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CLOSED k-STOP DISTANCE IN GRAPHS

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Abstract

The Traveling Salesman Problem (TSP) is still one of the most researched topics in computational mathematics, and we introduce a variant of it, namely the study of the closed k-walks in graphs. We search for a shortest closed route visiting k cities in a non complete graph without weights. This motivates the following definition. Given a set of k distinct vertices $\mathcal{S} = \{x_1, x_2, \ldots, x_k\}$ in a simple graph G, the closed k-stop-distance of set $\mathcal{S}$ is defined to be

$$d_k(\mathcal{S}) = \min_{\theta \in \mathcal{P}(\mathcal{S})} \Big(d(\theta(x_1), \theta(x_2)) + d(\theta(x_2), \theta(x_3)) + \cdots + d(\theta(x_k), \theta(x_1)) \Big),$$

where $\mathcal{P}(\mathcal{S})$ is the set of all permutations from $\mathcal{S}$ onto $\mathcal{S}$. That is the same as saying that $d_k(\mathcal{S})$ is the length of the shortest closed walk through the vertices $\{x_1, \ldots, x_k\}$. Recall that the Steiner distance $sd(\mathcal{S})$ is the number of edges in a minimum connected subgraph containing all of the vertices of $\mathcal{S}$. We note some relationships between Steiner distance and closed k-stop distance.

The closed 2-stop distance is twice the ordinary distance between two vertices. We conjecture that $rad_k(G) \leq diam_k(G) \leq \frac{k}{k-1} rad_k(G)$ for any connected graph G for $k \geq 2$. For $k = 2$, this formula reduces to the classical result $rad(G) \leq diam(G) \leq 2rad(G)$. We prove the conjecture in the cases when $k = 3$ and $k = 4$ for any graph G and for $k \geq 3$ when G is a tree. We consider the minimum number of vertices with each possible 3-eccentricity between $rad_3(G)$ and $diam_3(G)$. We also study the closed k-stop center and closed k-stop periphery of a graph, for $k = 3$.

Keywords: Traveling Salesman, Steiner distance, distance, closed k-stop distance.

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1. Definitions and Motivation

In this paper, all graphs are simple (i.e., no loops or multiple edges). For vertices u and v in a connected graph G, let $d(u, v)$ denote the standard distance from u to v (i.e., the length of the shortest path from u to v). Recall that the eccentricity $e(u)$ of a vertex u is the maximum distance $d(u, v)$ over all other vertices $v \in V(G)$. The radius $rad(G)$ of G is the minimum eccentricity $e(u)$ over all vertices $u \in V(G)$, and the diameter $diam(G)$ is the maximum eccentricity $e(u)$ taken over all vertices $u \in V(G)$.

Let $G = (V(G), E(G))$ be a graph of order n ($|V(G)| = n$) and size m ($|E(G)| = m$). Let $S \subseteq V(G)$. Recall ([2, 4, 5, 6, 7]) that a *Steiner tree* for S is a connected subgraph of G of smallest size (number of edges) that contains S. The size of such a subgraph is called the *Steiner distance* for S and is denoted by $sd(S)$. Then, the Steiner k-eccentricity $se_k(v)$ of a vertex v of G is defined by $se_k(v) = \max\{sd(S) | S \subseteq V(G), |S| = k, v \in S\}$. Then the Steiner k-radius and k-diameter are defined by $srad_k(G) = \min\{se_k(v) | v \in V(G)\}$ and $sdiam_k(G) = \max\{se_k(v) | v \in V(G)\}$.

