complexity of generalised colourings. A. Farrugia proved in [9] that if $\mathcal{P}$ and $\mathcal{Q}$ are additive induced-hereditary graph properties, then $(\mathcal{P}, \mathcal{Q})$-colouring is NP-hard, with the sole exception of graph 2-vertex colouring (the case where both $\mathcal{P}$ and $\mathcal{Q}$ are the set $\mathcal{O}$ of finite edgeless graphs). Moreover, $(\mathcal{P}, \mathcal{Q})$-colouring is NP-complete if and only if $\mathcal{P}$- and $\mathcal{Q}$-recognition are both in NP. These results show that reducible additive induced-hereditary properties are rather complicated mathematical structures.

In [6, 21] a general concept of *object-systems* have been introduced and some results on the structure of additive hereditary properties of object-systems and uniquely partitionable objects systems have been presented. The concept of an object-system is a common generalization of simple graphs, digraphs and hypergraphs. One can observe that the proofs of UFT for graphs and digraphs are technically quite complicated. On the other hand, it is not difficult to notice, that they follow the same idea and they are rather independent from the considered underlying combinatorial structures - graphs and digraphs. The aim of this paper is to generalise used methods and constructions for systems of objects and show that UFT is valid also for such general mathematical structures. Moreover we provide an application of Formal Concept Analysis (FCA) which allows us to eliminate some technical difficulties in the previous proofs. This paper is an extended version of [23] that shows an application of FCA in the simpler situation — the domain of graphs.

## 2. OBJECT-SYSTEMS OVER A CONCRETE CATEGORY

We shall use the basic elementary notions of category theory following e.g. [24]. A concrete category **C** is a collection of *objects* and *arrows* called *morphisms*, where an object is *a set with structure*. We shall denote the *ground-set* of the object $A$ by $V(A)$. In our investigation we will always assume that the objects and their ground-sets are finite and each object has at least two elements. The morphism between two objects is considered to be a *structure preserving mapping.* The examples of concrete categories are e.g. **FinSet** of finite sets, **Graph** of finite graphs, **FinPoset** of finite partially ordered sets, with structure preserving mappings, i.e., the homomorphisms of corresponding structures. We will consider concrete categories **C** with *isomorphisms* i.e., structure preserving bijections between the ground-sets of objects only. To avoid formal difficulties, in what follows we will assume that the elements of the finite ground-set $V(A)$ of each object $A$ of **C** are labeled by labels taken from a given countable set $\mathcal{V}$ (e.g. $\mathbb{N}$) of cardinality $\aleph_0$. Hence, any considered concrete category **C** will be *small* (a category where objects and arrows are sets, see [24]) and the objects of **C** are "labeled" objects. We also assume, that for each object $A$ of **C** a relabeling of the elements of $V(A)$ by labels from a set $V^* \subset \mathcal{V}$ yields to a "new" object $A^*$ with $V(A^*) = V^*$ belonging to **C** and isomorphic to $A$. This requirement is quite natural and it is satisfied e.g. if the concrete category **C** is one of the above mentioned categories. Let us call categories which satisfies these requierements *wide.*

Let **C** be a wide category. A *simple finite (countable) object-system* over category **C** is an ordered pair $S = (V, E)$, where $V = V(S)$ is a finite (countable) set (of vertices) and $E = E(S) = \{A_1, A_2, \ldots, A_m\}$ is a finite (countable) set of objects of **C**, called the objects of the object-system $S$. The ground-set $V(A_i)$ of each object $A_i \in E$ is a finite set with at least two elements (i.e., there are no loops) and $V \supseteq \bigcup_{i=1}^{m} V(A_i)$. The symbols $K_0$ and $K_1$ denotes the *null* system $K_0 = (\emptyset, \emptyset)$ and the system $K_1 = (\{v\}; \emptyset)$ consisting of one isolated element, respectively.

For example, graphs can be viewed as object-systems over the concrete category of two-element sets with bijections as arrows; digraphs are object-systems over the category of ordered pairs; hypergraphs are finite set systems, etc. Let us remark, that the relational $L$-structures, introduced by Fraïssé in [12] (see also [1, 13, 25]), that generalises graphs, digraphs and $k$-uniform hypergraphs are object-systems over a category of relations given by the signature $L$.

To generalise the results on generalised colourings of graphs to arbitrary simple object-systems we need to define *isomorphism of systems, disjoint union of systems* and *induced-subsystems*, respectively. We can do this in a natural way:

