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ADJACENT VERTEX DISTINGUISHING EDGE COLORINGS OF THE DIRECT PRODUCT OF A REGULAR GRAPH BY A PATH OR A CYCLE

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Abstract

In this paper we investigate the minimum number of colors required for a proper edge coloring of a finite, undirected, regular graph G in which no two adjacent vertices are incident to edges colored with the same set of colors. In particular, we study this parameter in relation to the direct product of G by a path or a cycle.

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1. Introduction

Let $G = (V, E)$ be a finite, simple and undirected graph. A proper edge coloring of G is a map f from E to a set of colors C such that $f((v,w)) \neq f((v,z))$ for every pair of adjacent edges $(v,w),(v,z)$.

The color set of a vertex $v \in V$ is the set $C(v)$ of colors of edges incident to v. A proper edge coloring of G is *adjacent vertex distinguishing* (for short *avd*) if $C(v) \neq C(w)$ whenever vertices v, w are adjacent (see [1], [2]). The same coloring is also called adjacent strong edge coloring [5]. The minimum number of colors for any *avd*-coloring of G is denoted $\chi'_a(G)$ [2] and called the *avd chromatic index* of G.

Let Δ denote the maximum degree of G; from the definition it follows that $\chi'_a(G) \geq \Delta$ and if G has two adjacent vertices of degree Δ, then $\chi'_a(G) \geq \Delta + 1$.

In [1] the authors prove that $\chi'_a(G) \leq 5$ for graphs of the maximum degree 3 and $\chi'_a(G) \leq \Delta + 2$ for bipartite graphs. In [5] the following conjecture was made:

If G is a simple, connected graph of at least 3 *vertices, of maximum degree Δ and different from C_5, then*

$$\Delta \leq \chi'_a(G) \leq \Delta + 2. \qquad (1)$$

The *direct product* $G \times H$ of two graphs $G = (V, E)$ and $H = (W, F)$ is the graph with vertex-set $V(G \times H) = V \times W$ and edge-set $E(G \times H) = \{(a, v)(b, w) \mid (a, b) \in E, (v, w) \in F\}$.

This product, also referred to as, for instance, the tensor product, the Kronecker product, the categorical product and the conjunction, has applications in engineering, computer science and related disciplines. It is commutative and associative. For terminologies not defined here we follow [4].

As usual C_n and P_n denote respectively a cycle and a path on n vertices. In relation to the direct product of cycles and paths, recall that each of $C_{2n+1} \times P_m$ and $C_{2n+1} \times C_m$ is a connected graph while each of $P_n \times P_m$, $C_{2n} \times P_m$ and $C_{2n} \times C_{2m}$ consists of two connected components. Moreover, the two components of $C_{2n} \times P_m$ are isomorphic [3].

The direct product of a bipartite graph and every other graph is bipartite; thus all the above mentioned products are bipartite except the products of cycles of odd length.

We introduce a notion which will be useful in the article.

Definition 1. A sequence $S_1, S_2, \ldots, S_m$ of d-subsets of a $(2d+1)$-set C, $d > 0$, is called an *avd* d-sequence of length m if the following properties hold, for $2 \leq i \leq m-1$:

- A_1: Every set S_i is disjoint from S_{i-1} and S_{i+1};
- A_2: The sets S_{i-1} and S_{i+1} are distinct.

An *avd* sequence of length m is called *cyclic* avd if the same properties hold for $1 \leq i \leq m$, where indices are modulo m.

Notice that in a cyclic *avd* d-sequence last set is disjoint from the first one and different from the second, while the first has to be different from the next to last.

In this article we prove that there exists an *avd* d-sequence of every length $m > 1$ and a cyclic *avd* sequence of every length $m \geq 2d+1$ and also of even length $4 < m \leq 2d$. This allows us to prove that for these values of m and a d-regular graph G, $\chi'_a(G \times P_m) = \chi'_a(G \times C_m) = 2d+1$ (Theorems 1, 4). Moreover, in Proposition 3 we prove that for odd $1 < m \leq 2d+1$ cyclic *avd* d-sequences of length m do not exist. This result does not imply that the corresponding *avd* chromatic index is different from $2d+1$, as proved, for instance, in relation to $C_3 \times C_3$ (Figure 1).