In this paper, we study an alternate but related method of defining the distance of a set of vertices. The closed k-stop distance was introduced by Gadzinski, Sanders, and Xiong [3] as k-stop-return distance. The closed k-stop-distance of a set of k vertices $\mathcal{S} = \{x_1, x_2, \ldots, x_k\}$, where $k \geq 2$, is defined to be

$$d_k(\mathcal{S}) = \min_{\theta \in \mathcal{P}(\mathcal{S})} \Big(d(\theta(x_1), \theta(x_2)) + d(\theta(x_2), \theta(x_3)) + \cdots + d(\theta(x_k), \theta(x_1)) \Big),$$

where $\mathcal{P}(\mathcal{S})$ is the set of all permutations from $\mathcal{S}$ onto $\mathcal{S}$. That is the same as saying that $d_k(\mathcal{S})$ is the length of the shortest closed walk through the vertices $\{x_1, \ldots, x_k\}$. The closed k-stop eccentricity $e_k(x)$ of a vertex x in G is $\max\{d_k(\mathcal{S}) | x \in \mathcal{S}, \mathcal{S} \subseteq V(G), |\mathcal{S}| = k\}$. The minimum closed k-stop eccentricity among the vertices of G is the closed *k-stop radius*, that is, $rad_k(G) = \min_{x \in V(G)} e_k(x)$. The maximum closed k-stop eccentricity among the vertices of G is the closed *k-stop diameter*, that is, $diam_k(G) = \max_{x \in V(G)} e_k(x)$.

Note that if $k = 2$, then $d_2(\{x_1, x_2\}) = 2d(x_1, x_2)$. We thus consider $k \geq 3$. In particular, the closed 3-stop distance of x, y and z ($x \neq y$, $x \neq z$, $y \neq z$) is

$$d_3(\{x, y, z\}) = d(x, y) + d(y, z) + d(z, x).$$

For simplicity, we will write $d_3(x, y, z)$ instead of $d_3(\{x, y, z\})$.

The closed *3-stop eccentricity* $e_3(x)$ of a vertex x in a graph G is the maximum closed 3-stop distance of a set of three vertices containing x, that is,

$$e_3(x) = \max_{y,z \in V(G)} \Big(d(x, y) + d(y, z) + d(z, x) \Big).$$

The central vertices of a graph G are the vertices with minimum eccentricity, and the center $C(G)$ of G is the subgraph induced by the central vertices. Similarly, we define the *closed k-stop central vertices* of G to be the vertices with minimum closed k-stop eccentricity and the *closed k-stop center* $C_k(G)$ of G to be the subgraph induced by the closed k-stop central vertices. A graph is closed k-stop self-centered if $C_k(G) = G$.

The peripheral vertices of a graph G are the vertices with maximum eccentricity, and the periphery $P(G)$ of G is the subgraph induced by the peripheral vertices. Similarly, we define the *closed k-stop peripheral vertices* of G to be the vertices with maximum closed k-stop eccentricity and the closed *k-stop periphery* $P_k(G)$ of G as the subgraph induced by the closed k-stop peripheral vertices. For simplicity in this paper, we will sometimes omit the words "closed" and "stop", so for instance, we will refer to the closed 3-stop eccentricity as the 3-eccentricity of a vertex.

Notice that for all values of $k \geq 2$, two times the k-Steiner distance is an upper bound on the closed k-stop distance of a set of vertices in a graph. (Given a Steiner tree for a set of k vertices, one possible closed walk through those vertices would trace each edge of the Steiner tree twice.) The k-Steiner distance plus one is always a lower bound for the closed k-stop distance, since the edges of a closed walk form a connected subgraph.

Necessarily, in a closed walk, either an edge is repeated or a cycle is formed, so at least one edge could be omitted without disconnecting the subgraph. That is, for a set S of $|S| = k \in \{1, 2, \ldots, n-1, n\}$ vertices, we have that

$$se_k(v) + 1 \leq e_k(v) \leq 2\, se_k(v), \forall v \in V(G), \tag{1}$$

$$srad_k(G) + 1 \leq rad_k(G) \leq 2\, srad_k(G), \text{ and} \tag{2}$$

$$sdiam_k(G) + 1 \leq diam_k(G) \leq 2\, sdiam_k(G). \tag{3}$$

For other graph theory terminology we refer the reader to [1]. In this paper we study the closed k-stop distance in graphs. Particularly, we present an inequality between the radius and diameter that generalizes the inequality for the standard distance. We also present a conjecture regarding this inequality that is verified to be true in trees. We also study the closed k-stop center and closed k-stop periphery of a graph, for $k = 3$.