Let $S_1 = (V_1, E_1)$ and $S_2 = (V_2, E_2)$ be two simple object-systems over a given concrete category $\mathbf{C}$. The object-system $S_1$ is a subobject-system of $S_2$ if $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$. The systems $S_1$ and $S_2$ are said to be *isomorphic* if there is a pair of mappings:

$$\phi : V_1 \longleftrightarrow V_2; \qquad\qquad \psi : E_1 \longleftrightarrow E_2,$$

such that if $\psi(A_{1i}) = A_{2j}$ then $\phi/V(A_{1i}) : V(A_{1i}) \longleftrightarrow V(A_{2j})$ is an isomorphism of the objects $A_{1i} \in E_1$ and $A_{2j} \in E_2$ in the category $\mathbf{C}$. The *homomorphism* of the systems can be defined in a similar way as object-preserving mappings. The *disjoint union* of the systems $S_1$ and $S_2$ is the system $S_1 \cup S_2 = (V_1 \cup V_2, E_1 \cup E_2)$, where we assume that $V_1 \cap V_2 = \emptyset$. A system is said to be *connected* if it cannot be expressed as a disjoint union of two systems. By $\mathcal{I}^\omega(\mathbf{C})$, $\mathcal{I}(\mathbf{C})$ and $\mathcal{I}^{conn}(\mathbf{C})$ we denote the class simple countable object-systems, simple finite object-systems and simple finite connected object-systems over $\mathbf{C}$, respectively.

The *subobject-system* $S[U]$ of the object-system $S = (V, E)$ *induced by the set* $U \subseteq V$ is the system $S[U] = (U, E[U])$ with objects $E([U]) := \{A_{1i} \in E | V(A_{1i}) \subseteq U\}$. $S_1$ is an induced-subsystem of $S$ if it is isomorphic to $S[U]$ for some $U \subseteq V$.

Using these definitions we can define, analogously as for graphs, that an *additive induced-hereditary property of simple object-systems over a category* $\mathbf{C}$ is any class of systems closed under isomorphism, induced-subsystems and disjoint union of systems. Thus a property $\mathcal{P}$ of object-systems is any isomorphism-closed nonempty subclass of $\mathcal{I}^\omega(\mathbf{C})$. It means that while investigating properties, in principle, we can restrict our considerations to unlabeled systems. Let us denote by $\mathbb{M}^a(\mathbf{C})$ the set of all additive induced-hereditary properties of simple finite object-systems over a category $\mathbf{C}$. In the following we shall call simple object-systems over a category $\mathbf{C}$ shortly *object-systems.*

In [21] it is proved, that the set $\mathbb{M}^a(\mathbf{C})$ of all additive induced-hereditary properties of simple finite object-systems over $\mathbf{C}$ partially ordered by set inclusion forms a complete and distributive lattice and the structure of the lattice $\mathbb{M}^a(\mathbf{C})$ is investigated. For a positive integer $k$ and a system $S$, the notation $k.S$ is used for the union of $k$ vertex disjoint copies of $S$. The *join* of object-systems $S, H$ is the object-system $S + H$ obtained from the disjoint union $S$ and $H$ by adding all possible objects of $\mathbf{C}$ having vertices belonging to the union of $V(S)$ and $V(H)$.

## 3. Compactness

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be given properties of object-systems. A vertex $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$-*colouring* (*partition*) of an object-system $S = (V, E)$ is a partition $(V_1, V_2, \ldots, V_n)$ of $V$ (every pair of $V_i$'s has empty intersection and the union of $V_i$'s forms $V$) such that each colour class $V_i$ induces a subobject-system $S[V_i]$ having property $\mathcal{P}_i$. For convenience, we allow empty partition classes in the partition sequence. An empty class induces the null object-system $K_0 = (\emptyset, \emptyset)$. If each of the $\mathcal{P}_i$'s, $i = 1, 2, \ldots, n$, is the property $\mathcal{O}$ of being object-less (i.e., $\mathcal{P} = \{S = (V, E) | E = \emptyset)$, we have the well-known proper vertex $n$-colouring. An object-system $S$ which have a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colouring is called $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-*colourable*, and in such a situation we say that $S$ has property $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$.

In 1951, de Bruijn and Erdős proved that an infinite graph $G$ is $k$-colourable if and only if every finite subgraph of $G$ is $k$-colourable. An analogous compactness theorem for generalised colourings of graphs was proved in [7]. The key concept for the Vertex Colouring Compactness Theorem of [7] is that of a property of being of *finite character*. Let $\mathcal{P}$ be an property of object-systems, $\mathcal{P}$ is of *finite character* if an object-system in $\mathcal{I}^{\omega}(\mathbf{C})$ has property $\mathcal{P}$ if and only if each its finite induced subobject-system has property $\mathcal{P}$. It is easy to see that if $\mathcal{P}$ is of finite character and an object-system has property $\mathcal{P}$ then so does every induced subobject-system and thus properties of finite character are induced-hereditary. However not all induced-hereditary properties are of finite character. Let us also remark that every property which is hereditary with respect to every subobject-system (we say simply *hereditary*) is induced-hereditary as well. The compactness theorem for $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colourings of graphs, where the $\mathcal{P}_i$'s are of finite character, have been proved using Rado's Selection Lemma in [7]. Using Theorem 3.1 of [7], on can easily see that the Vertex Colouring Compactness Theorem (VCCT) can be formulated also in the following form:

**Theorem 2** (Vertex Colouring Compactness Theorem)**.** *Let $S$ be an arbitrary object-system in $\mathcal{I}^{\omega}(\mathbf{C})$ and let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be properties of object-systems of finite character. Then $S$ is $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colourable if and only if every finite induced subobject-system of $S$ is $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colourable.*

Let us denote by $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$, $n \geq 2$ the set of all $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colourable object-systems. The binary operation $\circ$ is obviously commutative, associative on the class of object-system properties and $\Theta = \{K_0\}$ is its neutral element. The properties $\Theta$, $\mathcal{I}(\mathbf{C})$ and $\mathcal{I}^{\omega}(\mathbf{C})$ are said to be trivial. A nontrivial property of object-systems $\mathcal{P}$ is said to be *reducible* if there exist nontrivial object-system properties $\mathcal{P}_1, \mathcal{P}_2$, such that $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$; otherwise $\mathcal{P}$ is called *irreducible*. In what follows each property is considered to be nontrivial. Let us remark, that using Theorem 2 one can prove UFT for the class $\mathbb{M}^{\omega a}$ of the additive properties of countable graphs of finite character (see [15]).

## 4. Hereditary Properties in the Language of FCA

As we already mentioned, it is quite easy to prove that the sets $\mathbb{M}^a$,$\mathbb{M}^{\omega a}$, and $\mathbb{M}^a(\mathbf{C})$ of all additive induced-hereditary properties of finite graphs, graph properties of finite character and properties of object-systems over $\mathbf{C}$, partially ordered by set inclusion, form complete distributive lattices. The lattices of hereditary properties have been studied quite extensively, references may be found in [3, 16, 21]. In this section we shall present a new approach — Formal Concept Analysis — to study the structure of the lattice of additive induced-hereditary properties of an object-system.

Formal Concept Analysis (FCA) is a theory of data analysis which identifies conceptual structures among data sets. It was introduced by R. Wille in 1982 and since then has grown rapidly (for a comprehensive overview see [14]). The mathematical lattices that are used in FCA can be interpreted as classification systems. Formalised classification systems can be analysed according to the consistency of their relations. We shall use FCA in the proof of the Unique Factorization Theorem for reducible properties of object-systems. In order to proceed we need to introduce some concepts of FCA according to the book [14].

**Definition 1.** A **formal context** $\mathbb{K} := (O, M, I)$ consists of two sets $O$ and $M$ and a relation $I$ between $O$ and $M$. The elements of $O$ are called the **objects** and the elements of $M$ are called the **attributes** of the formal context.

For a set $A \subseteq O$ of objects we define

$$A' := \{m \in M | gIm \text{ for all } g \in A\}.$$

Analogously, for a set $B$ of attributes we define

$$B' := \{g \in O | gIm \text{ for all } m \in B\}.$$

A **formal concept** of the formal context $(O, M, I)$ is a pair $(A, B)$ with $A \subseteq O, B \subseteq M, A' = B$ and $B' = A$.

We call $A$ the **extent** and $B$ the **intent** of a formal concept $(A, B)$. $\mathbb{L}(O, M, I)$ denotes the set of all formal concepts of the formal context $(O, M, I)$.

If $(A_1, B_1)$ and $(A_2, B_2)$ are formal concepts of a formal context and $A_1 \subseteq A_2$ (which is equivalent to $B_2 \subseteq B_1$), we write $(A_1, B_1) \leq (A_2, B_2)$.

For an object $g \in O$ we write $\gamma g$ for the **object concept** $(g'', g')$, where $g'' = \{\{g\}'\}'$.

Let us mention that, according to the Basic Theorem on Concept Lattices, the set $\mathbb{L}(O, M, I)$ of all formal concepts of the formal context $(O, M, I)$ partially ordered by the relation $\leq$ (see Definition 1) is a complete lattice.

Let us present additive induced-hereditary object-system properties as formal concepts in a given formal context. Using FCA we can proceed in the following way: Let us define a formal context $(O, M, I)$ by setting objects to countable simple object-systems, i.e. $O = \mathcal{I}^{\omega}(\mathbf{C})$. For each connected finite simple object-system $F \in \mathcal{I}^{conn}(\mathbf{C})$ let us consider an attribute $m_F$: "do not contain an induced subobject-system isomorphic to $F$". Thus $SIm_F$ means that the object-system $S$ does not contain any induced subsystem isomorphic to $F$. We can immediately observe the following:

**Lemma 1.** *Let $O = \mathcal{I}^{\omega}(\mathbf{C})$ and $M = \{m_F, F \in \mathcal{I}^{conn}(\mathbf{C})\}$. Then the formal concepts of the formal context $\mathbb{K} = (O, M, I)$ are additive induced-hereditary properties of finite character and the concept lattice $(\mathbb{L}(O, M, I), \leq)$ is isomorphic to the lattice $(\mathbb{M}^{\omega a}(\mathbf{C}), \subseteq)$. Moreover, for each formal concept $\mathcal{P} = (A, B)$ there is an object — a countable object-system $S \in O$ such that $\mathcal{P} = \gamma S = (S'', S')$.*