The article is subdivided into five sections. In Section 2 we determine properties of *avd* d-sequences; in Section 3 we consider the problem of the *avd* chromatic index of the direct product of a regular graph G by a path; in Section 4 we consider a similar problem in relation to the direct product of G by a cycle and in Section 5 in relation to the direct product of two cycles.

2. *avd* d-sequences

In this section we establish the existence and some properties of *avd* d-sequences. We start with an example of an *avd* d-sequence of length $m > 1$; in particular, we consider the sequence Σ_m of d-subsets of $C = \{1, 2, \ldots, 2d+1\}$

$$Q_1, Q_2, \ldots, Q_m, \tag{2}$$

where every set Q_i is obtained by taking d cyclically consecutive elements of C; thus $Q_1 = \{1, 2, \ldots, d\}$, $Q_2 = \{d+1, \ldots, 2d\}$, $Q_3 = \{2d+1, 1, 2, \ldots, d-1\}$, and so on. It is easy to see that Σ_m satisfies A_1 and A_2.

Moreover it is immediate to prove the following Lemma.

Lemma 1. *In relation to the sequence* Σ_m*, the minimum integer* m *such that* $Q_{m+1} = Q_1$ *is* $m = 2d + 1$*.*

From (2) and Lemma 1, it follows

Proposition 1. *There exists an avd* d*-sequence of every length* $m > 1$*.*

Notice that for $m = 2d + 1$ the *avd* d-sequence (2) is also cyclic.

Lemma 2. *Let* $S_1, S_2, \ldots, S_m$ *be an avd* d*-sequence of length* m*; then, for every* $1 \leq i \leq m - 2$*,* $| S_i \cap S_{i+2} |= d - 1$*.*

Proof. S_{i+2} is disjoint from S_{i+1} and distinct from S_i, then $S_{i+2} = R \cup H$, where $R = C \setminus (S_i \cup S_{i+1})$ has cardinality 1 and $H \subseteq S_i$. Then $| S_i \cap S_{i+2} |=$ $| H |= d - 1$. ■

The *concatenation* of two d-sequences $R = (R_1, R_2, \ldots, R_r)$ and $T = (T_1, T_2, \ldots, T_q)$ is the d-sequence $RT = (R_1, R_2, \ldots, R_r, T_1, T_2, \ldots, T_q)$. If $R = T$, we write R^2. In an obvious way the definition may be extended to a greater number of d-sequences.

Assume that R and T are *avd.* In this case, if T_1 is disjoint from R_r and distinct from R_{r-1} and T_2 is distinct from R_r, then also RT is *avd.* In addition if T_q is disjoint from R_1, distinct from R_2 and T_{q-1} is distinct from R_1, then RT is cyclic *avd.*

Lemma 3. *If there exists a cyclic avd sequence of length* r*, then, for every integer* $t > 1$*, there exists a cyclic avd sequence of length* tr*.*

Proof. Let $W : T_1, T_2, \ldots, T_r$ be a cyclic *avd* sequence of length r; then the sequence W^t obtained by concatenating t times W is clearly cyclic *avd.* ■

Lemma 4. *For every* $d \geq 2$*, there is no cyclic avd* d*-sequence of length* 4*.*

Proof. By way of contradiction let us assume that there is a cyclic *avd* sequence of length 4: S_1, S_2, S_3, S_4. Let $S_1 = \{1, 2, \ldots, d\}$ and $S_2 = \{d + 1, \ldots, 2d\}$. Now S_3 has to be disjoint from S_2 and different from S_1. Thus $S_3 = \{2d + 1\} \cup H$, where $H \subseteq S_1$. Thus we obtain the impossible condition that S_4, which is disjoint from S_1 and S_3, has to coincide with S_2. ■

3. Direct Product of a Regular Graph by a Path

Let G be a simple, regular graph of degree d, having $n > 1$ vertices.