2. Possible Values of Closed 3-stop Eccentricities

It is well-known that the ordinary radius and diameter of a graph G are related by $rad(G) \leq diam(G) \leq 2rad(G)$. Furthermore, for every k such that $rad(G) < k \leq diam(G)$, a graph must have at least two vertices with eccentricity k, and at least one vertex with eccentricity $rad(G)$. In the case of closed 3-stop distance, there is at least one vertex with closed 3-stop eccentricity $rad_3(G)$, and there are at least three vertices with closed 3-stop eccentricity $diam_3(G)$.

Proposition 1. *A connected graph G of order at least 3 has at least three closed 3-stop peripheral vertices.*

Proof. Let $x \in V(P_3(G))$. Then there exist vertices x_1 and $x_2 \in V(G)$ such that $e_3(x) = d(x, x_1) + d(x_1, x_2) + d(x_2, x) = e_3(x_1) = e_3(x_2)$. Thus $x, x_1, x_2 \in V(P_3(G))$. ■

Recall that in a graph G, the following relation holds: $rad(G) \leq diam(G) \leq 2rad(G)$. We present a similar sharp inequality between the closed 3-stop radius and closed 3-stop diameter.

Proposition 2. *For a connected graph G, we have*

$$rad_3(G) \leq diam_3(G) \leq \tfrac{3}{2} rad_3(G).$$

Proof. The first inequality follows by definition. Let $u \in V(C_3(G))$, and let $y \in V(P_3(G))$. There are vertices w and x, necessarily also in the closed 3-stop periphery, such that $e_3(y) = d(y,w)+d(w,x)+d(x,y) = e_3(x) = e_3(w)$. Assume, without loss of generality, that $d(u,y)+d(y,x)+d(x,u) \leq d(u,w)+d(w,x)+d(x,u)$ and $d(u,w)+d(w,y)+d(y,u) \leq d(u,w)+d(w,x)+d(x,u)$. This gives $d(u,y) + d(y,x) \leq d(u,w) + d(w,x)$ and $d(w,y) + d(y,u) \leq d(w,x) + d(x,u)$.

Case I. $d(w,x) \leq 2d(u,y)$.
Using the inequalities above,

$$\begin{aligned} e_3(y) &= d(y,w) + d(w,x) + d(x,y) \\ &\leq d(w,x) + d(x,u) - d(y,u) + d(w,x) + d(u,w) + d(w,x) - d(u,y) \\ &= d(u,x) + d(x,w) + d(w,u) + 2(d(w,x) - d(u,y)) \\ &\leq e_3(u) + 2(d(w,x) - d(u,y)). \end{aligned}$$

Now, clearly, $d(w,x) \leq d(w,u) + d(u,x)$, and from our assumption for this case, $2d(w,x) \leq 4d(u,y)$. Thus, $4d(w,x) \leq d(w,u) + d(u,x) + d(w,x) + 4d(u,y)$, which simplifies to

$$\begin{aligned} 2(d(w,x) - d(u,y)) &\leq \frac{1}{2}\left(d(u,w) + d(w,x) + d(x,u)\right) \\ &\leq \frac{1}{2}e_3(u). \end{aligned}$$

Thus, $e_3(y) \leq \frac{3}{2}e_3(x)$.