***Proof.*** It is easy to verify that the extent of any formal concept $(A, B)$ of $\mathbb{K}$ forms an additive induced-hereditary property $\mathcal{P} = A$ of finite character. Obviously, each countable object-system $S = (V, E)$ in the formal context $\mathbb{K}$ leads to an "object concept" $\gamma S = (S'', S')$. On the other hand, from the additivity it follows that the disjoint union of all finite object-systems having a given additive induced-hereditary property $\mathcal{P} \in \mathbb{M}^{\omega a}(\mathbf{C})$ is a countable infinite object-system $K$ satisfying $\gamma K = (\mathcal{P}, \mathcal{I}^{conn} \setminus \mathcal{P})$. ■

In order to characterise additive induced-hereditary properties of finite graphs, mainly two different approaches were used: a characterization by generating sets and/or by minimal forbidden subgraphs (see [3] and [11]). While the extent $A$ of a formal concept $(A, B) \in \mathbb{L}(O, M, I)$ is related to a property of object-systems $\mathcal{P}$, the intent $B$ consists of forbidden connected subobject-systems of $\mathcal{P}$. The set $\boldsymbol{F}(\mathcal{P})$ of *minimal forbidden induced subobject-systems* for $\mathcal{P}$ consists of minimal elements of the poset $(B, \subseteq)$ of $B$ partially ordered by the partial order "'to be an induced subobject-system"'. For a given countable object-system $S \in \mathcal{I}^\omega$ let us denote by $age(S)$ the class of all finite object-systems isomorphic to finite induced subobject-system of $S$ (see e.g. [25]). Scheinerman in [26] showed, that for each additive induced-hereditary property $\mathcal{P}$ of finite graphs, there is an infinite countable graph $G$ such that $\mathcal{P} = age(G)$. This result is related to Lemma 1. On the other hand, it is worth to mention that $\gamma S = (\mathcal{P}, S')$ does not imply, in general, that $\mathcal{P} = age(S)$. Let us define a binary relation $\cong$ on $\mathcal{I}^\omega(\mathbf{C})$ by $S_1 \cong S_2$ whenever $\gamma S_1 = \gamma S_2$ in the formal context $\mathbb{K}$, and we say that $S_1$ is *congruent* with $S_2$ with respect to $\mathbb{K}$. Obviously, $\cong$ is an equivalence relation on $\mathcal{I}^\omega(\mathbf{C})$. The aim of the next section is to find appropriate representatives of congruence classes and to describe their properties.

## 5. UNIQUELY DECOMPOSABLE OBJECT-SYSTEMS

Let $\mathcal{R}$ be a reducible additive induced-hereditary property of object-systems. All the previous proofs of UFT are based on a construction of finite *uniquely $\mathcal{R}$-decomposable* object-systems that are defined as follows:

**Definition 2.** For given finite object-systems $S_1, S_2, \ldots, S_n$, $n \geq 2$, denote by $S_1 * S_2 * \cdots * S_n$ the set of object-systems

$$\left\{ H \in \mathcal{I}^\omega(\mathbf{C}) : \bigcup_{i=1}^{n} S_i \subseteq H \subseteq \sum_{i=1}^{n} S_i \right\},$$

where $\bigcup_{i=1}^n S_i$ denotes the disjoint union and $\sum_{i=1}^n S_i$ the join of the object-systems $S_1, S_2, \ldots, S_n$, respectively. For an object-system $S$, $k \geq 2$, $k \otimes S$ stands for the class $S * S * \cdots * S$, with $k$ copies of $S$.

Let $S$ be an object-system and $\mathcal{R}$ be an additive induced-hereditary property of object-systems. Then we define the $\mathcal{R}$-decomposability number of $S$ $dec_{\mathcal{R}}(S) = \max\{n :$ there exists a partition $\{V_1, V_2, \ldots, V_n\}$, $V_i \neq \emptyset$, of $V(S)$ such that for each $k \geq 1$, $k.S[V_1] * k.S[V_2] * \cdots * k.S[V_n] \subseteq \mathcal{R}\}$. We shall call such partition $\{V_1, V_2, \ldots, V_n\}$ **$\mathcal{R}$-decomposition** of $S$. If $S \notin \mathcal{R}$ we set $dec_{\mathcal{R}}(S)$ to zero.

An object-system $S$ is said to be **$\mathcal{R}$-decomposable** if $dec_{\mathcal{R}}(S) \geq 2$; otherwise $S$ is $\mathcal{R}$-indecomposable.

An object-system $S \in \mathcal{P}$ is called **$\mathcal{P}$-strict** if $S * K_1 \not\subseteq \mathcal{P}$. The class of all $\mathcal{P}$-strict object-systems is denoted by $\mathrm{S}(\mathcal{P})$. Put $dec(\mathcal{R}) = \min\{dec_{\mathcal{R}}(S) : S \in \mathrm{S}(\mathcal{R})\}$.