First consider the following result, where, in relation to a graph H and an integer $d > 1$, dH denotes the multigraph obtained from H by replacing every edge e by d edges having the same vertices of e.

Proposition 2. *For a d-regular graph G and an arbitrary graph H, we have*

$$\chi'_a(G \times H) \leq \chi'_a(dH).$$

Proof. Let α be a χ'_a-coloring of dH. We prove that we are able to construct an *avd* coloring of $G \times H$, using the colors of α. Let (u, v) be an edge of H; it is easy to see that $G \times (u, v)$ is an induced subgraph of $G \times H$, which turns out to be d-regular and bipartite. It follows that its chromatic index equals d. We determine a proper coloring of such a subgraph by using the d colors assigned by α to the d edges (u, v) of dH. By proceeding in the same way in relation to every edge of H, we obtain a proper coloring β of $G \times H$. Let (z_1, u_1) be a vertex of $G \times H$ and D_1 the set of colors assigned by β to the edges incident such a vertex. It follows that D_1 coincides with the set of colors assigned to the edges incident to u_1 in dH. Now let (z_2, u_2) a vertex of $G \times H$ adjacent to (z_1, u_1) and D_2 the similar set of colors. Because α is an *avd*-coloring it follows that $D_1 \neq D_2$. Then also the coloring β assigned to the edges of $G \times H$ is *avd* and the result follows. ■

Now consider the direct product $G \times P_m$, where $m > 1$.

Denote $V(P_m) = \{z_1, z_2, \ldots, z_m\}$. We see that $G \times P_m$ is the union of $m-1$ edge disjoint subgraphs $H_i = G \times (z_i, z_{i+1})$, $1 \leq i \leq m-1$. The edges of H_i are the pairs $((v_t, z_i), (v_j, z_{i+1}))$ and $((v_t, z_{i+1}), (v_j, z_i))$, where v_t, v_j are adjacent vertices of G. Note that H_i is bipartite; moreover, it is not connected and consists of two components isomorphic to G if and only if G is bipartite. In the case of the direct product of G by a cycle of m vertices C_m, the same previous partition holds, with the addition of the subgraph $H_m = G \times (z_m, z_1)$.

In any case, the maximum degree of H_i coincides with the maximum degree of G; thus, as G is regular of degree d, H_i is regular of degree d and $G \times P_m$ has maximum degree $2d$. For $m > 3$, there are adjacent vertices of degree $2d$ and $\chi'_a(G \times P_m) \geq 2d + 1$.

Lemma 5. *For $d > 0$ and $m > 3$ $\chi'_a(dP_m) = 2d+1$, while for $m = 3$ $\chi'_a(dP_3) = 2d$.*

Proof. Let us assume that $m > 3$. From Proposition 1 it follows that there exists an *avd* d-sequence of every length $m > 1$, denoted $Q_1, Q_2, \ldots Q_m$. Let $V(P_m) = (z_1, z_2, \ldots, z_m)$. If we assign to the d-edges (z_i, z_{i+1}) of dP_m, $1 \le i \le m-1$, the d elements of Q_i as colors, it is easy to see that we obtain an *avd* $(2d+1)$-coloring of dP_m. In the case of $m = 3$ it is sufficient to consider as sets of colors two disjoint d-sets Q_1, Q_2 and obtain an *avd* $2d$-coloring of P_3. ∎

Theorem 1. *Let G be a d-regular graph and $m > 2$ a positive integer. Then*

$$\chi'_a(G \times P_m) = \chi'_a(dP_m) = \begin{cases} 2d & for \quad m = 3, \\ 2d+1 & for \quad m > 3. \end{cases} \tag{3}$$

Proof. By Proposition 2 and Lemma 5 we obtain that

$$\chi'_a(G \times P_m) \le \chi'_a(dP_m) = \begin{cases} 2d & for \quad m = 3, \\ 2d+1 & for \quad m > 3. \end{cases} \tag{4}$$

Let $m = 3$. As the maximum degree of $G \times P_3$ holds $2d$, then $\chi'_a(G \times P_m) \ge 2d$ and by (4) we obtain the result.