Case II. $d(w,x) > 2d(u,y)$.
If we restrict the paths from y so that they must come and go through u, the resulting paths will be the same length or longer than they would be without the restriction. Thus, $e_3(y) \leq 2d(y,u) + e_3(u) < d(w,x) + e_3(u)$. Since $e_3(u) \geq d(u,w) + d(w,x) + d(x,u)$ and $d(w,x) \leq d(u,w) + d(x,u)$, it follows that $d(w,x) \leq \frac{1}{2}e_3(u)$. Thus, $e_3(y) \leq \frac{3}{2}e_3(u)$. ■

Recall that, for the standard eccentricity, $|e(u) - e(v)| \leq 1$ for adjacent vertices u and v in a connected graph. Gadzinski, Sanders and Xiong noted a similar relationship for the closed k-stop eccentricities of adjacent vertices. Suppose u and $v \in V(G)$ are adjacent. Let $x_2, x_3, \ldots, x_k$ be vertices such that $e_k(u) = d_k(\{u, x_2, x_3, \ldots, x_k\})$. One possible closed walk through $\{u, x_2, x_3, \ldots, x_k\}$ would be from u to v, followed by a shortest closed walk through $\{v, x_2, x_3, \ldots, x_k\}$, and then from v to u. Thus, $e_k(u) \leq e_k(v) + 2$. Similarly, $e_k(v) \leq e_k(u) + 2$.

Proposition 3 [3]**.** *If u and v are adjacent vertices in a connected graph, then* $|e_k(u) - e_k(v)| \leq 2$.

The following example shows that it is possible for every vertex between $rad_3(G)$ and $diam_3(G)$ to be realized as the closed 3-stop eccentricity of some vertex, though it is also possible that some values may only be achieved once. Let $V(G) = \{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k, w_1, w_2, \ldots, w_k, x_0, x_1, \ldots, x_k\}$ and $E(G) = \{u_iu_{i+1}, v_iv_{i+1}, w_iw_{i+1}, x_ix_{i+1} | 1 \leq i \leq k-1\} \cup \{x_0x_1, x_0u_1, x_0v_1, x_0w_1, u_1v_1, v_1w_1\}$. Then $rad_3(G) = e_3(x_0) = 4k$, $e_3(u_i) = e_3(x_i) = e_3(w_i) = 4k + 2i$, and $e_3(v_i) = 4k + 2i - 1$. Notice that all odd eccentricities larger than $4k + 2M - 1$ may be skipped by leaving out the vertices v_i for $i > M$. Thus, this construction also shows that not all integers between $rad_3(G)$ and $diam_3(G)$ must be realized. Figure 1 shows an example of this construction with $k = 3$.

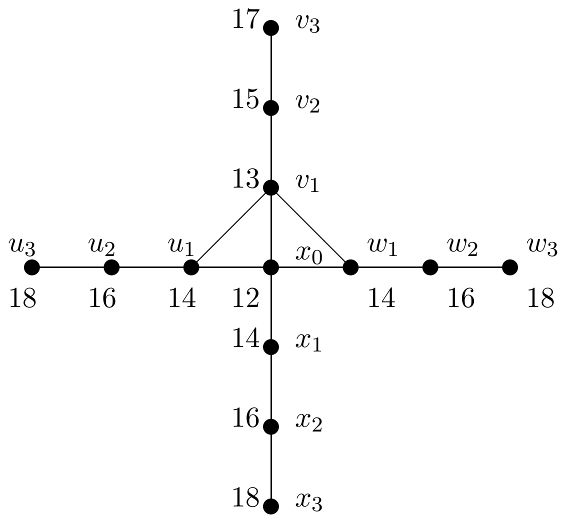

Figure 1. Graph with closed 3-stop eccentricities 12, 13, 14, 15, 16, 17, 18.

In any graph G, there is at least one vertex with closed 3-stop eccentricity $rad_3(G)$ and at least three vertices with closed 3-stop eccentricity $diam_3(G)$. From Proposition 3, we may conclude that, for any two consecutive integers k and $k+1$ with $rad_3(G) \leq k < diam_3(G)$, there must be a vertex with closed

3-stop eccentricity either k or $k+1$. In fact, for every pair of consecutive numbers between $rad_3(G)$ and $diam_3(G)$, there must be at least two vertices with closed 3-stop eccentricity equal to one of those numbers.