An $\mathcal{R}$-strict object-system $S$ with $dec_{\mathcal{R}}(S) = dec(\mathcal{R}) = n \geq 2$ is said to be **uniquely $\mathcal{R}$-decomposable** if there is exactly one $\mathcal{R}$-partition $\{V_1, V_2, \ldots, V_n\}$, $V_i \neq \emptyset$, such that for each $k \geq 1$, $k.S[V_1] * k.S[V_2] * \cdots * k.S[V_n] \subseteq \mathcal{R}$. We call the object-systems $S[V_1], S[V_2], \ldots, S[V_n]$ **ind-parts** of the uniquely decomposable object-system $S$.

These notions are motivated by the following observation: Let us suppose that $S \in \mathcal{R} = \mathcal{P} \circ \mathcal{Q}$ and let $(V_1, V_2)$ be a $(\mathcal{P}, \mathcal{Q})$-partition of $S$. Then by additivity of $\mathcal{P}$ and $\mathcal{Q}$ we have that $k.S[V_1] * k.S[V_2] \subseteq \mathcal{R}$ for every positive integer $k$. Thus, if the property $\mathcal{R}$ is reducible, every object-system $S \in \mathcal{R}$ with at least two vertices is $\mathcal{R}$-decomposable.

In [15, 20] we proved that for any reducible additive induced-hereditary graph property the converse assertion holds, too. We shall prove that this is valid also for object-systems, i.e., that an induced-hereditary additive property $\mathcal{R}$ is reducible if and only if all object-systems in $\mathcal{R}$ with at least two vertices are $\mathcal{R}$-decomposable.

Let us show that every object-system $S \in \mathcal{P}$ is an induced subobject-system of a $\mathcal{P}$-strict object-system. Obviously for any nontrivial property there exists an object-system $F \notin \mathcal{P}$. For an induced-hereditary property $\mathcal{P}$ we can therefore define $f(\mathcal{P})$ to be the least number of vertices of a forbidden subobject-system of $\mathcal{P}$, i.e., $f(\mathcal{P}) = \min\{|V(F)| : F \notin \mathcal{P}\}$. Now it is easy to see, that for every $S \in \mathcal{P}$ the class $S * K_1 * \cdots * K_1 \not\subseteq \mathcal{P}$ if the number of the $K_1$'s is $f(\mathcal{P}) - 1$ which means that if $S$ is not $\mathcal{P}$-strict, then repeating the operation $*$ with $K_1$'s after less than $f(\mathcal{P})$ steps we will obtain a $\mathcal{P}$-strict object-system $S'$ such that $S \leq S'$.

To present our main result we need some notions introduced in [10]:

**Definition 3.** Let $d_0 = \{U_1, U_2, \ldots, U_m\}$ be an $\mathcal{R}$-decomposition of an object-system $S \in \mathcal{R}$. An $\mathcal{R}$-decomposition $d_1 = \{V_1, V_2, \ldots, V_n\}, n \geq m$ of $S$ **respects** $d_0$ if no $V_i$ intersects two or more $U_j$'s; that is each $V_i$ is contained in some $U_j$. We say that the object-system $S^* \in s \otimes S$ **respects** $d_0$

if $S^* \in s.S[U_1] * s.S[U_2] * \cdots * s.S[U_m]$. For an object-system $S^* \in s \otimes S$, denote the copies of $S$ by $S^1, S^2, \ldots, S^s$. Then we say that an $\mathcal{R}$-decomposition $d = \{V_1, V_2, \ldots, V_n\}$ of $S^*$ **respects** $d_0$ **uniformly** whenever for each $i$ there is a $j$ such that for every $k$ we have $V_i \cap V(S^k) \subseteq U_j$. The decomposition of $S^k$ induced by $d$ is denoted $d|S^k$.

If $S$ is uniquely $\mathcal{R}$-decomposable, the vertices of its ind-parts *respect* $d_0$ if its unique $\mathcal{R}$-decomposition with $dec_{\mathcal{R}}(S)$ parts respects $d_0$. If $S^*$ is uniquely $\mathcal{R}$-decomposable, its ind-parts *respect* $d_0$ *uniformly* if:

(a) for some positive integer $s$, $S^* \in s \otimes S$;

(b) $S^*$ respects $d_0$; and

(c) $S^*$'s unique $\mathcal{R}$-decomposition with $dec_{\mathcal{R}}(S^*)$ parts respects $d_0$ uniformly.

The *extension of* $d_0$ to $S^*$ is the decomposition obtained by repeating $d_0$ on each copy of $S$. If $S^*$ respects $d_0$, or if it has an $\mathcal{R}$-decomposition that respects $d_0$ uniformly, then the extension of $d_0$ is also an $\mathcal{R}$-decomposition of $S^*$. In particular, $S^*$ is an object-system in $\mathcal{R}$.

We shall sometimes write $S^i \cap U_x$ (or just $U_x$ when it is clear that we are referring to $S^i$) to mean the vertices of $S^i$ that correspond to $U_x$, and $S^* \cap U_x$ (or just $S_x$, when it is clear from the formal context) to mean $S^*[\bigcup_i (S^i \cap U_x)]$.