Now let us assume that $m > 3$. Because $G \times P_m$ contains adjacent vertices of maximum degree $2d$, then $\chi'_a(G \times P_m) \ge 2d+1$ and by (4) the result still follows. ∎

4. Direct Product of a Regular Graph by a Cycle

In this section we investigate the problem of the existence of cyclic *avd* d-sequences of even and odd length; the results allow us to determine the *avd* chromatic index of the direct product of a d-regular graph G by a cycle.

Theorem 2. *For $d \ge 3$, there exists a cyclic avd d-sequence of every even length $m > 4$.*

Proof. For $d \ge 3$, consider the d-subsets $A = \{1, 2, \ldots, d-1\}$ and $B = \{d+1, d+2, \ldots, 2d-1\}$ of the set $S = \{1, 2, \ldots, 2d+1\}$ and the following d-sequence of length $6, 8, 10$ respectively:

$D_6 : A \cup \{d\}, B \cup \{2d\}, A \cup \{2d+1\}, B \cup \{d\}, A \cup \{2d\}, B \cup \{2d+1\}$,

$D_8 : A \cup \{d\}, B \cup \{2d\}, A \cup \{2d+1\}, B \cup \{d\}, (A \cup \{2d, 2d+1\}) \setminus \{1\}, B \cup \{1\}, (A \cup \{d, 2d\}) \setminus \{1\}, B \cup \{2d+1\}$,

$D_{10} : A \cup \{d\}, B \cup \{2d\}, A \cup \{2d+1\}, B \cup \{d\}, A \cup \{2d, 2d+1\} \setminus \{1\}, B \cup \{1\}, A \cup \{d, 2d, 2d+1\} \setminus \{1, 2\}, B \cup \{2\}, A \cup \{d, 2d\} \setminus \{2\}, B \cup \{2d+1\}$.

It is easy to see that D_6, D_8, D_{10} are cyclic *avd*. Notice that the first two sets of D_6, D_8, D_{10} coincide. This allows to concatenate D_6 by D_6, D_8, D_{10} and obtain cyclic *avd* sequences of length $12, 14, 16$. By repeating the procedure of concatenation we are able to obtain sequences of every possible even length. ■

For example, we may find for $d = 3$ the sequences:

$$D_6 : 123, 456, 127, 345, 126, 457,$$

$$D_8 : 123, 456, 127, 345, 267, 145, 236, 457,$$

$$D_{10} : 123, 456, 127, 345, 267, 145, 367, 245, 136, 457.$$

Theorem 3. *For $d > 2$, there exists a cyclic avd d-sequence of every odd length $m \geq 2d+1$.*

Proof. Notice that the d-sequence from (2), of length $m = 2d+1$, turns out to be cyclic. Denote such a sequence $C_{2d+1} : Q_1, Q_2, \ldots, Q_{2d+1}$. Now consider the d-sequence C_{2d+3} of length $2d+3$ obtained from C_{2d+1} with the replacement of Q_{2d} by $Q_{2d} \cup \{d+2\} \setminus \{3\}$ and Q_{2d+1} by $Q_{2d+1} \cup \{3\} \setminus d+2$ and the addition of the two sets $Q_{2d+2} = \{1, 2, 4, \ldots, d+1\}$ and $Q_{2d+3} = \{d+2, \ldots 2d+1\}$.

Moreover consider the d-sequence C_{2d+5} obtained from C_{2d+1} by replacing Q_{2d+1} by the set $(Q_{2d+1} \setminus \{2d\}) \cup \{1\}$ and the the addition of the four d-sets Q_{2d-2}, Q_{2d-1}, Q_{2d}, Q_{2d+1}.