Proposition 4. *Let G be a connected graph and let k be an integer such that $rad_3(G) < k < diam_3(G) - 1$. Then there are at least two vertices in G with closed 3-stop eccentricity either k or $k+1$.*

Proof. Suppose to the contrary that $v \in V(G)$ is the only vertex with closed 3-stop eccentricity either k or $k+1$. Let $A = \{u \in V(G) | e_3(u) < k\}$ and $B = \{u \in V(G) | e_3(u) > k+1\}$. Notice that both A and B are non-empty and $A \cup B \cup \{v\} = V(G)$. Consider any $x \in A$ and $y \in B$. Since $e_3(x) \leq k-1$ and $e_3(y) \geq k+2$, it follows from Proposition 3 that any x-y path must contain a vertex with eccentricity either k or $k+1$. However, v is the only such vertex. Thus, v is a cut-vertex and A and B are not connected in $G - v$. Let w and y be vertices such that $e_3(v) = d_3(v, w, y)$. Since $e_3(w) \geq e_3(v)$ and $e_3(y) \geq e_3(v)$, both w and y must be in B. Now, let $u \in A$. Every path from u to w or y must go through v, so $e_3(u) \geq d_3(u, w, y) = 2d(u, v) + d_3(v, w, y) = 2d(u, v) + e_3(v)$. But this contradicts the fact that $e_3(u) < e_3(v)$. ■

In every example that we have found, there are at least three vertices with closed 3-stop eccentricity either k or $k+1$ for $rad_3(G) < k < diam_3(G) - 1$.

Conjecture 5. Let G be a connected graph and let k be an integer such that

$$rad_3(G) < k < diam_3(G) - 1.$$

Then there are at least three vertices in G with closed 3-stop eccentricity either k or $k+1$.

3. Closed k-stop Radius and Closed k-stop Diameter

In this section we study closed k-stop eccentricity. Proposition 1 can be generalized for $k \geq 4$.

Proposition 6. *Let G be a connected graph of order at least k, $k \in \mathbb{N}$. Then G has at least k vertices that are closed k-stop peripheral.*

Proof. Let $x_1 \in V(P_k(G))$. Then there exist vertices $x_2, x_3, \ldots, x_k \in V(G)$ such that $e_k(x_1) = d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_k, x_1) = e_k(x_2) = e_k(x_3) = \cdots = e_k(x_k)$. Thus $x_1, x_2, \ldots, x_k \in V(P_k(G))$. ■

Also, Proposition 2 can be generalized for $k = 4$.

Proposition 7. *For any connected graph G, we have*

$$rad_4(G) \leq diam_4(G) \leq \frac{4}{3} rad_4(G).$$

Proof. Let G be a connected graph. Suppose $u \in V(C_4(G))$ and $v \in V(P_4(G))$. Furthermore, suppose that $e_4(v)$ is attained by visiting w, x, and y, not necessarily in that order. We must have w, x, and $y \in V(P_4(G))$, and $e_4(v) = e_4(w) = e_4(x) = e_4(y) = d_4(\{v, w, x, y\})$.

Without loss of generality, we may assume that the minimum distance among $d(v, w)$, $d(v, x)$, $d(v, y)$, $d(w, x)$, $d(x, y)$, and $d(w, y)$ is $d(v, w)$. If we now distinguish v and w from x and y, we may assume, without loss of generality, that the distance from $\{v, w\}$ to $\{x, y\}$, that is, the minimum distance among $d(v, x)$, $d(v, y)$, $d(w, x)$, and $d(w, y)$, is $d(v, y)$. Thus, v is the vertex in common in these two distances. Now,

$$\begin{aligned}
&(4) \quad rad_4(G) = e_4(u) \\
&(5) \quad \geq d_4(u, w, x, y) \\
&(6) \quad = \min(d(u, w) + d(w, x) + d(x, y) + d(y, u), d(u, x) + d(x, w) \\
&(7) \quad + d(w, y) + d(y, u), d(u, w) + d(w, y) + d(y, x) + d(x, u)) \\
&(8) \quad \geq d(w, y) + d(w, x) + d(x, y).
\end{aligned}$$