Based on the construction given in [20] and a refinement given in [10] we can prove:

**Theorem 3.** *Let $S$ be an $\mathcal{R}$-strict object-system with $dec_{\mathcal{R}}(S) = dec(\mathcal{R}) = n \geq 2$ and let $d_0 = \{U_1, U_2, \ldots, U_m\}$ be a fixed $\mathcal{R}$-decomposition of $S$. Then there is a uniquely $\mathcal{R}$-decomposable finite object-system $S^* \in s \otimes S$, for some positive integer $s$, that respects $d_0$, and moreover any $\mathcal{R}$-decomposition of $S^*$ with $n$ parts respects $d_0$ uniformly.*

***Proof.*** Let $d_i = (V_{i,1}, V_{i,2}, \ldots, V_{i,n})$, $i = 1, \ldots, r$, be the $\mathcal{R}$-decompositions of $S$ with $n$ parts which do not respect $d_0$. Since $S$ is a finite object-system, $r$ is a nonnegative integer. If $r = 0$, take $S^* = S$; otherwise we shall construct an object-system $S^* = S^*(r) \in s \otimes S$ as below, denoting the $s$ copies of $S$ by $S^1, \ldots, S^s$.

If the resulting $S^*$ has an $\mathcal{R}$-decomposition $d$ with $n$ parts, then, since $S$ is $\mathcal{R}$-strict, $d|S^i$ will also have $n$ parts. The aim of the construction is to add new objects $E^* = E^*(r)$ to $sS$ to exclude the possibility that $d|S^i = d_j$, for any $1 \leq i \leq s$, $1 \leq j \leq r$.

Whenever we add an object $e$, if $e$ intersects $S^i \cap U_x$, it will also intersect some $S^j, i \neq j$, and some $U_y, x \neq y$; thus $S^*$ will respect $d_0$, and the object-systems constructed will always be in $\mathcal{P}$.

We shall use two types of constructions.

**Construction 1.** $S^i \Rightarrow S^j$.

First let us construct an object-system $S^i \Rightarrow S^j$ in $2 \otimes S$ such that, if $d$ is an $\mathcal{R}$-decomposition of $S^i \Rightarrow S^j$ and $d|S^i$ respects $d_0$, then $d|S^j$ respects $d_0$; moreover, $d$ respects $d_0$ uniformly on $S^i \Rightarrow S^j$.

Since $S$ is $\mathcal{R}$-strict, there is an object-system $S' \in (S * K_1) \setminus \mathcal{R}$. Let $E'$ be the set of objects of $S'$ that contain $z \in V(K_1)$. For $x = 1, 2, \ldots, m$, let $E'_x$ be the set of objects from $E'$ that contain only $z$ and vertices of $U_x$, while $E'_{\overline{x}}$ is the set of objects from $E'$ that contain some vertex of $V(S) \setminus U_x$. Let $S^i, S^j, i \neq j$, be disjoint copies of $S$; for every $x$, and every vertex $v \in U_x \cap V(S^j)$, we add the objects of $E'_{\overline{x}}$ (with $v$ taking the place of $z \in V(K_1)$, and $S^i$ taking the place of $S$). Note that $S^i \Rightarrow S^j \in 2S[U_1] * 2S[U_2] * \cdots * 2S[U_m]$. Since $d_0$ is an $\mathcal{R}$-decomposition of $S$, this implies that $(S^i \Rightarrow S^j) \in \mathcal{R}$.

If $d = (V_1, V_2, \ldots, V_l)$ would be an $\mathcal{R}$-decomposition of $T = (S^i \Rightarrow S^j)$ such that $d|S^i$ respects $d_0$, but $d|S^j$ does not respect $d_0$ (or at least, not in the same manner, i.e., $d$ does not respect $d_0$ uniformly). Then there exist $k$ and $x \neq y$ such that $V_k \cap S^i \subseteq U_y$, but some $v \in V_k \cap S^j$ belongs to $U_x$. We can add the objects corresponding to $E'_{\overline{x}}$ (with $v$ taking the place of $z \in V(K_1)$, and $S^i$ taking the place of $S$) because they contain at least one vertex $w$ of $S^i \cap U_x$ (so $w \notin V_k$). But then, $S'$ is an induced subobject-system of an object-system in $T[V_1] * T[V_2] * \cdots * T[V_l]$, which implies $S' \in \mathcal{R}$, a contradiction.

**Construction 2.** $m \bullet k_t S$.

For an $\mathcal{R}$-decomposition $d_t = (V_{t,1}, V_{t,2}, \ldots, V_{t,dec_{\mathcal{R}}(S)})$ of $S$ that does not respect $d_0$, $m \bullet k_t S$ is an object-system in $(mk_t) \otimes S$ having no $\mathcal{R}$-decomposition $d = (W_1, W_2, \ldots, W_{dec_{\mathcal{R}}(S)})$ such that, for all of the $mk_t$ induced copies $S^i$ of $S$, $d|S^i = d_t$.