It is not difficult to prove that C_{2d+3} and C_{2d+5} are cyclic *avd*. Notice that the first two sets in the sequences C_{2d+1}, C_{2d+5}, C_{2d+7} coincide and also coincide with the same sets of the cyclic *avd* sequences of length even, as proved in Theorem 2. This allows to concatenate C_{2d+1} by the *avd* sequences of even length $h \geq 6$, thus obtaining, together with the sequences C_{2d+1}, C_{2d+3} and C_{2d+5}, *avd* sequences of every odd length $m \geq 2d+1$. ■

For $d = 3$, an example of *avd* 3-sequences of length 7, 9, 11 is the following:

$$C_7 : 123, 456, 127, 345, 167, 234, 567,$$

$$C_9 : 123, 456, 127, 345, 167, 245, 367, 124, 567,$$

$$C_{11} : 123, 456, 127, 345, 167, 234, 156, 237, 145, 236, 457.$$

Lemma 6. *For $d > 2$ and a positive integer $m > 4$, when even, or $m \geq 2d + 1$, when odd, $\chi'_a(dC_m) = 2d + 1$.*

Proof. By Theorem 2 in the case of m even and Theorem 3 in the case of m odd, there exists a cyclic *avd* d-sequence, denoted $(Q_1, Q_2, \ldots, Q_m)$. Let $V(C_m) = \{v_1, v_2, \ldots, v_m\}$; if we assign to the d edges (v_i, v_{i+1}), $1 \leq i \leq m$, the d colors of Q_i we obtain an *avd* coloring of dC_m. ■

Theorem 4. *Let G be a d-regular graph, where $d > 2$, and a positive integer $m > 4$, when even, or $m \geq 2d + 1$, when odd. Then*

$$\chi'_a(G \times C_m) = 2d + 1. \tag{5}$$

Proof. By Proposition 2 and Lemma 6 we have $\chi'_a(G \times C_m) \leq 2d + 1$. By the condition that $G \times C_m$ contains adjacent vertices of degree $2d$, then $\chi'_a(G \times C_m) \geq 2d + 1$ and the result follows. ■

For odd values of $7 \leq m \leq 2d - 1$ we could have $\chi'_a(G \times C_m) = 2d + 1$; but the coloring is not be represented by a cyclic *avd* d-sequence.

Proposition 3. *Let $1 < m < 2d + 1$ be an odd integer and $d > 2$; there is not a cyclic avd d-sequence of length m.*

Proof. Let us assume that there exists a cyclic *avd* d-sequence S of odd length m, where $1 < m < 2d + 1$, whose elements belong to a $(2d + 1)$-set C. Notice that, because S is cyclic, every element $a \in C$ belongs to at most $\frac{m-1}{2}$ sets of S. Then the number of elements involved in S is at most $\frac{m-1}{2} \cdot (2d + 1)$; by the condition on S we also have that the number of elements involved in S is $m \cdot d$. Thus we obtain the impossible inequality $md \leq \frac{m-1}{2} \cdot (2d + 1)$. ■

5. Direct Product of Two Cycles

In this section we investigate the case of the direct product of two cycles, which turns out the case of $d = 2$ excluded in previous section.

Lemma 7. *There is not a cyclic avd* 2*-sequence of length* 7*.*

Proof. Assume to the contrary that $D_1, \ldots, D_7$ is a cyclic *avd* sequence of length 7, where $D_i \subseteq \{1, 2, \ldots, 5\}$, for $1 \leq i \leq 7$. Notice that every element has to appear at most 3 times in the subsets which form W, because otherwise there exist two non-disjoint consecutive sets. Without loss of generality we may assume that $D_1 = \{1, 2\}$, $D_2 = \{3, 4\}$ and $1, 2, 3, 4$ appear 3 times; then $1, 2$ have to appear in $D_i, 3 \leq i \leq 6$, two times. Let $1 \in D_3$. If $1 \in D_6$ it is not possible to arrange 2 in two non consecutive sets. Thus $1 \in D_5$ and $2 \in D_4, D_6$. Now we see that $3, 4$ have to belong two times to $D_j, 4 \leq j \leq 7$. One of them belongs to D_4. Assume that $3 \in D_4$; then it follows that $3 \in D_7$. Now we have the impossible condition that $4 \in D_5, D_6$. ■

Notice that previous Lemma does not imply that $\chi'_a(C_n \times C_7) > 5$. The claim only states that for $m = 7$ there is not a cyclic *avd* 2-sequence.