The last inequality follows by applying the triangle inequality to each of terms in the minimum. Thus, $4rad_4(G) \geq 4d(w, y) + 4d(w, x) + 4d(x, y)$. On the other hand, $3diam_4(G) = 3e_4(v) = 3\min(d(v, w) + d(w, x) + d(x, y) + d(y, v), d(v, w) + d(w, y) + d(y, x) + d(x, v), d(v, x) + d(x, w) + d(w, y) + d(y, v)) \leq 3d(v, w) + 3d(w, x) + 3d(x, y) + 3d(y, v)$.

From our initial assumptions, $3d(v, w) \leq d(x, y) + 2d(w, y)$ and $3d(y, v) \leq d(w, x) + 2d(w, y)$. Thus, we have $3diam_4(G) \leq 3d(v, w) + 3d(w, x) + 3d(x, y) + 3d(y, v) \leq 4d(x, y) + 4d(w, x) + 4d(w, y) \leq 4rad_4(G)$. ■

Conjecture 8. For any integer $k \geq 2$ and any connected graph G, we have

$$rad_k(G) \leq diam_k(G) \leq \frac{k}{k-1} rad_k(G).$$

Notice that for $k = 2$, this conjecture reduces to the classical result for ordinary distance that $rad(G) \leq diam(G) \leq 2rad(G)$. We have shown that the conjecture is true for $k = 3$ and $k = 4$. However, for higher values of k, the proof would have to take into account the order of the eccentric vertices w, x, and y of the peripheral vertex v in the last step of equation 8. Suppose, for instance, that the vertices $v_1, v_2, \ldots, v_k$ are arranged so that the length of a closed walk is minimized, that is, $d(v_1, v_2) + d(v_2, v_3) + \cdots + d(v_{k-1}, v_k) + d(v_k, v_1)$ is as small as possible. If another vertex v is included, we may wonder if the minimum length closed walk for $\{v_1, v_2, \ldots, v_k, v\}$ can always be achieved by inserting v in some location in the list $v_1, v_2, \ldots, v_k$ or if the original vertices may also have to be rearranged. If $k \leq 3$, the minimum can always be achieved by simply inserting v. However, consider the example in Figure 2 for $k = 4$. A minimum closed walk containing $\{v_1, v_2, v_3, v_4\}$ has length 8 and visits these four vertices in order v_1, v_2, v_3, v_4, v_1 or in reverse order v_1, v_4, v_3, v_2, v_1. However, a minimum closed walk containing $\{v_1, v_2, v_3, v_4, v\}$ has length 11 and visits the vertices in one of the following orders: $v_1, v_3, v_2, v, v_4, v_1$, $v_1, v_3, v_4, v, v_2, v_1$, $v_1, v_2, v, v_4, v_3, v_1$, or $v_1, v_4, v, v_2, v_3, v_1$.

4. Closed k-stop Distance in Trees

In this section we study the closed k-stop distance in trees. We start with some observations and illustrations concerning closed k-stop distance.

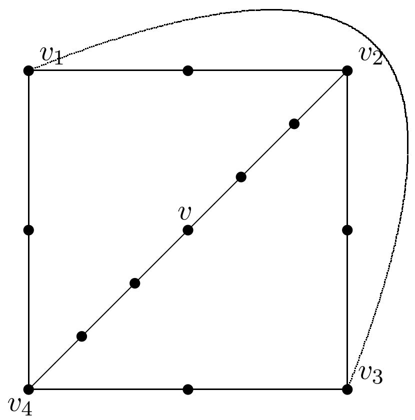

Figure 2. The shortest closed walk including v_1, v_2, v_3, v_4, v cannot be formed by inserting v into the shortest closed walk including v_1, v_2, v_3, v_4.