Let $A_{i,j}(t)$ denote $U_i \cap V_{t,j}, 1 \leq i \leq m, 1 \leq j \leq n$. Since $d_t$ does not respect $d_0$, at least $n+1$ sets $A_{i,j}(t)$ are nonempty. Because $dec_{\mathcal{R}}(S) = n$, there exists a positive integer $k_t$ such that $k_t S[A_{1,1}(t)] * k_t S[A_{1,2}(t)] * \cdots *$

$k_tS[A_{m,n}(t)] \not\subset \mathcal{R}$. Fix an object-system $T_t \in (k_tS[A_{1,1}(t)] * k_tS[A_{1,2}(t)] * \cdots * k_tS[A_{m,n}(t)]) \setminus \mathcal{R}$. Note that $T_t$ differs from $k_tS$ only in the objects that intersect at least two different $U_i$'s, or at least two different $V_j$'s.

The $k_iU_i$'s form an $\mathcal{R}$-decomposition of $k_tS$, so we can replace the objects of $k_tS$ that intersect at least two $U_i$'s, with the objects of $T_t$ that intersect at least two $U_i$'s, and still remain in $\mathcal{R}$. If, in the resulting object-system $\tilde{W}$, the $V_j$'s also formed an $\mathcal{R}$-decomposition, we could replace the objects of $\tilde{W}$ that intersect at least two different $V_j$'s with the objects of $T_t$ that intersect at least two different $V_j$'s, and still remain in $\mathcal{R}$. But this is impossible because we would then have $T_t \in \mathcal{R}$.

The only problem with $\tilde{W}$ is that, in order to construct it, we altered objects *inside* the $k_t$ copies that we had of $S$. We therefore construct $m \bullet k_tS$ by taking $m$ disjoint copies of $W = k_tS$, denoted by $W^j, j = 1, 2, \ldots, m$, and adding objects between $W^1 \cap U_1, W^2 \cap U_2, \ldots, W^m \cap U_m$. Specifically, suppose an object of $T_t$ intersects $U_{a_1}, \ldots, U_{a_r}$ $(1 \le a_1 < \cdots < a_r \le m$, $r \ge 2)$; then in $m \bullet k_tS$ we put a corresponding object that intersects $W^{a_1} \cap U_{a_1}, \ldots, W^{a_r} \cap U_{a_r}$.

Suppose $d = (X_1, X_2, \ldots, X_{dec_{\mathcal{R}}(S)})$ is an $\mathcal{R}$-decomposition of $m \bullet k_tS$ such that, for every one of the $mk_t$ induced copies $S^i$ of $S$, $d|S^i = d_t$. Then $W^1 \cap U_1, \ldots, W^m \cap U_m$ induce a copy of the object-system $\tilde{W}$ from which we could obtain $T_t$ while still remaining in $\mathcal{R}$, thus getting a contradiction as above.

We now construct $S^*$ as follows. First let $S(0) := S$ and $S(1) := m \bullet k_1S$. For $1 < l \le r$, construct $S(l)$ by taking $mk_l$ disjoint copies $S(l-1)^1, \ldots, S(l-1)^{mk_l}$ of $S(l-1)$. For each copy of $S$ in $S(l-1)^i$ and each copy of $S$ in $S(l-1)^j, i \neq j$ we add the objects between them that are between the $i^{th}$ and $j^{th}$ copies of $S$ in $m \bullet k_lS$.

Finally, from $S(r)$, which is in, say, $s \otimes S$, consisting of copies $S^1$, $S^2$, $\ldots$, $S^s$ of $S$, we create $S^*$ by adding two more copies $S^+$ and $S^-$ of $S$. We add objects between $S^+$ and $S^-$ to create the object-system $S^- \Rightarrow S^+$, and, for each $i = 1, \ldots, s$, we add objects to obtain $S^i \Rightarrow S^-$ and $S^+ \Rightarrow S^i$.

Let $d$ be an $\mathcal{R}$-decomposition of $S^*$ with $n$ parts (it might happen that none such decomposition exists and it this case we are immediately done). For $1 \le l \le r$, if every copy of $S(l-1)$ in $S(l)$ contains a copy of $S$ for which $d|S = d_l$, then we would have $mk_l$ such copies of $S$ inducing a copy of $m \bullet k_lS$, what is impossible. So by induction from $r$ to 1, there is a copy $S^p$ of $S$ for which $d|S^p$ is none of $d_1, d_2, \ldots, d_r$. Thus, $d$ is the unique $\mathcal{R}$-decomposition of $S^*$, $d|S^p$ respects $d_0$. But $S^p \Rightarrow S^-$ is an induced subobject-system of $S^*$, so $d|S^- = d_0$ (and in fact $d$ respects $d_0$ uniformly on these two copies of $S$). Similarly, $d|S^+$ respects $d_0$ and, in the same way, $d$ respects $d_0$ uniformly, as it is required. ■

Using Theorem 3 we can prove:

**Theorem 4.** *Let $\mathcal{R} \in \mathbb{M}^{\omega a}$ be a reducible property of object-systems of finite character. Then there exists a uniquely $\mathcal{R}$-decomposable infinite countable object-system $H$ such that $\gamma H = (\mathcal{R}, H')$ and $age(H) = \mathcal{R} \cap \mathcal{I}$.*

***Proof.*** Analogously as in [26], a *composition sequence* of a class $\mathcal{R}$ of finite object-systems is a sequence of finite object-systems $H_1, H_2, \ldots, H_n, \ldots$ such that $H_i \in \mathcal{R}, H_i < H_{i+1}$ for all $i \in \mathbb{N}$ and for all $S \in \mathcal{R}$ there exists a $j$ such that $S \leq H_j$. Because of additivity of $\mathcal{R}$, using the same arguments as in [26], $\mathcal{R}$ has a composition sequence $H_1, H_2, \ldots, H_n, \ldots$. According to Theorem 3, we can easily find a composition sequence $H_1^*, H_2^*, \ldots, H_n^*, \ldots$ of $\mathcal{R} \cap \mathcal{I}$ consisting of finite uniquely $\mathcal{R}$-decomposable object-systems $H_i^*$ containing $H_i$. Without loss of generality, we may assume that if $i < j$, $i, j \in \mathbb{N}$, then $V(H_i^*) \subset V(H_j^*)$. Let $V(H) = \bigcup_{i \in \mathbb{N}} V(H_i^*)$ and $A_i \in E(H)$ if and only if $A_i \in E(H_j^*)$ for some $j \in \mathbb{N}$. It is easy to see that $age(H) = \mathcal{R} \cap \mathcal{I}$, implying $\gamma H = (\mathcal{R}, H')$. Let us remark that, according to Theorem 2, $H$ is $\mathcal{R}$-decomposable if every finite induced subobject-system of $H$ is $\mathcal{R}$-decomposable. In order to verify, that $H$ is uniquely $\mathcal{R}$-decomposable it is sufficient to verify that if $\{V_{j_1}, V_{j_2}, \ldots, V_{j_n}\}$, $V_{j_i} \neq \emptyset$ is the unique $\mathcal{R}$-decomposition of $H_j$, $j \in \mathbb{N}$, then $\{U_1, U_2, \ldots, U_n\}$, where $U_k = \bigcup_{j \in \mathbb{N}} V_{j_k}$, $k = 1, 2, \ldots, n$, is the unique $\mathcal{R}$-decomposition of $H$. Indeed, this is because the existence of other $\mathcal{R}$-decomposition of $H$ would imply the existence of other decomposition of some $H_i$ and it provides a contradiction. ■

## 6. The Unique Factorization Theorem for Properties of Object-Systems

In [15], based on Theorem 2 and Theorem 1 we proved for graph properties of finite character, that every reducible additive graph property $\mathcal{R}$ of finite character is uniquely factorisable into finite number of irreducible factors.

Here we present a proof of the Unique Factorization Theorem for object-systems based on Theorem 4 in the formal context $\mathbb{K}$.

**Theorem 5.** *Every reducible additive property of object-systems $\mathcal{R}$ of finite character is uniquely factorisable into finite number of irreducible factors.*

***Proof.*** According to Theorem 4, let $H$ be a uniquely $\mathcal{R}$-decomposable infinite countable object-system such that $\gamma H = (\mathcal{R}, H')$ and let $d_H = \{W_1, W_2, \ldots, W_n\}$ be the unique $\mathcal{R}$-decomposition of $H$. Let $\mathcal{P}_i = \gamma H[W_i]$ for $i = 1, 2, \ldots, n = dec(\mathcal{R})$. Then obviously we have $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$. If there would be some other factorization of $\mathcal{R}$ into $n$ irreducible factors then $H$ would have another $\mathcal{R}$-decomposition, which contradicts the fact that $H$ is uniquely $\mathcal{R}$-decomposable. Since $dec(H) = dec(\mathcal{R}) = n$, there is no factorization of $\mathcal{R}$ into more then $n$ factors. Thus we have to prove that $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ is the unique factorization of $\mathcal{R}$. Further, let $\mathcal{R} = \mathcal{Q}_1 \circ \mathcal{Q}_2 \circ \cdots \circ \mathcal{Q}_m, m < n$ and $d_0 = \{U_1, U_2, \ldots, U_m\}$ be a $(\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_m)$-partition of $H$. Then, by Theorem 3, there is an $s \in \mathbb{N}$ such that $s \otimes H$ respects $d_0$ uniformly. Thus, since $m < n$, there exists an index $j$ such that $H[U_j] \in H[W_r] * H[W_s]$, implying $\mathcal{Q}_j$ is reducible. ■

**Acknowledgments**

We would like to thank to the anonymous referee and Alastair Farrugia (for his stimulating discussions during our common work on the Unique Factorization Theorem for hypergraphs) for their substantional help in preparation of this paper.

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Received 7 April 2010
Revised 21 August 2010
Accepted 27 August 2010