Proposition 4. *There exist cyclic avd* 2*-sequences of length* $m \geq 5$*, except for* $m = 7$*.*

Proof. For $m = 5, 6, 8, 9$ we may consider the following sequences, where W_i denote a cyclic *avd* sequence of length i:

$$W_5 : 12, 34, 51, 23, 45,$$

$$W_6 : 12, 34, 25, 13, 24, 35,$$

$$W_8 : 12, 34, 15, 23, 45, 13, 24, 35,$$

$$W_9 : 12, 34, 15, 23, 45, 13, 25, 14, 35.$$

The case of $m = 7$ follows from the previous Lemma. Notice that all the sequences have the same first two sets. Then by concatenating these sequences we obtain the result. ■

If we augment the number of colors we are able to determine suitable *avd* cyclic sequences. Indeed, for $m = 7$ we have the following cyclic *avd* sequence of 2-subsets of a 6-set:

$$12, 34, 56, 12, 34, 25, 63.$$

Therefore

$$5 \leq \chi'_a(C_n \times C_7) \leq 6 \tag{6}$$

which is consistent with Conjecture 1.

By previous results we have that when n or m are even and greater than 4 or both odd and greater than 5, but different from 7, then $\chi'_a(C_n \times C_m) = 5$. In Figure 1 we show that the equality holds also for $n = m = 3$. For $n = m = 4$, first we prove the following Lemma.

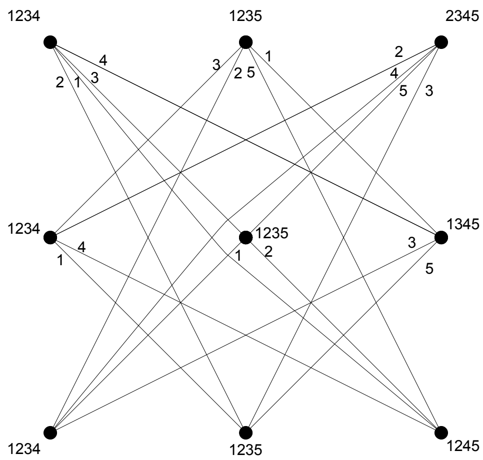

Figure 1. AVD coloring of $C_3 \times C_3$.

Lemma 8. *The graph $C_4 \times C_4$ consists of two subgraphs isomorphic to $K_{4,4}$.*

Proof. Let G_1 and G_2 two copies of C_4 and $\{v_1, v_2, v_3, v_4\}$ and $\{w_1, w_2, w_3, w_4\}$ their sets of vertices respectively. Consider the sets of vertices $A_1 = \{(v_1, w_1), (v_1, w_3), (v_3, w_1), (v_3, w_3)\}$ and $B_1 = \{(v_2, w_2), (v_2, w_4), (v_4, w_2), (v_4, w_4)\}$. Notice that the vertices of these sets are independent. Moreover, all the vertices of A_1 are adjacent to all the vertices of B_1. Thus H_1 is isomorphic to $K_{4,4}$. In a similar way the sets $A_2 = \{(v_1, w_2), (v_1, w_4), (v_3, w_2), (v_3, w_4)\}$ and $B_2 = \{(v_2, w_1), (v_2, w_3), (v_4, w_1), (v_4, w_3)\}$ turn out to be the partite sets of a subgraph H_2 isomorphic to $K_{4,4}$. ■

In [5] it was proved that $\chi'_a(K_{n,n}) = n + 2$; this implies that $\chi'_a(K_{4,4}) = 6$ and therefore that $\chi'_a(C_4 \times C_4) = 6$, thus obtaining a case of a direct product by a cycle satisfying the upper bound of Conjecture 1.

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