Proposition 9. *If G is a graph, and T is a spanning tree of G, then for any vertices $x_1, x_2, \ldots, x_k \in V(G)$, $d_k(\{x_1, x_2, \ldots, x_k\})$ in G is at most $d_k(\{x_1, x_2, \ldots, x_k\})$ in T.*

As a result of Proposition 9 we have that $rad_k(G) \leq rad_k(T)$ and $diam_k(G) \leq diam_k(T)$. For this reason we study trees next.

In a tree T, the upper inequalities (1), (2), and (3) actually become equalities, so $e_k(v) = 2se_k(v)$ for all $v \in V(T)$, $rad_k(T) = 2srad_k(T)$ and $diam_k(T) = 2sdiam_k(T)$, where the $srad_k(T)$ and $sdiam_k(T)$ are the Steiner radius and diameter, respectively. A closed walk containing a set of vertices traces every edge of a Steiner tree for those vertices twice. As a consequence, we have the following observation, also noted independently in [3].

Observation 10. *Let T be a tree and let $k \geq 2$ be an integer. Then $e_k(v)$ is even, for all $v \in V(T)$.*

For any positive integer $k \geq 2$ and connected graph G, the Steiner k-center of G, $sC_k(G)$, is the subgraph induced by the vertices v such that $se_k(v) = srad_k(G)$. Notice that since the Steiner distance of two vertices is simply the usual distance, $sC_2(G)$=$C(G)$. Oellermann and Tian found the following relationship between Steiner k-centers of trees.

Theorem 11 [7]**.** *Let $k \geq 3$ be an integer and T a tree of order greater than k. Then $sC_{k-1}(T) \subseteq sC_k(T)$.*

Similarly, the Steiner k-periphery of a graph G, $sP_k(G)$, is the subgraph induced by the vertices v such that $se_k(v) = sdiam_k(G)$. When $k = 2$, notice that $sP_2(G)$ is the usual periphery $P(G)$. Henning, Oellermann, and Swart found a relationship similar to the one above for the Steiner k-peripheries of trees.

Theorem 12 [4]**.** *Let $k \geq 3$ be an integer and T a tree of order greater than k. Then $sP_{k-1}(T) \subseteq sP_k(T)$.*

Since $rad_k(T) = 2srad_k(T)$ and $diam_k(T) = 2sdiam_k(T)$ for a tree T, we have $sC_k(T) = C_k(T)$ and $sP_k(T) = P_k(T)$. Thus, the results above produce the following corollary.

Corollary 13. *Let T be a tree of order n. Then $C(T) \subseteq C_3(T)$ and $P(T) \subseteq P_3(T)$. Furthermore, for any k with $3 \leq k \leq n$, we have $C_k(T) \subseteq C_{k+1}(T)$ and $P_k(T) \subseteq P_{k+1}(T)$.*

We next present the only tree that is closed 3-stop self-centered.

Proposition 14. *Let T be a tree. T is closed 3-stop self-centered if and only if $T \cong P_n$ $(n \geq 3)$.*

Proof. If $T \cong P_n$ $(n \geq 3)$, the result follows. For the converse, let $T \ncong P_n$ be a tree of order $n \geq 3$. Then T has three end-vertices $x, y, z \in V(P_3(T))$ such that $diam_3(T) = d_3(x, y, z)$. Let $x = x_0, x_1, \ldots, x_p = y$ be the geodesic from x to y in T. Then $e_3(x) = d(x, y) + d(y, z) + d(z, x)$, and $e_3(x_1) = d(x_1, y) + d(y, z) + d(z, x_1) < e_3(x)$, and so T is not closed 3-stop self-centered. ■

As a quick corollary of the above proof we have the following result.

Corollary 15. *Let T be a tree. T is closed 3-stop self-peripheral if and only if $T \cong P_n$ $(n \geq 3)$.*

As we have seen already, the path P_n has many special properties. The next result shows that P_n is the only tree that has the same closed k-stop eccentricity for each vertex and for any k with $1 \leq k \leq n-1$. This result follows as the path has only two end vertices and a unique path between them.

Proposition 16. *Let T be a tree of order n. Then $e_k(v) = 2n$, for all $v \in V(T)$, and for all $k \in \{1, 2, \ldots, n-1\}$ if and only if $T = P_n$, the path of order n.*

The following is a consequence of the Steiner distance in trees.

Proposition 17. *Let T be a tree and k an integer with $1 \leq k \leq n$. Then T has at most $k-1$ end vertices if and only if T is closed k-stop self-centered.*

Proof. Let T be a tree with at most $k-1$ end vertices, say they form the set $S = \{x_1, x_2, \ldots, x_{k-1}\}$, $k \geq 3$. Then for all $v \in V(G)$,

$$e_k(v) = \min_{\theta \in \mathcal{P}(\mathcal{S})} \Big(d(\theta(v), \theta(x_1)) + d(\theta(x_1), \theta(x_2)) + d(\theta(x_2), \theta(x_3)) + \cdots + d(\theta(x_{k-1}), \theta(v)) \Big),$$

where $\mathcal{P}(\mathcal{S})$ is the set of all permutations from $\mathcal{P}(\mathcal{S})$ onto $\mathcal{P}(\mathcal{S})$. Since T is a tree with $k-1$ end vertices, it follows that $e_k(v) = 2m$, $\forall v \in V(G)$.

For the converse, assume that T is closed k-stop self-centered, and assume to the contrary, that T has at least k end vertices, say $y_1, y_2, \ldots, y_t$, for $t \geq k \geq 3$. Let z_1 be the support vertex of y_1 and let $S = \{y_2, y_3, \ldots, y_{k-1}\}$, $k \geq 3$. Then

$$e_k(z_1) = \min_{\theta \in \mathcal{P}(\mathcal{S})} \Big(d(\theta(z_1), \theta(y_2)) + d(\theta(y_2), \theta(y_3)) + d(\theta(y_3), \theta(y_4)) + \cdots + d(\theta(y_{k-1}), \theta(z_1)) \Big),$$

where $\mathcal{P}(\mathcal{S})$ is the set of all permutations from $\mathcal{P}(\mathcal{S})$ onto $\mathcal{P}(\mathcal{S})$. However, $e_k(y_1) = 2 + e_k(z_1)$, which is a contradiction to T being closed k-stop self-centered. ■

As a quick corollary of the above proof we have the following result.

Corollary 18. *Let T be a tree and k an integer with $1 \leq k \leq n$. Then T has at most $k-1$ end vertices if and only if T is closed k-stop self-peripheral.*

5. Further Research

As seen in Section 3, Proposition 2 can be generalized for $k = 4$. The following conjecture was posed in Section 3.

Conjecture (Section 3): For any integer $k \geq 2$ and any connected graph G, we have

$$rad_k(G) \leq diam_k(G) \leq \frac{k}{k-1} rad_k(G).$$

Chartrand, Oellermann, Tian, and Zou showed a similar result for Steiner radius and diameter for trees.

Theorem 19 [2]**.** *If $k \geq 2$ is an integer and T is a tree of order at least k, then*

$$srad_k(T) \leq sdiam_k(T) \leq \frac{k}{k-1} srad_k(T).$$

Since $e_k(v) = 2se_k(v)$ for any vertex v in a tree, we have the corollary.

Corollary 20. *If $k \geq 2$ is an integer and T is a tree of order at least k, then*

$$rad_k(T) \leq diam_k(T) \leq \frac{k}{k-1} rad_k(T).$$

We have also been able to verify this conjecture for $k = 3$ and $k = 4$ for arbitrary connected graphs. As an interesting side note, Chartrand, Oellermann, Tian and Zou conjectured that $srad_k(G) \leq sdiam_k(G) \leq \frac{k}{k-1} srad(G)$ for any connected graph G [2]. This conjecture was disproven in [5], but our conjecture for closed k-stop distance holds for the class of graphs used as a counterexample to the Steiner conjecture.

We propose the extension of the study of centrality and eccentricity for closed k-stop distance in graphs for $k \geq 4$.

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